Acta Univ. Sapientiae, Mathematica, **4**, 1 (2012) 11–19

Balancing diophantine triples

Murat Alp
Niğde University,
Department of Mathematics,
Niğde, Turkey
email: muratalp@nigde.edu.tr

Nurettin Irmak
Niğde University,
Department of Mathematics,
Niğde, Turkey
email: nirmak@nigde.edu.tr

László Szalay
University of West Hungary,
Institute of Mathematics,
Sopron, Hungary
email: laszalay@emk.nyme.hu

Abstract. In this paper, we show that there are no three distinct positive integers a, b and c such that $ab+1, ac+1, bc+1$ all are balancing numbers.

1 Introduction

A *diophantine* m*-tuple* is a set $\{a_1, \dots, a_m\}$ of positive integers such that $a_i a_j + 1$ is square for all $1 \leq i < j \leq m$. Diophantus investigated first the problem of finding *rational* quadruples, and he provided one example: $\{1/16, 33/16, 68/16, 105/16\}$. The first integer quadruple, $\{1, 3, 8, 120\}$ was found by Fermat. Infinitely many diophantine quadruples of integers are known and it is conjectured that there is no integer diophantine quintuple. This was almost proved by Dujella [2], who showed that there can be at most finitely many diophantine quintuples and all of them are, at least in theory, effectively computable.

2010 Mathematics Subject Classification: 11D09, 11B39
Key words and phrases: diophantine triples, balancing numbers

The following variant of the diophantine tuples problem was treated by [4]. Let A and B be two nonzero integers such that $D = B^2+4A \neq 0$. Let $(u_n)_{n=0}^{\infty}$ be a binary recursive sequence of integers satisfying the recurrence

$$u_{n+2} = Au_{n+1} + Bu_n \qquad \text{for all } n \geq 0.$$

It is well-known that if we write α and β for the two roots of the *characteristic equation* $x^2 - Ax - B = 0$, then there exist constants $\gamma, \delta \in \mathbb{Q}[\alpha]$ such that

$$u_n = \gamma\alpha^n + \delta\beta^n \qquad \text{for all } n \geq 0.$$

Assume further that the sequence $(u_n)_{n=0}^{\infty}$ is *non-degenerate* which means that $\gamma\delta \neq 0$ and α/β are not root of unity. We shall also make the convention that $|\alpha| \geq |\beta|$.

A diophantine triple with values in the set $U = \{u_n : n \geq 0\}$, is a set of three distinct positive integers $\{a, b, c\}$, such that $ab+1, ac+1, bc+1$ are all in U. Note that if $u_n = 2^n + 1$ for all $n \geq 0$, then there are infinitely many such triples (namely, take a, b, c to be any distinct powers of two). The main result in [4] shows that only similar sequences can possess this property. The precise result proved there is the following.

Theorem 1 *Assume that* $(u_n)_{n=0}^{\infty}$ *is a non-degenerate binary recurrence sequence with* $D > 0$*, and suppose that there exist infinitely many nonnegative integers* a, b, c *with* $1 \leq a < b < c$*, and* x, y, z *such that*

$$ab + 1 = u_x, \quad ac + 1 = u_y, \quad bc + 1 = u_z.$$

Then $\beta \in \{\pm 1\}$, $\delta \in \{\pm 1\}$, $\alpha, \gamma \in \mathbb{Z}$*. Furthermore, for all but finitely many of sixtuples* $(a, b, c; x, y, z)$ *as above one has* $\delta\beta^z = \delta\beta^y = 1$ *and one of the followings holds:*

(i) $\delta\beta^x = 1$*. In this case, one of* δ *or* $\delta\alpha$ *is a perfect square;*

(ii) $\delta\beta^x = -1$*. In this case,* $x \in \{0, 1\}$.

No finiteness result was proved for the case when $D < 0$.

