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Some sequence spaces in 2-normed spaces defined by Musielak-Orlicz function

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Abstract. In the present paper we introduce some sequence spaces combining lacunary sequence, invariant means in 2-normed spaces defined by Musielak-Orlicz function $\mathcal{M} = (M_k)$. We study some topological properties and also prove some inclusion results between these spaces.

1 Introduction and preliminaries

The concept of 2-normed space was initially introduced by Gahler [2] as an interesting linear generalization of a normed linear space which was subsequently studied by many others see ([3], [9]). Recently a lot of activities have started to study sumability, sequence spaces and related topics in these linear spaces see ([4], [10]).

Let X be a real vector space of dimension d, where $2 \leq d < \infty$. A 2-norm on X is a function $\|.,.\| : X \times X \to \mathbb{R}$ which satisfies

(i) $\|x, y\| = 0$ if and only if x and y are linearly dependent

(ii) $\|x, y\| = \|y, x\|$

(iii) $\|\alpha x, y\| = |\alpha|\|x, y\|$, $\alpha \in \mathbb{R}$

(iv) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$ for all $x, y, z \in X$.

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The pair $(X, \|.,.\|)$ is then called a 2-normed space see [3]. For example, we may take $X = \mathbb{R}^2$ equipped with the 2-norm defined as $\|x, y\|$ = the area of the parallelogram spanned by the vectors x and y which may be given explicitly by the formula

$$\|x_1, x_2\|_E = \text{abs}\left(\begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} \right).$$

Then, clearly $(X, \|.,.\|)$ is a 2-normed space. Recall that $(X, \|.,.\|)$ is a 2-Banach space if every cauchy sequence in X is convergent to some x in X.

Let σ be the mapping of the set of positive integers into itself. A continuous linear functional φ on l_∞,is said to be an invariant mean or σ-mean if and only if

(i) $\varphi(x) \geq 0$ when the sequence $x = (x_k)$ has $x_k \geq 0$ for all k,

(ii) $\varphi(e) = 1$, where $e = (1, 1, 1, \dots)$ and

(iii) $\varphi(x_{\sigma(k)}) = \varphi(x)$ for all $x \in l_\infty$.

If $x = (x_n)$, write $Tx = Tx_n = (x_{\sigma(n)})$. It can be shown in [11] that

$$V_\sigma = \left\{ x \in l_\infty \mid \lim_k t_{kn}(x) = l, \ \text{ uniformly } in \ n, \ l = \sigma - \lim x \right\},$$

where

$$t_{kn}(x) = \frac{x_n + x_{\sigma^1 n} + \dots + x_{\sigma^k n}}{k+1}.$$

In the case σ is the translation mapping $n \to n+1$, σ-mean is often called a Banach limit and V_σ, the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences see [6].

By a lacunary sequence $\theta = (k_r)$ where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$. We write $h_r = k_r - k_{r-1}$. The ratio $\dfrac{k_r}{k_{r-1}}$ will be denoted by q_r. The space of lacunary strongly convergent sequence was defined in [1].

Let X be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if

(i) $p(x) \geq 0$, for all $x \in X$

(ii) $p(-x) = p(x)$, for all $x \in X$

(iii) $p(x + y) \leq p(x) + p(y)$, for all $x, y \in X$

(iv) if (σ_n) is a sequence of scalars with $\sigma_n \to \sigma$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\sigma_n x_n - \sigma x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [12], Theorem 10.4.2, P-183).

An orlicz function $M : [0, \infty) \to [0, \infty)$ is a continuous, non-decreasing and convex function such that $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.

Lindenstrauss and Tzafriri [5] used the idea of Orlicz function to define the following sequence space. Let w be the space of all real or complex sequences $x = (x_k)$, then

$$l_M = \left\{ x \in w \mid \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \right\}$$

which is called a Orlicz sequence space. Also l_M is a Banach space with the norm

$$\|x\| = \inf\left\{ \rho > 0 \mid \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}.$$

Also, it was shown in [5] that every Orlicz sequence space l_M contains a subspace isomorphic to $l_p (p \geq 1)$. The Δ_2- condition is equivalent to $M(Lx) \leq LM(x)$, for all L with $0 < L < 1$. An Orlicz function M can always be represented in the following integral form

$$M(x) = \int_0^x \eta(t) dt$$

where η is known as the kernel of M, is right differentiable for $t \geq 0, \eta(0) = 0, \eta(t) > 0$, η is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$.

