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Mills' ratio: Reciprocal convexity and functional inequalities

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Dedicated to my children Boróka and Koppány

Abstract. This note contains sufficient conditions for the probability density function of an arbitrary continuous univariate distribution, supported on $(0,\infty)$, such that the corresponding Mills ratio to be reciprocally convex (concave). To illustrate the applications of the main results, the reciprocal convexity (concavity) of Mills ratio of the gamma distribution is discussed in details.

1 Introduction

By definition (see [7]) a function $f:(0,\infty)\to\mathbb{R}$ is said to be (strictly) reciprocally convex if $x\mapsto f(x)$ is (strictly) concave and $x\mapsto f(1/x)$ is (strictly) convex on $(0,\infty)$. Merkle [7] showed that f is reciprocally convex if and only if for all $x,y>0$ we have

$$f\left(\frac{2xy}{x+y}\right)\leq\frac{f(x)+f(y)}{2}\leq f\left(\frac{x+y}{2}\right)\leq\frac{xf(x)+yf(y)}{x+y}. \tag{1}$$

We note here that in fact the third inequality follows from the fact that the function $x\mapsto f(1/x)$ is convex on $(0,\infty)$ if and only if $x\mapsto xf(x)$ is convex on $(0,\infty)$. In what follows, similarly as in [7], a function $g:(0,\infty)\to\mathbb{R}$ is said

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to be (strictly) reciprocally concave if and only if $-g$ is (strictly) reciprocally convex, i.e. if $x \mapsto g(x)$ is (strictly) convex and $x \mapsto g(1/x)$ is (strictly) concave on $(0, \infty)$. Observe that if f is differentiable, then $x \mapsto f(1/x)$ is (strictly) convex (concave) on $(0, \infty)$ if and only if $x \mapsto x^2 f'(x)$ is (strictly) increasing (decreasing) on $(0, \infty)$.

As it was shown by Merkle [7], reciprocally convex functions defined on $(0, \infty)$ have a number of interesting properties: they are increasing on $(0, \infty)$ or have a constant value on $(0, \infty)$, they have a continuous derivative on $(0, \infty)$ and they generate a sequence of quasi-arithmetic means, with the first one between harmonic and arithmetic mean and others above the arithmetic mean. Some examples of reciprocally convex functions related to the Euler gamma function were given in [7].

By definition (see [9]) a function $f : (0, \infty) \to \mathbb{R}$ is said to be completely monotonic, if f has derivatives of all orders and satisfies

$$(-1)^n f^{(n)}(x) \geq 0$$

for all $x > 0$ and $n \in \{0, 1, \dots\}$. Note that strict inequality always holds above unless f is constant. It is known (Bernstein's Theorem) that f is completely monotonic if and only if [9, p. 161]

$$f(x) = \int_0^\infty e^{-xt} d\nu(t),$$

where ν is a nonnegative measure on $[0, \infty)$ such that the integral converges for all $x > 0$. An important subclass of completely monotonic functions consists of the Stieltjes transforms defined as the class of functions $g : (0, \infty) \to \mathbb{R}$ of the form

$$g(x) = \alpha + \int_0^\infty \frac{d\nu(t)}{x + t},$$

where $\alpha \geq 0$ and ν is a nonnegative measure on $[0, \infty)$ such that the integral converges for all $x > 0$.

It was pointed out in [7] that if a function $h : (0, \infty) \to \mathbb{R}$ is a Stieltjes transform, then $-h$ is reciprocally convex, i.e. h is reciprocally concave. We note that some known reciprocally concave functions comes from probability theory. For example, the Mills ratio of the standard normal distribution is a reciprocally concave function. For this let us see some basics. The probability density function $\varphi : \mathbb{R} \to (0, \infty)$, the cumulative distribution function $\Phi : \mathbb{R} \to (0, 1)$ and the reliability function $\overline{\Phi} : \mathbb{R} \to (0, 1)$ of the standard normal law, are defined by