The first definition of balancing numbers is essentially due to Finkelstein [3], although he called them numerical centers. A positive integer n is called balancing number if

$$1 + 2 + \cdots + (n - 1) = (n + 1) + (n + 2) + \cdots + (n + r)$$

holds for some positive integer r. Then r is called balancer corresponding to the balancing number n. The n^{th} term of the sequence of balancing numbers is denoted by B_n. The balancing numbers satisfy the recurrence relation

$$B_{n+2} = 6B_{n+1} - B_n,$$

where the initial conditions are $B_0 = 0$ and $B_1 = 1$. Let α and β denote the roots of the characteristic polynomial $b(x) = x^2 - 6x + 1$. Then the explicit formula for the terms B_n is given by

$$B_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{(3+2\sqrt{2})^n - (3-2\sqrt{2})^n}{4\sqrt{2}}. \tag{1}$$

The first few terms of the balancing sequence are

$$0,\ 1,\ 6,\ 35,\ 204,\ 1189,\ 6930,\ 40391,\ 235416, \ldots.$$

Let denote the half of the associate sequence of the balancing numbers by C_n. Clearly, $C_n = (\alpha^n + \beta^n)/2$ satisfies $C_n = 6C_{n-1} - C_{n-2}$. Note that the terms C_n are odd positive integers:

$$1,\ 3,\ 17,\ 99,\ 577,\ 3363,\ 19601,\ 114243,\ 665857, \ldots.$$

Although Theorem 1 guarantees that there are at most finitely many Fibonacci and Lucas diophantine triples, it does not give a hint to find all of them. Luca and Szalay described a method to determine diophantine triples for Fibonacci numbers and Lucas numbers ([6] and [7], respectively). In this paper, we follow their method, although some new types of problems appeared when we proved the following theorem.

Theorem 2 *There do no exist positive integers* $a < b < c$ *such that*

$$ab + 1 = B_x, \qquad ac + 1 = B_y, \qquad bc + 1 = B_z, \tag{2}$$

where $0 < x < y < z$ *are natural numbers and* $(B_n)_{n=0}^{\infty}$ *is the sequence of balancing numbers.*

The main idea in the proof of Theorem 2 coincides the principal tool of [6], the details are different since the balancing numbers have less properties have been known then in case of Fibonacci and Lucas numbers.

2 Preliminary results

The proof of Theorem 2 uses the next lemma.

Lemma 1 *The following identities hold.*

1. $B_n = 35B_{n-2} - 6B_{n-3}$;
2. *If* $n \geq m$ *then* $(B_n - B_m)(B_n + B_m) = B_{n-m}B_{n+m}$, *especially* $(B_n - 1)(B_n + 1) = B_{n-1}B_{n+1}$;
3. $\gcd(B_n, B_m) = B_{\gcd(n,m)}$, *especially* $\gcd(B_n, B_{n-1}) = 1$;
4. $\gcd(B_n, C_n) = 1$;
5. $B_{n+m} = B_nC_m + C_nB_m$;
6. $B_{2n+1} - 1 = 2B_nC_{n+1}$.

Proof. The first property is a double application of the recurrence relation of balancing numbers. The second identity is Theorem 2.4.13 in [9], the next one is a specific case of a general statement described by [5]. The fourth feature can be found in the proof of Theorem VII in [1], the fifth property is given in [8]. Finally, the last one is coming easily from the explicit formulae for B_n and C_n. □

Lemma 2 *Any integer* $n \geq 2$ *satisfies the relation* $\gcd(B_n - 1, B_{n-2} - 1) \leq 34$.