A sequence $\mathcal{M} = (M_k)$ of Orlicz function is called a Musielak-Orlicz function see ([7], [8]). A sequence $\mathcal{N} = (N_k)$ is called a complementary function of a Musielak-Orlicz function $\mathcal{M}$

$$N_k(v) = \sup\left\{ |v|u - M_k \mid u \geq 0 \right\},\ k = 1, 2, \ldots$$

For a given Musielak-Orlicz function $\mathcal{M}$, the Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows

$$\begin{aligned} t_{\mathcal{M}} &= \left\{ x \in w \mid I_{\mathcal{M}}(cx) < \infty, \text{ for some } c > 0 \right\}, \\ h_{\mathcal{M}} &= \left\{ x \in w \mid I_{\mathcal{M}}(cx) < \infty, \text{ for all } c > 0 \right\}, \end{aligned}$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k), x = (x_k) \in t_{\mathcal{M}}.$$

We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$\|x\| = \inf\left\{k > 0 \mid I_{\mathcal{M}}\left(\frac{x}{k}\right) \leq 1\right\}$$

or equipped with the Orlicz norm

$$\|x\|^0 = \inf\left\{\frac{1}{k}\left(1 + I_{\mathcal{M}}(kx)\right) \mid k > 0\right\}.$$

Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $(X, \|., .\|)$ be a 2-normed space and $p = (P_k)$ be any sequence of strictly positive real numbers. By $S(2-X)$ we denote the space of all sequences defined over $(X, \|., .\|)$. We now define the following sequence spaces:

$$w_{\sigma}^{o}[\mathcal{M}, p, \|., .\|]_{\theta} = \left\{x \in S(2-X) \mid \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \left[M_k\left(\left\|\frac{t_{kn}(x)}{\rho}, z\right\|\right)\right]^{p_k} = 0,\right.$$
$$\left.\rho > 0, \text{ uniformly in } n\right\},$$

$$w_{\sigma}[\mathcal{M}, p, \|., .\|]_{\theta} = \left\{x \in S(2-X) \mid \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \left[M_k\left(\left\|\frac{t_{kn}(x-l)}{\rho}, z\right\|\right)\right]^{p_k} = 0,\right.$$
$$\left.\rho > 0, \text{ uniformly in } n\right\}, \text{ and}$$

$$w_{\sigma}^{\infty}[\mathcal{M}, p, \|., .\|]_{\theta} = \left\{x \in S(2-X) \mid \sup_{r,n} \frac{1}{h_r} \sum_{k\in I_r} \left[M_k\left(\left\|\frac{t_{kn}(x)}{\rho}, z\right\|\right)\right]^{p_k} < \infty,\right.$$
$$\left.\text{for some } \rho > 0\right\}.$$

When $\mathcal{M}(x) = x$ for all k, the spaces $w_{\sigma}^{o}\big[\mathcal{M}, p, \|., .\|\big]_{\theta}$, $w_{\sigma}\big[\mathcal{M}, p, \|., .\|\big]_{\theta}$ and $w_{\sigma}^{\infty}\big[M_k, p, \|., .\|\big]_{\theta}$ reduces to the spaces $w_{\sigma}^{o}\big[p, \|., .\|\big]_{\theta}$, $w_{\sigma}\big[, p, \|., .\|\big]_{\theta}$ and $w_{\sigma}^{\infty}\big[p, \|., .\|\big]_{\theta}$ respectively.

If $p_k = 1$ for all k, the spaces $w_{\sigma}^{o}\big[\mathcal{M}, p, \|., .\|\big]_{\theta}$, $w_{\sigma}\big[\mathcal{M}, p, \|., .\|\big]_{\theta}$ and $w_{\sigma}^{\infty}\big[\mathcal{M}, p, \|., .\|\big]_{\theta}$ reduces to $w_{\sigma}^{o}\big[\mathcal{M}, \|., .\|\big]_{\theta}$, $w_{\sigma}\big[\mathcal{M}, \|., .\|\big]_{\theta}$ and $w_{\sigma}^{\infty}\big[\mathcal{M}, \|., .\|\big]_{\theta}$ respectively.