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2},$$

$$\Phi(x) = \int_{-\infty}^{x} \varphi(t)dt$$

and

$$\overline{\Phi}(x) = 1 - \Phi(x) = \int_{x}^{\infty} \varphi(t)dt.$$

The function $m : \mathbb{R} \to (0, \infty)$, defined by

$$m(x) = \frac{\overline{\Phi}(x)}{\varphi(x)} = e^{x^2/2} \int_{x}^{\infty} e^{-t^2/2}dt = \int_{0}^{\infty} e^{-xt} e^{-t^2/2}dt,$$

is known in literature as Mills' ratio [8, Sect. 2.26] of the standard normal law, while its reciprocal $r = 1/m$, defined by $r(x) = 1/m(x) = \varphi(x)/\overline{\Phi}(x)$, is the so-called failure (hazard) rate. For Mills' ratio of other distributions, like gamma distribution, we refer to [6] and to the references therein.

It is well-known that Mills' ratio of the standard normal distribution is convex and strictly decreasing on $\mathbb{R}$, at the origin takes on the value $m(0) = \sqrt{\pi/2}$. Moreover, it can be shown (see [2]) that $x \mapsto m'(x)/m(x)$ is strictly increasing and $x \mapsto x^2 m'(x)$ is strictly decreasing on $(0, \infty)$. With other words, the Mills ratio of the standard normal law is strictly reciprocally concave on $(0, \infty)$. Some other monotonicity properties and interesting functional inequalities involving the Mills ratio of the standard normal distribution can be found in [2]. The following complements the above mentioned results.

Theorem 1 *Let m be the Mills ratio of the standard normal law. Then the function $x \mapsto m(\sqrt{x})/\sqrt{x}$ is a Stieltjes transform and consequently it is strictly completely monotonic and strictly reciprocally concave on $(0, \infty)$. In particular, if $x, y > 0$, then the following chain of inequalities holds*

$$\sqrt{\frac{x+y}{2xy}}\, m\left(\sqrt{\frac{2xy}{x+y}}\right) \geq \frac{\sqrt{y}m(\sqrt{x}) + \sqrt{x}m(\sqrt{y})}{2\sqrt{xy}}$$
$$\geq \sqrt{\frac{2}{x+y}}\, m\left(\sqrt{\frac{x+y}{2}}\right) \geq \frac{\sqrt{x}m(\sqrt{x}) + \sqrt{y}m(\sqrt{y})}{x+y}.$$

In each of the above inequalities equality holds if and only if $x = y$.

Proof. For $x > 0$ the Mills of the standard normal distribution can be represented as [5, p. 145]

$$m(x) = \int_{-\infty}^{\infty} \frac{x}{x^2 + t^2} \varphi(t)dt = 2\int_{0}^{\infty} \frac{x}{x^2 + t^2} \varphi(t)dt.$$

From this we obtain that

$$\frac{m(\sqrt{x})}{\sqrt{x}} = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{1}{x+s} \frac{e^{-s/2}}{\sqrt{s}} ds,$$

which shows that the function $x \mapsto m(\sqrt{x})/\sqrt{x}$ is in fact a Stieltjes transform and owing to Merkle [7, p. 217] this implies that the function $x \mapsto -m(\sqrt{x})/\sqrt{x}$ is reciprocally convex on $(0,\infty)$, i.e. the function $x \mapsto m(\sqrt{x})/\sqrt{x}$ is reciprocally concave on $(0,\infty)$. The rest of the proof follows easily from (1). We note that the strictly complete monotonicity of the function $x \mapsto m(\sqrt{x})/\sqrt{x}$ can be proved also by using the properties of completely monotonic functions. Mills ratio m of the standard normal distribution is in fact a Laplace transform and consequently it is strictly completely monotonic (see [2]). On the other hand, it is known (see [9]) that if u is strictly completely monotonic and v is nonnegative with a strictly completely monotone derivative, then the composite function $u \circ v$ is also strictly completely monotonic. Now, since the function m is strictly completely monotonic on $(0,\infty)$ and $x \mapsto 2(\sqrt{x})' = 1/\sqrt{x}$ is strictly completely monotonic on $(0,\infty)$, we obtain that $x \mapsto m(\sqrt{x})$ is also strictly completely monotonic on $(0,\infty)$. Finally, by using the fact that the product of completely monotonic functions is also completely monotonic, the function $x \mapsto m(\sqrt{x})/\sqrt{x}$ is indeed strictly completely monotonic on $(0,\infty)$. □