Proof. Using the common tools in evaluating the greatest common divisor, the recurrence relation of balancing numbers, and Lemma 1 the statement is implied by the following rows. Put $Q_1 = \gcd(B_n - 1, B_{n-2} - 1)$. Then

$$\begin{aligned}
Q_1 &= \gcd(B_n - 1, B_n - B_{n-2}) = \gcd(B_n - 1, 6B_{n-1} - 2B_{n-2}) \leq \\
&\leq 2\gcd(B_n - 1, 3B_{n-1} - B_{n-2}) \leq 2\gcd(B_{n-1}B_{n+1}, 3B_{n-1} - B_{n-2}) \leq \\
&\leq 2\gcd(B_{n-1}, 3B_{n-1} - B_{n-2})\gcd(B_{n+1}, 3B_{n-1} - B_{n-2}) = \\
&= 2\gcd(B_{n-1}, B_{n-2})\gcd(35B_{n-1} - 6B_{n-2}, 3B_{n-1} - B_{n-2}) = \\
&= 2\gcd(-B_{n-1} + 6B_{n-2}, 3B_{n-1} - B_{n-2}) = \\
&= 2\gcd(-B_{n-1} + 6B_{n-2}, 17B_{n-2}) \leq \\
&\leq 34\gcd(-B_{n-1} + 6B_{n-2}, B_{n-2}) = 34\gcd(-B_{n-1}, B_{n-2}) = 34.
\end{aligned}$$

□

Lemma 3 *For any integer* $n \geq 2$ *we have* $\gcd(B_{2n-3}-1, B_n-1) \leq 1190$.

Proof. Similarly to the previous lemma, put $Q_2 = \gcd(B_{2n-3}-1, B_n-1)$. Then

$$\begin{aligned} Q_2 &= \gcd(2B_{n-2}C_{n-1}, B_n-1) \leq 2\gcd(B_{n-2}, B_n-1)\gcd(C_{n-1}, B_n-1) \leq \\ &\leq 2\gcd(B_{n-2}, B_{n-1}B_{n+1})\gcd(C_{n-1}, B_{n-1}B_{n+1}) \leq \\ &\leq 2\gcd(B_{n-2}, B_{n-1})\gcd(B_{n-2}, B_{n+1})\gcd(C_{n-1}, B_{n-1})\gcd(C_{n-1}, B_{n+1}) \leq \\ &\leq 2\cdot 1\cdot 35\cdot 1\cdot 17 = 1190. \end{aligned}$$

For explaining that $\gcd(C_{n-1}, B_{n+1}) \leq 17$, by Lemma 1 we write

$$\gcd(C_{n-1}, B_{n+1}) = \gcd(C_{n-1}, B_{n-1}C_2 + C_{n-1}B_2) = \gcd(C_{n-1}, 17B_{n-1}) \leq 17.$$

□

Remark 1 For our purposes, it is sufficient to have upper bounds given by Lemma 2 and Lemma 3. Without proof we state that the possible values for Q_1 are only 1, 2 and 34, while $Q_2 \in \{1, 2, 5, 34\}$.

Lemma 4 *Let* $u_0 \geq 3$ *be a positive integer. Then for all integers* $u \geq u_0$ *the inequalities*

$$\alpha^{u-0.9831} < B_u < \alpha^{u-0.983} \tag{3}$$

hold.

Proof. Let $c_0 = 4\sqrt{2}$. Since $0 < \beta < 1 < \alpha$ then the inequalities $u \geq u_0 \geq 3$ imply

$$B_u \geq \frac{\alpha^u - \beta^{u_0}}{c_0} = \alpha^u\left(\frac{1-\frac{\beta^{u_0}}{\alpha^u}}{c_0}\right) \geq \alpha^u\left(\frac{1-\left(\frac{\beta}{\alpha}\right)^{u_0}}{c_0}\right) \geq \alpha^{u-0.9831}.$$

For any non-negative integer u,

$$B_u \leq \frac{\alpha^u}{c_0} < \alpha^{u-0.983}.$$

□

Lemma 5 *All positive integer solutions to the system (2) satisfy* $z \leq 2y-1$.

Proof. The last two equations of the system (2) imply

$$c|\gcd(B_y - 1, B_z - 1). \tag{4}$$

Obviously, $B_z = bc + 1 < c^2$, hence $\sqrt{B_z} < c$. This, together with (4) gives $\sqrt{B_z} < B_y$. By (3) we obtain

$$\sqrt{\alpha^{z-0.9831}} < \sqrt{B_z} < B_y < \alpha^{y-0.983}.$$

It leads to

$$\alpha^{z-0.9831} < \alpha^{2y-1.966},$$

and then $z \leq 2y - 1$. □

3 Proof of Theorem 2

Suppose that the integers $0 < a < b < c$ and $0 < x < y < z$ satisfy (2). Thus $1 \cdot 2 + 1 \leq ab + 1 = B_x$ implies $2 \leq x$. Thus $3 \leq y$. The proof is split into two parts.