The following inequality will be used throughout the paper. If $0 \leq p_k \leq \sup p_k = H$, $K = \max(1, 2^{H-1})$ then

$$|a_k + b_k|^{p_k} \leq K\{|a_k|^{p_k} + |b_k|^{p_k}\} \tag{1}$$

for all k and $a_k, b_k \in \mathbb{C}$. Also $|a|^{p_k} \leq \max(1, |a|^H)$ for all $a \in \mathbb{C}$.

In the present paper we study some topological properties of the above sequence spaces.

2 Main results

Theorem 1 *Let $\mathcal{M} = (M_k)$ be Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers, then the classes of sequences $w^o_\sigma[\mathcal{M}, p, \|.,.\|]_\theta$, $w_\sigma[\mathcal{M}, p, \|.,.\|]_\theta$ and $w^\infty_\sigma[\mathcal{M}, p, \|.,.\|]_\theta$ are linear spaces over the field of complex numbers.*

Proof. Let $x, y \in w^o_\sigma[\mathcal{M}, p, \|.,.\|]_\theta$ and $\alpha, \beta \in \mathbb{C}$. In order to prove the result we need to find some ρ_3 such that

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \left[M_k\left(\left\|\frac{t_{kn}(\alpha x + \beta y)}{\rho_3}, z\right\|\right)\right]^{p_k} = 0, \text{ uniformly in } n.$$

Since $x, y \in w^o_\sigma[\mathcal{M}, p, \|.,.\|]_\theta$, there exist positive ρ_1, ρ_2 such that

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \left[M_k\left(\left\|\frac{t_{kn}(x)}{\rho_1}, z\right\|\right)\right]^{p_k} = 0, \text{ uniformly in } n$$

and

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \left[M_k\left(\left\|\frac{t_{kn}(y)}{\rho_2}, z\right\|\right)\right]^{p_k} = 0, \text{ uniformly in } n.$$

Define $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since (M_k) is non-decreasing and convex

$$\begin{aligned}
\frac{1}{h_r} \sum_{k\in I_r} \left[M_k\left(\left\|\frac{t_{kn}(\alpha x + \beta y)}{\rho_3}, z\right\|\right)\right]^{p_k} &\leq \frac{1}{h_r} \sum_{k\in I_r} \left[M_k\left(\left\|\frac{t_{kn}(\alpha x)}{\rho_3}, z\right\| + \left\|\frac{t_{kn}(\beta y)}{\rho_3}, z\right\|\right)\right] \\
&\leq \frac{1}{h_r} \sum_{k\in I_r} \left[M_k\left(\left\|\frac{t_{kn}(x)}{\rho_1}, z\right\| + \left\|\frac{t_{kn}(y)}{\rho_2}, z\right\|\right)\right] \\
&\leq K\frac{1}{h_r} \sum_{k\in I_r} \left[M_k\left(\left\|\frac{t_{kn}(x)}{\rho_1}, z\right\|\right)\right] + \\
&\quad + K\frac{1}{h_r} \sum_{k\in I_r} \left[M_k\left(\left\|\frac{t_{kn}(y)}{\rho_2}, z\right\|\right)\right] \\
&\to 0 \text{ as } r \to \infty, \text{ uniformly in } n.
\end{aligned}$$

So that $\alpha x + \beta y \in w^{o}_{\sigma}\big[\mathcal{M}, p, \|., .\|\big]_{\theta}$. This completes the proof. Similarly, we can prove that $w_{\sigma}\big[\mathcal{M}, p, \|., .\|\big]_{\theta}$ and $w^{\infty}_{\sigma}\big[\mathcal{M}, p, \|., .\|\big]_{\theta}$ are linear spaces. □

Theorem 2 *Let $\mathcal{M} = (M_k)$ be Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers. Then $w^{o}_{\sigma}\big[\mathcal{M}, p, \|., .\|\big]_{\theta}$ is a topological linear spaces paranormed by*

$$g(x) = \inf\left\{\rho^{\frac{p_r}{H}} : \left(\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\left\|\frac{t_{kn}(x)}{\rho}, z\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq 1, r = 1, 2, \ldots,\ \ n = 1, 2, \ldots\right\},$$

where $H = \max(1, \sup_k p_k < \infty)$.