Now, since the Mills ratio of the standard normal distribution is reciprocally concave a natural question which arises here is the following: under which conditions does the Mills ratio of an arbitrary continuous univariate distribution, having support $(0,\infty)$, will be reciprocally convex (concave)? The goal of this paper is to find some sufficient conditions for the probability density function of an arbitrary continuous univariate distribution, supported on the semi-infinite interval $(0,\infty)$, such that the corresponding Mills ratio to be reciprocally convex (concave). The main result of this paper, namely Theorem 2 in section 2, is based on some recent results of the author [3] and complement naturally the results from [2, 3]. To illustrate the application of the main result, the Mills ratio of the gamma distribution is discussed in details in section 3.

We note that although the reciprocal convexity (concavity) of Mills ratio is interesting in his own right, the convexity of the Mills ratio of continuous distributions has important applications in monopoly theory, especially in static pricing problems. For characterizations of the existence or uniqueness of global maximizers we refer to [4] and to the references therein. Another application can be found in [10], where the convexity of Mills ratio is used to show that the price is a sub-martingale.

2 Reciprocal convexity (concavity) of Mills ratio

In this section our aim is to find some sufficient conditions for the probability density function such that the corresponding Mills ratio to be reciprocally convex (concave). As in [3] the proof is based on the monotone form of l'Hospital's rule [1, Lemma 2.2].

Theorem 2 *Let* $f : (0,\infty) \to (0,\infty)$ *be a probability density function and let* $\omega : (0,\infty) \to \mathbb{R}$, *defined by* $\omega(x) = f'(x)/f(x)$, *be the logarithmic derivative of* f. *Let also* $\overline{F} : (0,\infty) \to (0,1)$, *defined by* $\overline{F}(x) = \int_x^\infty f(t)dt$, *be the survival function and* $m : (0,\infty) \to (0,\infty)$, *defined by* $m(x) = \overline{F}(x)/f(x)$, *be the corresponding Mills ratio. Then the following assertions are true:*

(a) *If* $f(x)/\omega(x) \to 0$ *as* $x \to \infty$, ω'/ω^2 *is (strictly) decreasing (increasing) on* $(0,\infty)$ *and the function*

$$x \mapsto \frac{x^3\omega'(x)}{x\omega^2(x) - x\omega'(x) - 2\omega(x)}$$

is (strictly) increasing (decreasing) on $(0,\infty)$, *then Mills ratio* m *is (strictly) reciprocally convex (concave) on* $(0,\infty)$.

(b) *If* $xf(x)/(1-x\omega(x)) \to 0$, $f(x)/\omega(x) \to 0$ *as* $x \to \infty$, ω'/ω^2 *is (strictly) decreasing (increasing) on* $(0,\infty)$ *and the function*

$$x \mapsto \frac{x^2\omega'(x) - x\omega(x) + 2}{x\omega^2(x) - x\omega'(x) - 2\omega(x)}$$

is (strictly) increasing (decreasing) on $(0,\infty)$, *then Mills ratio* m *is (strictly) reciprocally convex (concave) on* $(0,\infty)$.

Proof. (a) By definition Mills ratio m is (strictly) reciprocally convex (concave) if m is (strictly) concave (convex) and $x \mapsto m(1/x)$ is (strictly) convex (concave). It is known (see [3, Theorem 2]) that if $f(x)/\omega(x)$ tends to zero as x tends to infinity and the function ω'/ω^2 is (strictly) increasing (decreasing), then m is (strictly) convex (concave). Thus, we just need to find conditions for the (strict) convexity (concavity) of the function $x \mapsto m(1/x)$. This function is (strictly) convex (concave) on $(0,\infty)$ if and only if the function $x \mapsto x^2m'(x)$ is (strictly) increasing (decreasing) on $(0,\infty)$. On the other hand, observe that Mills ratio m satisfies the differential equation