I. $z \leq 449$.

In this case, we ran an exhaustive computer search to detect all positive integer solutions to the system (2). Observe that we have

$$a = \sqrt{\frac{(B_x - 1)(B_y - 1)}{(B_z - 1)}}, \qquad 2 \leq x < y < z \leq 449.$$

Going through all the eligible values for x, y and z, and checking if the above number a is an integer, we found no solution to the system (2).

II. $z > 449$.

Put $Q = \gcd(B_z - 1, B_y - 1)$. From the proof of Lemma 5 we know that $\sqrt{B_z} < Q$. Applying now Lemma 1,

$$\begin{aligned} Q &\leq \gcd(B_{z-1}B_{z+1}, B_{y-1}B_{y+1}) \\ &\leq \prod_{i,j\in\{\pm 1\}} \gcd(B_{z-i}, B_{y-j}) = \prod_{i,j\in\{\pm 1\}} B_{\gcd(z-i,y-j)}. \end{aligned} \tag{5}$$

Let $\gcd(z - i, y - j) = \frac{z-i}{k_{ij}}$. Suppose that $k_{ij} \geq 8$, for all the four possible pairs (i, j) in (5). Then Lemma 4, together with the previous two estimates, provides

$$\alpha^{\frac{z-0.9831}{2}} < \sqrt{B_z} < Q \leq \left(B_{(z-1)/8}\right)^2 \left(B_{(z+1)/8}\right)^2 < \alpha^{4\cdot\left(\frac{z+1}{8} - 0.983\right)}$$

which leads to a contradiction if one compares the exponents of α.

Assume now that $k_{ij} \leq 7$ fulfills for some i and j, let denote k this k_{ij}. Suppose further that

$$\frac{z-i}{k} = \frac{y-j}{l}$$

holds for a suitable positive integer l coprime to k.

If $l > k$, then according to $y < z$, the relation $z-i < y-j$ implies $z = y+1$. But this is impossible since

$$Q = \gcd(B_{y+1}-1, B_y-1) \leq \gcd(B_{y+2}B_y, B_{y+1}B_{y-1}) = \gcd(B_{y+2}, B_{y-1}) \leq B_3$$

follows in the virtue of Lemma 1. Thus

$$\alpha^{\frac{z-0.9831}{2}} < \sqrt{B_z} < Q \leq B_3 = 35$$

leads to a contradiction by $z < 5.1$.

Suppose now that $k = l = 1$. Now $z - i = y - j$ can hold only if $z = y + 2$. Thus, by Lemma 3, we have

$$Q = \gcd(B_{y+2}-1, B_y-1) \leq 34 < B_3.$$

Hence, as in the previous part, we arrived at a contradiction.

In the sequel, we assume $l < k$. First suppose $3 \leq k$. Taking any pair $(i_0, j_0) \neq (i, j)$ from the remaining three cases of $(-1,-1)$, $(-1,1)$, $(1,-1)$ and $(1,1)$, we have

$$y - j_0 = \frac{l}{k}(z-i) + j - j_0 = \frac{lz - li + kj - kj_0}{k}. \tag{6}$$

Thus

$$\begin{aligned}
\gcd(z-i_0, y-j_0) &= \gcd\left(z-i_0, \frac{lz-li+kj-kj_0}{k}\right) \\
&\leq \gcd(lz-li_0, lz-li+kj-kj_0) \\
&= \gcd(lz-li_0, li_0-li+kj-kj_0).
\end{aligned}$$

Since $li_0 - li + kj - kj_0$ does not vanish, it follows that

$$\gcd(lz-li_0, li_0-li+kj-kj_0) \leq |li_0-li+kj-kj_0| \leq 2(k+l) \leq 26.$$

Indeed, it is easy to see that $li_0 - li + kj - kj_0 = 0$, or equivalently $l(i_0 - i) = k(j_0-j)$ leads to a contradiction since $2 \leq k \leq 7$ and $1 \leq l \leq k-1$ are coprime, further $i_0 - i$ and $j_0 - j$ are in the set $\{0, \pm 2\}$ meanwhile at least one of them is non-zero.