Proof. Clearly $g(x) \geq 0$ for $x = (x_k) \in w^{o}_{\sigma}\big[\mathcal{M}, p, \|., .\|\big]_{\theta}$. Since $M_k(0) = 0$, we get $g(0) = 0$.

Conversely, suppose that $g(x) = 0$, then

$$\inf\left\{\rho^{\frac{p_r}{H}} : \left(\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\left\|\frac{t_{kn}(x)}{\rho}, z\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq 1, r \geq 1,\ n \geq 1\right\} = 0.$$

This implies that for a given $\epsilon > 0$, there exists some $\rho_\epsilon (0 < \rho_\epsilon < \epsilon)$ such that

$$\left(\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\left\|\frac{t_{kn}(x)}{\rho_\epsilon}, z\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq 1.$$

Thus

$$\left(\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\left\|\frac{t_{kn}(x)}{\epsilon}, z\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq \left(\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\left\|\frac{t_{kn}(x)}{\rho_\epsilon}, z\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq 1,$$

for each r and n. Suppose that $x_k \neq 0$ for each $k \in N$. This implies that $t_{kn}(x) \neq 0$, for each $k, n \in N$. Let $\epsilon \to 0$, then $\left\|\frac{t_{kn}(x)}{\epsilon}, z\right\| \to \infty$. It follows that

$$\left(\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\left\|\frac{t_{kn}(x)}{\epsilon}, z\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}} \to \infty$$

which is a contradiction.

Therefore, $t_{kn}(x) = 0$ for each k and thus $x_k = 0$ for each $k \in \mathbb{N}$. Let $\rho_1 > 0$ and $\rho_2 > 0$ be such that

$$\left(\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\left\|\frac{t_{kn}(x)}{\rho_1}, z\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq 1$$

and

$$\left(\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\left\|\frac{t_{kn}(y)}{\rho_2}, z\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq 1$$

for each r . Let $\rho = \rho_1 + \rho_2$. Then, we have

$$\begin{aligned}
&\left(\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\left\|\frac{t_{kn}(x+y)}{\rho}, z\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}} \\
&\quad \leq \left(\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\left\|\frac{t_{kn}(x)+t_{kn}(y)}{\rho_1+\rho_2}, z\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}} \\
&\quad \leq \left(\frac{1}{h_r}\sum_{k\in I_r}\left[\frac{\rho_1}{\rho_1+\rho_2}M_k\left(\left\|\frac{t_{kn}(x)}{\rho_1}, z\right\|\right) + \frac{\rho_2}{\rho_1+\rho_2}M_k\left(\left\|\frac{t_{kn}(y)}{\rho_2}, z\right\|\right)\right]^{p^k}\right)^{\frac{1}{H}} \\
&\quad \leq \left(\frac{\rho_1}{\rho_1+\rho_2}\right)\left(\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\left\|\frac{t_{kn}(x)}{\rho_1}, z\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}} \\
&\quad + \left(\frac{\rho_2}{\rho_1+\rho_2}\right)\left(\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\left\|\frac{t_{kn}(y)}{\rho_2}, z\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq 1.
\end{aligned}$$

(by Minkowski's inequality)

Since $\rho's$ are non-negative, so we have

$$g(x+y) =$$

$$= \inf\left\{\rho^{\frac{p_r}{H}} \mid \left(\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\left\|\frac{t_{kn}(x)+t_{kn}(y)}{\rho}, z\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}} \le 1, r\ge 1, n\ge 1\right\}$$

$$\le \inf\left\{\rho_1^{\frac{p_r}{H}} \mid \left(\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\left\|\frac{t_{kn}(x)}{\rho_1}, z\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}} \le 1, r\ge 1, n\ge 1\right\} +$$

$$+ \inf\left\{\rho_2^{\frac{p_r}{H}} \mid \left(\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\left\|\frac{t_{kn}(x)}{\rho_2}, z\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}} \le 1, r\ge 1, n\ge 1\right\}.$$

Therefore,

$$g(x+y) \le g(x) + g(y).$$

Finally, we prove that the scalar multiplication is continuous. Let λ be any complex number. By definition,

$$g(\lambda x)=\inf\left\{\rho^{\frac{p_r}{H}} \mid \left(\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\left\|\frac{t_{kn}(\lambda x)}{\rho}, z\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}} \le 1, r\ge 1, n\ge 1\right\}.$$