$$m'(x) = -\omega(x)m(x) - 1.$$

Thus, by using the monotone form of l'Hospital's rule (see [1, Lemma 2.2]) to prove that the function

$$x \mapsto x^2 m'(x) = -\frac{\left(\overline{F}(x) + f(x)/\omega(x)\right) - \lim\limits_{x\to\infty}\left(\overline{F}(x) + f(x)/\omega(x)\right)}{f(x)/(x^2\omega(x)) - \lim\limits_{x\to\infty} f(x)/(x^2\omega(x))}$$
$$= -\frac{\overline{F}(x) + f(x)/\omega(x)}{f(x)/(x^2\omega(x))}$$

is (strictly) increasing (decreasing) on $(0,\infty)$ it is enough to show that

$$x \mapsto -\frac{\left(\overline{F}(x) + f(x)/\omega(x)\right)'}{\left(f(x)/(x^2\omega(x))\right)'} = \frac{x^3\omega'(x)}{x\omega^2(x) - x\omega'(x) - 2\omega(x)}$$

is (strictly) increasing (decreasing) on $(0,\infty)$.

(b) Observe that according to [7, Lemma 2.2] the function $x \mapsto m(1/x)$ is (strictly) convex (concave) if and only if $x \mapsto xm(x)$ is (strictly) convex (concave) on $(0,\infty)$. Now, by using the monotone form of l'Hospital's rule (see [1, Lemma 2.2]) the function

$$x \mapsto (xm(x))' = m(x) - x - x\omega(x)m(x) = \frac{\overline{F}(x) - xf(x)/(1-x\omega(x))}{f(x)/(1-x\omega(x))}$$
$$= \frac{\left(\overline{F}(x) - xf(x)/(1-x\omega(x))\right) - \lim\limits_{x\to\infty}\left(\overline{F}(x) - xf(x)/(1-x\omega(x))\right)}{(f(x)/(1-x\omega(x))) - \lim\limits_{x\to\infty}(f(x)/(1-x\omega(x)))}$$

is (strictly) increasing (decreasing) on $(0,\infty)$ if the function

$$x \mapsto \frac{\left(\overline{F}(x) - xf(x)/(1-x\omega(x))\right)'}{\left(f(x)/(1-x\omega(x))\right)'} = \frac{x^2\omega'(x) - x\omega(x) + 2}{x\omega^2(x) - x\omega'(x) - 2\omega(x)}$$

is (strictly) increasing (decreasing) on $(0,\infty)$. Note that we used tacitly the fact that if $xf(x)/(1-x\omega(x)) \to 0$ as $x \to \infty$, then $f(x)/(1-x\omega(x)) \to 0$ as $x \to \infty$. □

We note here that the reciprocal concavity of the Mills ratio of the standard normal distribution can be verified easily by using part (a) or part (b) of Theorem 2. More precisely, in the case of the standard normal distribution we have $\omega(x) = -x$, $\omega'(x) = -1$. Consequently $\varphi(x)/\omega(x) = -\varphi(x)/x \to 0$ as $x \to \infty$, the function $x \mapsto \omega'(x)/\omega^2(x) = -1/x^2$ is strictly increasing and

$$x \mapsto \frac{x^3\omega'(x)}{x\omega^2(x) - x\omega'(x) - 2\omega(x)} = -\frac{x^2}{x^2+3}$$

is strictly decreasing on $(0,\infty)$. This is turn implies that by using part (a) of Theorem 2 the Mills ratio of the standard normal distribution is strictly reciprocally concave on $(0,\infty)$.

Similarly, since $\varphi(x)/(1+x^2)\to 0$, $x\varphi(x)/(1+x^2)\to 0$, $-\varphi(x)/x\to 0$ as $x\to\infty$, the function $x\mapsto \omega'(x)/\omega^2(x)=-1/x^2$ is strictly increasing and

$$x\mapsto \frac{x^2\omega'(x)-x\omega(x)+2}{x\omega^2(x)-x\omega'(x)-2\omega(x)}=\frac{2}{x^3+3x}$$

is strictly decreasing on $(0,\infty)$, part (b) of Theorem 2 also implies that the Mills ratio of the standard normal distribution is strictly reciprocally concave on $(0,\infty)$.

