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On imbalances in oriented multipartite graphs

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Abstract. An oriented k-partite graph(multipartite graph) is the result of assigning a direction to each edge of a simple k-partite graph. Let $D(V_1, V_2, \cdots, V_k)$ be an oriented k-partite graph, and let $d^+_{v_{ij}}$ and $d^-_{v_{ij}}$ be respectively the outdegree and indegree of a vertex v_{ij} in V_i. Define $b_{v_{ij}}$ (or simply b_{ij} as $b_{ij} = d^+_{v_{ij}} - d^-_{v_{ij}}$ as the imbalance of the vertex v_{ij}. In this paper, we characterize the imbalances of oriented k-partite graphs and give a constructive and existence criteria for sequences of integers to be the imbalances of some oriented k-partite graph. Also, we show the existence of an oriented k-partite graph with the given imbalance set.

1 Introduction

A digraph without loops and without multi-arcs is called a simple digraph. Mubayi et al. [1] defined the imbalance of a vertex v_i in a digraph as b_{v_i} (or simply b_i) $= d^+_{v_i} - d^-_{v_i}$, where $d^+_{v_i}$ and $d^-_{v_i}$ are respectively the outdegree and indegree of v_i. The imbalance sequence of a simple digraph is formed by

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listing the vertex imbalances in non-increasing order. A sequence of integers $F = [f_1, f_2, \cdots, f_n]$ with $f_1 \geq f_2 \geq \cdots \geq f_n$ is feasible if it has sum zero and satisfies $\sum_{i=1}^{k} f_i \leq k(n-k)$, for $1 \leq k < n$.

The following result [1] provides a necessary and sufficient condition for a sequence of integers to be the imbalance sequence of a simple digraph.

Theorem 1 *A sequence is realizable as an imbalance sequence if and only if it is feasible.*

The above result is equivalent to saying that a sequence of integers $B = [b_1, b_2, \cdots, b_n]$ with $b_1 \geq b_2 \geq \cdots \geq b_n$ is an imbalance sequence of a simple digraph if and only if for $1 \leq k < n$

$$\sum_{i=1}^{k} b_i \leq k(n-k),$$

with equality when $k = n$.

On arranging the imbalance sequence in non-decreasing order, we have the following observation.

Theorem 2 *A sequence of integers* $B = [b_1, b_2, \cdots, b_n]$ *with* $b_1 \leq b_2 \leq \cdots \leq b_n$ *is an imbalance sequence of a simple digraph if and only if for* $1 \leq k < n$

$$\sum_{i=1}^{k} b_i \geq k(n-k),$$

with equality when $k = n$.

Various results for imbalances in digraphs and oriented graphs can be found in [2, 3, 4, 5].

2 Imbalance sequences in oriented multipartite graphs

An oriented multipartite (k-partite) graph is the result of assigning a direction to each edge of a simple multipartite (k-partite) graph, $k \geq 2$. Throughout this paper we denote an oriented k-partite graph by k-OG, unless otherwise stated. Let $V_i = \{v_{i1}, v_{i2}, \cdots, v_{in_i}\}$, $1 \leq i \leq k$, be k parts of k-OG $D(V_1, V_2, \cdots, V_k)$, and let $d^+_{v_{ij}}$ and $d^-_{v_{ij}}$, $1 \le j \le n_i$, be respectively the outdegree and indegree of a vertex v_{ij} in V_i. Define $b_{v_{ij}}$ (or simply b_{ij} as $b_{ij} = d^+_{v_{ij}} - d^-_{v_{ij}}$ as the imbalance of the vertex v_{ij}. The sequences $B_i = [b_{i1}, b_{i2}, \cdots, b_{in_i}]$, $1 \le i \le k$, in nondecreasing order are called the imbalance sequences of $D(V_1, V_2, \cdots, V_k)$.

The k sequences of integers $B_i = [b_{i1}, b_{i2}, \cdots, b_{in_i}]$, $1 \le i \le k$, in nondecreasing order are said to be realizable if there exists an k-OG with imbalance sequences B_i, $1 \le i \le k$. Various criterions for imbalance sequences in k-OG can be found in [2].