Then

$$g(\lambda x)= \inf\left\{(|\lambda| t)^{\frac{p_r}{H}} \mid \left(\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\left\|\frac{t_{kn}(x)}{t}, z\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}} \le 1, r\ge 1, n\ge 1\right\}.$$

where $t = \dfrac{\rho}{|\lambda|}$. Since $|\lambda|^{p_r} \le \max(1, |\lambda|^{\sup p_r})$, we have

$$\begin{aligned} g(\lambda x) \;\le\; & \max(1, |\lambda|^{\sup p_r}) \\ & \inf\left\{t^{\frac{p_r}{H}} \mid \left(\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\left\|\frac{t_{kn}(x)}{t}, z\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}} \le 1, r\ge 1, n\ge 1\right\}. \end{aligned}$$

So, the fact that scalar multiplication is continuous follows from the above inequality.

This completes the proof of the theorem. □

Theorem 3 *Let $\mathcal{M} = (M_k)$ be Musielak-Orlicz function. If*

$$\sup_k \left[M_k(t)\right] p_k < \infty \text{ for all } t > 0,$$

then

$$w_\sigma \left[\mathcal{M}, p, \|.,.\|\right]_\theta \subset w_\sigma^\infty \left[\mathcal{M}, p, \|.,.\|\right]_\theta .$$

Proof. Let $x \in w_\sigma\left[\mathcal{M}, p, \|.,.\|\right]_\theta$. By using inequality (1), we have

$$\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\left\|\frac{t_{kn}(x)}{\rho}, z\right\|\right)\right]^{p_k} \le \frac{K}{h_r}\sum_{k\in I_r}\left[M_k\left(\left\|\frac{t_{kn}(x-l)}{\rho}, z\right\|\right)\right]^{p_k} + \frac{K}{h_r}\sum_{k\in I_r}\left[M_k\left(\left\|\frac{l}{\rho}, z\right\|\right)\right]^{p_k}.$$

Since $\sup_k \left[M_k(t)\right]^{p_k} < \infty$, we can take that $\sup_k \left[M_k(t)\right]^{p_k} = T$. Hence we get $x \in w_\sigma^\infty\left[\mathcal{M}, p, \|.,.\|\right]_\theta$. □

Theorem 4 *Let $\mathcal{M} = (M_k)$ be Musielak-Orlicz function which satisfies Δ_2-condition for all k, then*

$$w_\sigma \left[p, \|.,.\|\right]_\theta \subset w_\sigma \left[\mathcal{M}, p, \|.,.\|\right]_\theta .$$

Proof. Let $x \in w_\sigma \left[p, \|.,.\|\right]_\theta$. Then we have

$$\mathcal{T}_r = \frac{1}{h_r}\sum_{k\in I_r}\|t_{kn}(x-l), z\|^{p_k} \to \infty \text{ as } r \to \infty \text{ uniformly in } n, \text{ for some } l.$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M_k(t) < \epsilon$ for $0 \le t \le \delta$ for all k. So that

$$\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\left\|\frac{t_{kn}(x-l)}{\rho}, z\right\|\right)\right]^{p_k} = \frac{1}{h_r}\sum^{1}_{\substack{k \in I_r \\ \|t_{kn}(x-l)z\| \le \delta}}\left[M_k\left(\left\|\frac{t_{kn}(x-l)}{\rho}, z\right\|\right)\right]^{p_k} + \frac{1}{h_r}\sum^{2}_{\substack{k \in I_r \\ \|t_{kn}(x-l)z\| \le \delta}}\left[M_k\left(\left\|\frac{t_{kn}(x-l)}{\rho}, z\right\|\right)\right]^{p_k}.$$

For the first summation in the right hand side of the above equation, we have $\sum^{1} \le \epsilon^H$ by using continuity of M_k for all k. For the second summation, we write

$$\|t_{kn}(x-l), z\| \le 1 + \|\frac{t_{kn}(x-l)}{\delta}, z\|.$$

Since M_k is non-decreasing and convex for all k, it follows that

$$\begin{aligned} M_k(\|t_{kn}(x-l), z\|) &< M_k\left(1 + \left\|\frac{t_{kn}(x-l)}{\delta}, z\right\|\right) \\ &\leq \frac{1}{2}M_k(2) + \frac{1}{2}M_k\left((2)\left\|\frac{t_{kn}(x-l)}{\delta}, z\right\|\right). \end{aligned}$$