Then (5), together with Lemma 4, yields

$$\alpha^{\frac{z-0.9831}{2}} < B_{\frac{z+1}{3}} \cdot B_{26}^3 < \alpha^{\frac{z+1}{3}-0.983}\left(\alpha^{25.017}\right)^3.$$

Consequently, $z < 449.4$. It contradicts the condition separating Case 2 and 1.

Assume now that $k = k_{ij} = 2$ fulfills for some eligible pair (i, j). Thus $l = 1$. First suppose that $\gcd(z-1, y-1) = (z-1)/2$. It yields $z = 2y - 1$, and we go back to the system

$$\begin{aligned} ab + 1 &= B_x, \\ ac + 1 &= B_y, \\ bc + 1 &= B_{2y-1}. \end{aligned}$$

First we obtain

$$\frac{B_{2y-1}}{B_y} = \frac{bc+1}{ac+1} < \frac{b}{a}$$

since $0 < a < b < c$. On the other hand, by Lemma 4,

$$\frac{B_{2y-1}}{B_y} > \frac{\alpha^{2y-1-0.9831}}{\alpha^{y-0.983}} = \alpha^{y-1.001}$$

follows. Consequently,

$$a\alpha^{y-1.001} < b,$$

and

$$a^2\alpha^{y-1.001} \le ab = B_x - 1 < B_x < \alpha^{x-0.983}.$$

Thus we arrived at a contradiction by

$$a^2 < \alpha^{x-y+0.018} \le \alpha^{-0.982} < 0.2.$$

If $\gcd(z-1, y+1) = (z-1)/2$ then $z = 2y+3$ contradicting Lemma 5. Similarly, $\gcd(z+1, y+1) = (z+1)/2$ leads to $z = 2y+1$. Finally, $\gcd(z+1, y-1) = (z+1)/2$ gives $z = 2y-3$, which is possible. But, in this case, by Lemma 3 we have

$$\alpha^{\frac{z-0.9831}{2}} < \sqrt{B_z} < c \le \gcd(B_{2y-3} - 1, B_y - 1) \le 1190,$$

and it results $z \le 9$ in the virtue of Lemma 4.

The proof of Theorem 2 is completed.

References

[1] R. D. Carmichael, On the numerical factors of the arithmetic function $\alpha^n \pm \beta^n$, *Annals Math., 2nd Ser.*, **15** (1913-1914), 30–48.

[2] A. Dujella, There are finitely many diophantine quintuples, *J. Reine Angew. Math.* **556** (2004), 183–214.

[3] R. P. Finkelstein, The house problem, *American Math. Monthly*, **72** (1965), 1082–1088.

[4] C. Fuchs, F. Luca, L. Szalay, Diophantine triples with values in binary recurrences, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **5** (2008), 579–608.

[5] P. Horak, L. Skula, A characterization of the second-order strong divisibility sequences, *Fibonacci Quart.*, **23** (1985), 126–132.

[6] F. Luca, L. Szalay, Fibonacci diophantine triples, *Glasnik Math.*, **43** (2008), 253–264.

[7] F. Luca, L. Szalay, Lucas diophantine triples, *Integers*, **9** (2009), 441–457.

[8] G. K. Panda, Sequence balancing and cobalancing numbers, *Fibonacci Quart.*, **45** (2007), 265-271.

[9] P. K. Ray, *Balancing and Cobalancing Numbers*, PhD. thesis, Rourkela, India, 2009.

Received: August 11, 2012