For any two vertices v_{ij} in V_i and v_{lm} in V_l ($i \neq l$, $1 \le i \le l \le k$, $1 \le j \le n_i$, $1 \le m \le n_l$) of k-OG $D(V_1, V_2, \cdots, V_k)$, we have one of the following possibilities.

(i). An arc directed from v_{ij} to v_{lm}, denoted by $v_{ij}(1-0)v_{lm}$.

(ii). An arc directed from v_{lm} to v_{ij}, denoted by $v_{ij}(0-1)v_{lm}$.

(iii). There is no arc from v_{ij} to v_{lm} and there is no arc from v_{lm} to v_{ij} and this is denoted by $v_{ij}(0-0)v_{lm}$.

A triple in k-OG is an induced suboriented graph of three vertices with exactly one vertex from each part. For any three vertices v_{ij}, v_{lm} and v_{pq} in k-OG D, the triples of the form $v_{ij}(1-0)v_{lm}(1-0)v_{pq}(1-0)v_{ij}$, or $v_{ij}(1-0)v_{lm}(1-0)v_{pq}(0-0)v_{ij}$ are said to be oriented intransitive, while as the triples of the form $v_{ij}(1-0)v_{lm}(1-0)v_{pq}(0-1)v_{ij}$, or $v_{ij}(1-0)v_{lm}(0-1)v_{pq}(0-0)v_{ij}$, or $v_{ij}(1-0)v_{lm}(0-0)v_{pq}(0-1)v_{ij}$, or $v_{ij}(1-0)v_{lm}(0-0)v_{pq}(0-0)v_{ij}$, or $v_{ij}(0-0)v_{lm}(0-0)v_{pq}(0-0)v_{ij}$ are said to be oriented transitive. An k-OG is said to be oriented transitive if all its triples are oriented transitive, otherwise oriented intransitive.

We have the following observation.

Theorem 3 *Let D and D'be two k-OG with the same imbalance sequences. Then D can be transformed to D' by successively transforming appropriate triples in one of the following ways. Either (a) by changing a cyclic triple $v_{ij}(1-0)v_{lm}(1-0)v_{pq}(1-0)v_{ij}$ to an oriented transitive triple $v_{ij}(0-0)v_{lm}(0-0)v_{pq}(0-0)v_{ij}$ which has the same imbalance sequences, or vice versa, or (b) by changing an oriented intransitive triple $v_{ij}(1-0)v_{lm}(1-0)v_{pq}(0-0)v_{ij}$ to an oriented transitive triple $v_{ij}(0-0)v_{lm}(0-0)v_{pq}(0-1)v_{ij}$ which has the same imbalance sequences, or vice versa.*

Proof. Let B_i be the imbalance sequences of k-OG D whose parts are V_i, $1 \le i \le k$ and $|V_i| = n_i$. Let D' be k-OG with parts V'_i, $1 \le i \le k$. To prove the result, it is sufficent to show that D' can be obtained from D by successively transforming triples in any one of the ways as given in (a), or (b).

We fix n_i, $2 \le i \le k$ and use induction on n_1. For $n_1 = 1$, the result is obvious. Assume that the result holds when there are fewer than n_1 vertices in the first part. Let $j_2, j_3, \cdots, j_k$ be such that for $l_2 > j_2, l_3 > j_3, \cdots, l_k > j_k$, $1 \le j_2 < l_2 \le n_2, 1 \le j_3 < l_3 \le n_3, \cdots, 1 \le j_k < l_k \le n_k$, the corresponding arcs have the same orientations in D and D′. For $j_2, j_3, \cdots, j_k$ and $2 \le i, p, q \le k$, $p \ne q$, we have three cases to consider.