Since M_k satisfies Δ_2-condition for all k, we can write

$$\begin{aligned} M_k\Big(\|t_{kn}(x-l), z\|\Big) &\leq \frac{1}{2}L\left\|\frac{t_{kn}(x-l)}{\delta}, z\right\|M_k(2) + \frac{1}{2}L\left\|\frac{t_{kn}(x-l)}{\delta}, z\right\|M_k(2) \\ &= L\left\|\frac{t_{kn}(x-l)}{\delta}, z\right\|M_k(2). \end{aligned}$$

So we write

$$\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\left\|\frac{t_{kn}(x-l)}{\rho}, z\right\|\right)\right]^{p_k} \leq \epsilon^H + [\max(1, LM_k(2))\delta]^H \mathcal{T}_r.$$

Letting $r \to \infty$, it follows that $x \in w_\sigma\big[\mathcal{M}, p, \|.,.\|\big]_\theta$.
This completes the proof. $\square$

Theorem 5 *Let $\mathcal{M} = (M_k)$ be Musielak-Orlicz function. Then the following statements are equivalent:*

(i) $w_\sigma^\infty\big[p, \|.,.\|\big]_\theta \subset w_\sigma^o\big[\mathcal{M}, p, \|.,.\|\big]_\theta$,

(ii) $w_\sigma^o\big[p, \|.,.\|\big]_\theta \subset w_\sigma^o\big[\mathcal{M}, p, \|.,.\|\big]_\theta$,

(iii) $\sup_r \frac{1}{h_r}\sum_{k\in I_r}\big[M_k(t)\big]^{p_k} < \infty$ *for all* $t > 0$.

Proof. (i) $\Longrightarrow$ (ii) We have only to show that $w_\sigma^o\big[p, \|.,.\|\big]_\theta \subset w_\sigma^\infty\big[p, \|.,.\|\big]_\theta$. Let $x \in w_\sigma^o\big[p, \|.,.\|\big]_\theta$. Then there exists $r \geq r_o$, for $\epsilon > 0$, such that

$$\frac{1}{h_r}\sum_{k\in I_r}\left\|\frac{t_{kn}(x)}{\rho}, z\right\|^{p_k} < \epsilon.$$

Hence there exists $H > 0$ such that

$$\sup_{r,n}\frac{1}{h_r}\sum_{k\in I_r}\left\|\frac{t_{kn}(x)}{\rho}, z\right\|^{p_k} < H$$

for all n and r. So we get $x \in w_\sigma^\infty\left[p, \|.,.\|\right]_\theta$.

(ii) $\Longrightarrow$ (iii) Suppose that (iii) does not hold. Then for some $t > 0$

$$\sup_r \frac{1}{h_r} \sum_{k \in I_r} [M_k(t)]^{p_k} = \infty$$

and therefore we can find a subinterval $I_{r(m)}$ of the set of interval I_r such that

$$\frac{1}{h_{r(m)}} \sum_{k \in I_{r(m)}} \left[M_k(\frac{1}{m})\right]^{p_k} > m, \quad m = 1, 2, \tag{2}$$

Let us define $x = (x_k)$ as follows, $x_k = \frac{1}{m}$ if $k \in I_{r(m)}$ and $x_k = 0$ if $k \notin I_{r(m)}$. Then $x \in w_\sigma^o\left[p, \|.,.\|\right]_\theta$ but by eqn. (2), $x \notin w_\sigma^\infty\left[\mathcal{M}, p, \|.,.\|\right]_\theta$. which contradicts (ii). Hence (iii) must hold. (iii) $\Longrightarrow$ (i) Suppose (i) not holds, then for $x \in w_\sigma^\infty\left[p, \|.,.\|\right]_\theta$, we have

$$\sup_{r,n} \frac{1}{h_r} \sum_{k \in I_r} \left[M_k\left(\left\|\frac{t_{kn}(x)}{\rho}, z\right\|\right)\right]^{p_k} = \tag{3}$$