(i). $v_{1n_1}(1-0)v_{ij_p}(1-0)v_{ij_q}$ and $v'_{1n_1}(0-0)v'_{ij_p}(0-0)v'_{ij_q}$, (ii). $v_{1n_1}(0-0)v_{ij_p}(0-1)v_{ij_q}$ and $v'_{1n_1}(1-0)v'_{ij_p}(0-0)v'_{ij_q}$ and (iii). $v_{1n_1}(1-0)v_{ij_p}(0-0)v_{ij_q}$ and $v'_{1n_1}(0-0)v'_{ij_p}(0-1)v'_{ij_q}$.

Case (i). Since v_{1n_1} and v'_{1n_1} have equal imbalances, we have $v_{1n_1}(0-1)v_{ij_q}$ and $v'_{1n_1}(0-0)v'_{ij_q}$, or $v_{1n_1}(0-0)v_{ij_q}$ and $v'_{1n_1}(1-0)v'_{ij_q}$. Thus there is a triple $v_{1n_1}(1-0)v_{ij_p}(1-0)v_{ij_q}(1-0)v_{1n_1}$, or $v_{1n_1}(1-0)v_{ij_p}(1-0)v_{ij_q}(0-0)v_{1n_1}$ in D, and corresponding to these $v'_{1n_1}(0-0)v'_{ij_p}(0-0)v'_{ij_q}(0-0)v'_{1n_1}$, or $v'_{1n_1}(0-0)v'_{ij_p}(0-0)v'_{ij_q}(0-1)v'_{1n_1}$ respectively is a triple in D′.

Case (ii). Since v_{1n_1} and v'_{1n_1} have equal imbalances, we have $v_{1n_1}(1-0)v_{ij_q}$ and $v'_{1n_1}(0-0)v'_{ij_q}$. Thus there is a triple $v_{1n_1}(0-0)v_{ij_p}(0-1)v_{ij_q}(0-1)v_{1n_1}$ in D and corresponding to this $v'_{1n_1}(1-0)v'_{ij_p}(0-0)v'_{ij_q}(0-0)v'_{1n_1}$ is a triple in D′.

Case (iii). Since v_{1n_1} and v'_{1n_1} have equal imbalances, therefore we have $v_{1n_1}(0-1)v_{ij_q}$ and $v'_{1n_1}(0-0)v'_{ij_q}$. Thus $v_{1n_1}(1-0)v_{ij_p}(0-0)v_{ij_q}(1-0)v_{1n_1}$ is a triple in D, and corresponding to this $v'_{1n_1}(0-0)v'_{ij_p}(0-1)v'_{ij_q}(0-0)v'_{1n_1}$ is a triple in D′.

Therefore from (i), (ii) and (iii) it follows that there is an k-OG that can be obtained from D by any one of the transformations (a) or (b) with the imbalances remaining unchanged. Hence the result follows by induction. □

Corollary 1 *Among all k-OG with given imbalance sequences, those with the fewest arcs are oriented transitive.*

A transmitter is a vertex with indegree zero. In a transitive oriented k-OG with imbalance sequences $B_i = [b_{i1}, b_{i2}, \cdots, b_{in_i}]$, $1 \le i \le k$, any of the vertices with imbalances b_{in_i}, can act as a transmitter.

The next result provides a useful recursive test of checking whether the sequences of integers are the imbalance sequences of k-OG.

Theorem 4 *Let* $B_i = [b_{i1}, b_{i2}, \cdots, b_{in_i}]$, $1 \le i \le k$, *be* k *sequences of integers in non-decreasing order with* $b_{1n_1} > 0$ *and* $b_{jn_j} \le \sum_{r=1, r\ne j}^{k} n_r$, *for all* j, $2 \le i \le k$. *Let* B'_1 *be obtained from* B_1 *by deleting one entry* b_{1n_1}, *and let*

$B'_2, B'_3, \cdots, B'_k$, *be obtained from* $B_2, B_3, \cdots, B_k$ *by increasing* b_{1n_1} *smallest entries of* $B_2, B_3, \cdots, B_k$ *by one each. Then* B_i *are imbalance sequences of some* k-*OG if and only if* B'_i *are imbalance sequences.*

Proof. Suppose B'_i be the imbalance sequences of some k-OG D' with parts V'_i, $1 \le i \le k$. Then k-OG D with imbalance sequences B_i can be obtained by adding a vertex v_{1n_1} in V'_1 such that $v_{1n_1}(1-0)v_{ij}$ for those vertices v_{ij} in V'_i, $i \neq 1$ whose imbalances are increased by one in going from B_i to B'_i.

Conversely, let B_i be the imbalance sequences of k-OG D with parts V_i, $1 \le i \le k$. By Corollary 4, any of the vertices v_{in_i} in V_i with imbalances b_{in_i}, $1 \le i \le k$ can be a transmitter. Assume that the vertex v_{1n_1} in V_1 with imbalance b_{1n_1} be a transmitter. Clearly, $d^+_{v_{1n_1}} > 0$ and $d^-_{v_{1n_1}} = 0$ so that $b_{1n_1} = d^+_{v_{1n_1}} - d^-_{v_{1n_1}} > 0$. Also, $d^+_{v_{jn_j}} \le \sum_{r=1, r\neq j}^{k} n_r$ and $d^-_{v_{jn_j}} \ge 0$ for $2 \le i \le k$ so that $b_{jn_j} = d^+_{v_{jn_j}} - d^-_{v_{jn_j}} \le \sum_{r=1, r\neq j}^{k} n_r$.

Let U be the set of v_{1n_1} vertices of smallest imbalances in V_j, $2 \le i \le k$ and let $W = V_2 \cup V_3 \cup \cdots \cup V_k - U$. Now construct D such that $v_{1n_1}(1-0)u$ for all u in U. Clearly $D - \{v_{1n_1}\}$ realizes V'_i, $1 \le i \le k$. $\square$

Theorem 5 provides an algorithm for determining whether or not the sequences B_i, $1 \le i \le k$ of integers in non-decreasing order are the imbalance sequences and for constructing a corresponding k-OG.

Suppose $B_i = [b_{i1}, b_{i2}, \cdots, b_{in_i}]$, $1 \le i \le k$, be imbalance sequences of k-OG with parts $V_i = \{v_{i1}, v_{i2}, \cdots, v_{in_i}\}$, where $b_{1n_1} > 0$ and $b_{jn_j} \le \sum_{r=1, r\neq j}^{k} n_r$, $2 \le i \le k$. Deleting b_{1n_1} and increasing b_{1n_1} smallest entries of $B_2, B_3, \cdots, B_k$ by 1 each to form $B'_2, B'_3, \cdots, B'_k$. Then arcs are defined by $v_{1n_1}(1-0)v_{ij}$, for which $b'_{v_{ij}} = b_{v_{ij}} + 1$, where $i \neq 1$. If at least one of the conditions $b_{1n_1} > 0$, or $b_{jn_j} \le \sum_{r=1, r\neq j}^{k} n_r$ does not hold, then we delete b_{in_i} for that i for which the conditions get satisfied and the same argument is used for defining arcs. If this method is applied recursively, then (i) it tests whether B_i are the imbalance sequences, and if B_i are the imbalance sequences (ii) k-OG $\triangle(B_i)$ with imbalance sequences B_i is constructed.

We illustrate this reduction and resulting construction as follows.

Consider the four sequences $B_1 = [1, 3, 4]$, $B_2 = [-3, 2, 2]$, $B_3 = [-4, -3]$ and $B_4 = [-3, 1]$.

(i) $[1, 3, 4]$, $[-3, 2, 2]$, $[-4, -3]$, $[-3, 1]$

(ii) $[1, 3]$, $[-2, 2, 2]$, $[-3, -2]$, $[-2, 1]$, $v_{13}(1-0)v_{21}$, $v_{13}(1-0)v_{31}$, $v_{13}(1-0)v_{32}$, $v_{13}(1-0)v_{41}$

(iii) $[1]$, $[-1, 2, 2]$, $[-2, -1]$, $[-2, 1]$, $v_{12}(1-0)v_{21}$, $v_{12}(1-0)v_{31}$, $v_{12}(1-0)v_{32}$