Let $t = \left\|\frac{t_{kn}(x)}{\rho}, z\right\|$ for each k and fixed n, so that eqn. (3) becomes

$$\sup_r \frac{1}{h_r} \sum_{k \in I_r} [M_k(t)]^{p_k} = \infty$$

which contradicts (iii). Hence (i) must hold. $\square$

Theorem 6 *Let $\mathcal{M} = (M_k)$ be Musielak-Orlicz function. Then the following statements are equivalent:*

(i) $w_\sigma^o\left[\mathcal{M}, p, \|.,.\|\right]_\theta \subset w_\sigma^o\left[p, \|.,.\|\right]_\theta$,

(ii) $w_\sigma^o\left[\mathcal{M}, p, \|.,.\|\right]_\theta \subset w_\sigma^\infty\left[p, \|.,.\|\right]_\theta$,

(iii) $\inf_r \sum_{k \in I_r} \left[M_k(t)\right]^{p_k} > 0$ *for all* $t > 0$.

Proof. (i) $\Longrightarrow$ (ii) : It is easy to prove.

(ii) $\Longrightarrow$ (iii) Suppose that (iii) does not hold. Then

$$\inf_r \frac{1}{h_r} \sum_{k \in I_r} [M_k(t)]^{p_k} = 0 \;\; \text{for some} \;\; t > 0,$$

and we can find a subinterval $I_{r(m)}$ of the set of interval I_r such that

$$\frac{1}{h_r}\sum_{k\in I_{r(m)}}[M_k(m)]^{p_k} < \frac{1}{m}, \quad m = 1, 2, \dots \tag{4}$$

Let us define $x_k = m$ if $k \in I_{r(m)}$ and $x_k = 0$ if $k \notin I_{r(m)}$. Thus by eqn.(4), $x \in w_\sigma^o\big[\mathcal{M}, p, \|., .\|\big]_\theta$ but $x \notin w_\sigma^\infty\big[p, \|., .\|\big]_\theta$ which contradicts (ii). Hence (iii) must hold.

(iii) $\Longrightarrow$ (i) It is obvious. □

Theorem 7 *Let $\mathcal{M} = (M_k)$ be Musielak-Orlicz function. Then $w_\sigma^\infty\big[\mathcal{M}, p, \|., .\|\big]_\theta \subset w_\sigma^o\big[p, \|., .\|\big]_\theta$ if and only if*

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\big[M_k(t)\big]^{p_k} = \infty \tag{5}$$

Proof. Let $w_\sigma^\infty\big[\mathcal{M}, p, \|., .\|\big]_\theta \subset w_\sigma^o\big[p, \|., .\|\big]_\theta$. Suppose that eqn. (5) does not hold. Therefore there is a subinterval $I_{r(m)}$ of the set of interval I_r and a number $t_o > 0$, where $t_o = \left|\frac{t_{kn}(x)}{\rho}, z\right|$ for all k and n, such that

$$\frac{1}{h_{r(m)}}\sum_{k\in I_{r(m)}}[M_k(t_o)]^{p_k} \le M < \infty, \quad m = 1, 2, \dots \tag{6}$$

Let us define $x_k = t_o$ if $k \in I_{r(m)}$ and $x_k = 0$ if $k \notin I_{r(m)}$. Then, by eqn. (6), $x \in w_\sigma^\infty\,[M_k, p, \|., .\|]_\theta$. But $x \notin w_\sigma^o\,[p, \|., .\|]_\theta$. Hence eqn. (5) must hold.

Conversely, suppose that eqn. (5) hold and that $x \in w_\sigma^\infty\,[M_k, p, \|., .\|]_\theta$. Then for each r and n

$$\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\left\|\frac{t_{kn}(x)}{\rho}, z\right\|\right)\right]^{p_k} \le M < \infty. \tag{7}$$

Now suppose that $x \notin w_\sigma^o\,[p, \|., .\|]_\theta$. Then for some number $\epsilon > 0$ and for a subinterval I_{ri} of the set of interval I_r, there is k_o such that $\|t_{kn}(x), z\|^{p_k} > \epsilon$ for $k \ge k_o$. From the properties of sequence of Orlicz functions, we obtain

$$\left[M_k\left(\frac{\epsilon}{\rho}\right)\right]^{p_k} \le \left[M_k\left(\left\|\frac{t_{kn}(x)}{\rho}, z\right\|\right)\right]^{p_k}$$

which contradicts eqn.(6), by using eqn. (7). This completes the proof. □

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