(iv) $\emptyset$, $[-1, 2, 2]$, $[-2, -1]$, $[-2, 1]$, $v_{11}(1-0)v_{31}$

(v) $\emptyset$, $[-1,2]$, $[0,-1]$, $[-1,1]$, $v_{23}(1-0)v_{31}$, $v_{23}(1-0)v_{41}$ or, $\emptyset$, $[-1,2]$, $[-1,0]$, $[-1,1]$

(vi)$\emptyset$, $[-1]$, $[0,0]$, $[0,1]$, $v_{22}(1-0)v_{32}$, $v_{22}(1-0)v_{41}$

(vii)$\emptyset$, $[0]$, $[0,0]$, $[0,0]$, $v_{42}(1-0)v_{21}$.

Clearly 4-OG with parts $V_1 = \{v_{11}, v_{12}, v_{13}\}$, $V_2 = \{v_{21}, v_{22}, v_{23}\}$, $V_3 = \{v_{31}, v_{32}\}$ and $V_4 = \{v_{41}, v_{42}\}$ in which $v_{13}(1-0)v_{21}$, $v_{13}(1-0)v_{31}$, $v_{13}(1-0)v_{32}$, $v_{13}(1-0)v_{41}$, $v_{12}(1-0)v_{21}$, $v_{12}(1-0)v_{31}$, $v_{12}(1-0)v_{32}$, $v_{11}(1-0)v_{31}$, $v_{23}(1-0)v_{31}$, $v_{23}(1-0)v_{41}$, $v_{22}(1-0)v_{32}$, $v_{22}(1-0)v_{41}$, $v_{42}(1-0)v_{21}$ are arcs has imbalance sequences $[1,3,4]$, $[-3,2,2]$, $[-4,-3]$ and $[-3,-1]$.

The next result gives a combinatorial criterion for determining whether k sequences of integers are realizable as imbalances.

Theorem 5 *Let* $B_i = [b_{i1}, b_{i2}, \cdots, b_{in_i}]$, $1 \le i \le k$, *be* k *sequences of integers in non-decreasing order. Then* B_i *are the imbalance sequences of some* k-*OG if and only if*

$$\sum_{i=1}^{k}\sum_{j=1}^{m_i} b_{ij} \ge 2\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} m_i m_j - \sum_{i=1}^{k} n_i \sum_{j=1}^{k} m_j - \sum_{i=1}^{k} m_i n_i, \qquad (1)$$

for all sets of k *integers* m_i, $0 \le m_i \le n_i$ *with equality when* $m_i = n_i$.

Proof. The necessity of the condition follows from the fact that the k-OG induced by m_i vertices for $1 \le i \le k$, $1 \le m_i \le n_i$ has a sum of imbalances $2\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} m_i m_j - \sum_{i=1}^{k} n_i \sum_{j=1}^{k} m_j - \sum_{i=1}^{k} m_i n_i$.

For sufficiency, assume that $B_i = [b_{i1}, b_{i2}, \cdots, b_{in_i}]$, $1 \le i \le k$ be the sequences of integers in non-decreasing order satisfying conditions (1) but are not the imbalance sequences of any k-OG. Let these sequences be chosen in such a way that n_i, $1 \le i \le k$ are the smallest possible and b_{11} is the least for the choice of n_i. We consider the following two cases.

Case (i). Suppose equality in (1) holds for some $m_j \le n_j$, $1 \le i \le k-1$, $m_k \le n_k$, so that

$$\sum_{i=1}^{k}\sum_{j=1}^{m_i} b_{ij} = 2\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} m_i m_j - \sum_{i=1}^{k} n_i \sum_{j=1}^{k} m_j - \sum_{i=1}^{k} m_i n_i.$$

By the minimality of n_i, $1 \le i \le k$ the sequences $B'_i = [b_{i1}, b_{i2}, \cdots, b_{im_i}]$ are the imbalance sequences of some k-OG $D'(V'_1, V'_2, \cdots, V'_k)$.

Define $B''_i = [b_{i(m_i+1)}, b_{i(m_i+2)}, \cdots, b_{i(n_i)}]$, $1 \le i \le k$.

Consider the sum

$$\begin{aligned}\sum_{i=1}^{k}\sum_{j=1}^{f_i} b_{i(m_i+j)} &= \sum_{i=1}^{k}\sum_{j=1}^{m_i+f_i} b_{ij} - \sum_{i=1}^{k}\sum_{j=1}^{m_i} b_{ij}\\
&\geq 2\sum_{i=1}^{k-1}\sum_{j=i+1}^{k}(m_i+f_i)(m_j+f_j) - \sum_{i=1}^{k} n_i \sum_{j=1}^{k}(m_j+f_j)\\
&- \sum_{i=1}^{k}(m_i+f_i)n_i - 2\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} m_i m_j + \sum_{i=1}^{k} n_i \sum_{j=1}^{k} m_j + \sum_{i=1}^{k} m_i n_i\\
&= 2\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} m_i m_j + 2\sum_{i=1}^{k-1}\sum_{j=i+1}^{k}(m_i f_j + f_i m_j + f_i f_j) - \sum_{i=1}^{k} n_i \sum_{j=1}^{k} f_j\\
&- \sum_{i=1}^{k} m_i n_i - \sum_{i=1}^{k} f_i n_i - 2\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} m_i m_j + \sum_{i=1}^{k} n_i \sum_{j=1}^{k} m_j +\\
&+ \sum_{i=1}^{k} m_i n_i \geq 2\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} f_i f_j - \sum_{i=1}^{k} n_i \sum_{j=1}^{k} f_j - \sum_{i=1}^{k} f_i n_i,\end{aligned}$$

for $1 \leq f_i \leq n_i - m_i$, with equality when $f_i = n_i - m_i$ for all i, $1 \leq i \leq k$. So by the minimality of n_i, $1 \leq i \leq k$, the sequences B''_i form the imbalance sequence of some k-OG $D''(V''_1, V''_2, \cdots, V''_k)$.

Construct a new k-OG $D(V_1, V_2, \cdots, V_k)$ as follows. Let $V_1 = V'_1 \cup V''_1, V_2 = V'_2 \cup V''_2, \cdots V_k = V'_k \cup V''_k$ with $V'_i \cap V''_i = \emptyset$ and the arc set containing those arcs which are among $V'_1, V'_2, \cdots, V'_k$ and among $V''_1, V''_2, \cdots, V''_k$. Then $D(V_1, V_2, \cdots, V_k)$ has imbalance sequences B_i, $1 \leq i \leq k$, which is a contradiction.

Case (ii). Assume that the strict inequality holds in (1) for some $m_i \neq n_i$, $1 \leq i \leq k$. Let $B'_1 = [b_{11}-1, b_{12}, \cdots, b_{1n_1-1}, b_{1n_1}]$ and let $B'_j = [b_{j1}, b_{j2}, \cdots, b_{jn_j}]$ for all j, $2 \leq j \leq k$. Clearly the sequences B'_i, $1 \leq i \leq k$ satisfy conditions (1). Therefore, by the minimality of b_{11}, the sequences B'_i, $1 \leq i \leq k$ are the imbalance sequences of some k-OG $D'(V'_1, V'_2, \cdots, V'_k)$. Let $b_{v_{11}} = b_{11} - 1$ and $b_{v_{1n_1}} = b_{1n_1} + 1$. Since $b_{v_{1n_1}} > b_{v_{11}} + 1$, there exists a vertex v_{ij} either in V_i, $1 \leq i \leq k$, $1 \leq j \leq n_i$, such that $v_{1n_1}(0-0)v_{ij}(1-0)v_{11}$, or $v_{1n_1}(1-0)v_{ij}(0-0)v_{11}$, or $v_{1n_1}(1-0)v_{ij}(1-0)v_{11}$, or $v_{1n_1}(0-0)v_{ij}(0-0)v_{11}$ in $D'(V'_1, V'_2, \cdots, V'_k)$, and if these are changed to $v_{1n_1}(0-1)v_{ij}(0-0)v_{11}$, or $v_{1n_1}(0-0)v_{ij}(0-1)v_{11}$, or $v_{1n_1}(0-0)v_{ij}(0-0)v_{11}$, or $v_{1n_1}(0-1)v_{ij}(0-1)v_{11}$ respectively, the result is k-OG with imbalance sequences B_i, which is a contradiction. This completes the proof. $\square$

3 Imbalance sets in oriented multipartite graphs

The set of distinct imbalances of the vertices in k-OG is called its imbalance set. Now we give the existence of k-OG with a given imbalance set.

Theorem 6 *Let* $S = \{s_1, s_2, \cdots, s_n\}$ *and* $T = \{-t_1, -t_2, \cdots, -t_n\}$*, where* $s_1, s_2, \cdots, s_n, t_1, t_2, \cdots, t_n$ *are positive integers with* $s_1 < s_2 < \cdots < s_n$ *and* $t_1 < t_2 < \cdots < t_n$*. Then there exists* k*-OG with imbalance set* $S \cup T$*.*

Proof. First assume that $k \geq 2$ is even. Construct k-OG $D(V_1, V_2, \cdots, V_k)$ as follows. Let

$$V_1 = V_{11} \cup V_{12} \cup \cdots \cup V_{1n},$$

$$V_2 = V_{21} \cup V_{22} \cup \cdots \cup V_{2n},$$

$$\cdots$$

$$V_k = V_{k1} \cup V_{k2} \cup \cdots \cup V_{kn},$$

with $V_{ij} \cap V_{lm} = \emptyset$, $|V_{ij}| = t_i$ for all odd i, $1 \leq i \leq k-1$, $1 \leq j \leq n$ and $|V_{ij}| = s_i$ for all even i, $2 \leq i \leq k$, $1 \leq j \leq n$. Let there be an arc from each vertex of V_{ij} to every vertex of $V_{(i+1)}j$ for all odd i, $1 \leq i \leq k-1$, $1 \leq j \leq n$ so that we obtain k-OG with imbalance of vertices as follows.

For odd i, $1 \leq i \leq k-1$ and $1 \leq j \leq n$

$$b_{v_{ij}} = |V_{(i+1)j}| - 0 = s_i,$$

for all $v_{ij} \in V_{ij}$; and for even i, $2 \leq i \leq k$ and $1 \leq j \leq n$

$$b_{v_{ij}} = 0 - |V_{(i+1)j}| = -t_i,$$

for all $v_{ij} \in V_{ij}$

Therefore imbalance set of $D(V_1, V_2, \cdots, V_k)$ is $S \cup T$.

Now assume $k \geq 3$ is odd. Construct k-OG $D(V_1, V_2, \cdots, V_k)$ as below. Let

$$V_1 = V_{11} \cup V'_{11} \cup V_{12} \cup V'_{12} \cup \ldots \cup V_{1n} \cup V'_{1n},$$
$$V_2 = V_{21} \cup V_{22} \cup \ldots \cup V_{2n},$$
$$\cdots$$
$$V_{k-1} = V_{(k-1)1} \cup V_{(k-1)2} \cup \ldots \cup V_{(k-1)n},$$
$$V_k = V'_{k1} \cup V'_{k2} \cup \ldots \cup V'_{kn},$$

with $V_{ij} \cap V_{lm} = \emptyset$, $V'_{ij} \cap V'_{lm} = \emptyset$, $V_{ij} \cap V'_{lm} = \emptyset$, $|V_{ij}| = t_i$ for all i, $1 \le i \le k-2$, $1 \le j \le n$, $|V_i j| = s_i$ for all even i, $2 \le i \le k-1$, $1 \le j \le n$, $|V'_{ij}| = t_i$ for all j, $1 \le j \le n$ and $|V'_{kj}| = s_j$ for all j, $1 \le j \le n$. Let there be an arc from each vertex of V_{ij} to every vertex of $V_{(i+1)j}$ for all i, $1 \le i \le k-2$, $1 \le j \le n$ and let there be an arc from each vertex of V'_{1j} to every vertex of V'_{kj} for all j, $1 \le j \le n$, so that we obtain k-OG with imbalance set $S \cup T$, as above. $\square$

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