Thus, Theorem 2 in fact generalizes some of the main results of [2].

3 Reciprocal convexity (concavity) of Mills ratio of the gamma distribution

The gamma distribution has support $(0,\infty)$, probability density function, cumulative distribution function and survival function as follows

$$f(x)=f(x;\alpha)=\frac{x^{\alpha-1}e^{-x}}{\Gamma(\alpha)},$$

$$F(x)=F(x;\alpha)=\frac{\gamma(\alpha,x)}{\Gamma(\alpha)}=\frac{1}{\Gamma(\alpha)}\int_0^x t^{\alpha-1}e^{-t}dt$$

and

$$\overline{F}(x)=\overline{F}(x;\alpha)=\frac{\Gamma(\alpha,x)}{\Gamma(\alpha)}=\frac{1}{\Gamma(\alpha)}\int_x^\infty t^{\alpha-1}e^{-t}dt,$$

where Γ is the Euler gamma function, $\gamma(\cdot,\cdot)$ and $\Gamma(\cdot,\cdot)$ denote the lower and upper incomplete gamma functions, and $\alpha>0$ is the shape parameter. As we can see below, *the Mills ratio of the gamma distribution* $m:(0,\infty)\to(0,\infty)$, defined by

$$m(x)=m(x;\alpha)=\frac{\Gamma(\alpha,x)}{x^{\alpha-1}e^{-x}},$$

is reciprocally convex on $(0,\infty)$ *for all* $0<\alpha\leq 1$ *and reciprocally concave on* $(0,\infty)$ *for all* $1\leq\alpha\leq 2$. In [3] it was proved that if $\alpha\geq 1$, then the Mills ratio m is decreasing and log-convex, and consequently convex on $(0,\infty)$. We

note that the convexity of Mills ratio of the gamma distribution actually can be verified directly (see [10]), since

$$m(x) = \int_x^\infty \left(\frac{t}{x}\right)^{\alpha-1} e^{x-t}dt = \int_1^\infty xu^{\alpha-1}e^{(1-u)x}du$$

and

$$\begin{aligned} m'(x) &= \int_1^\infty \Big((\alpha-1)u^{\alpha-2}\Big)\Big((1-u)e^{(1-u)x}\Big)\,du \\ &= \int_1^\infty u^{\alpha-1}e^{(1-u)x}du + \int_1^\infty xu^{\alpha-1}(1-u)e^{(1-u)x}du, \end{aligned}$$

where the last equality follows from integration by parts. From this we clearly have that

$$m''(x) = \int_1^\infty (\alpha-1)(1-u)^2u^{\alpha-2}e^{(1-u)x}du$$

and consequently m is convex on $(0,\infty)$ if $\alpha \geq 1$ and is concave on $(0,\infty)$ if $0 < \alpha \leq 1$. The concavity of the function m can be verified also by using [3, Theorem 2]. Namely, if let $\omega(x) = f'(x)/f(x) = (\alpha-1)/x - 1$, then $f(x)/\omega(x)$ tends to zero as x tends to infinity and the function $x \mapsto \omega'(x)/\omega^2(x) = (1-\alpha)/(\alpha-1-x)^2$ is decreasing on $(0,\infty)$ for all $0 < \alpha \leq 1$. Consequently in view of [3, Theorem 2] m is indeed concave on $(0,\infty)$ for all $0 < \alpha \leq 1$.

Now let us focus on the reciprocal convexity (concavity) of the Mills ratio of gamma distribution. Since

$$\frac{x^3\omega'(x)}{x\omega^2(x) - x\omega'(x) - 2\omega(x)} = \frac{(1-\alpha)x^2}{(\alpha-1-x)^2 + 2x + 1 - \alpha},$$

we obtain that

$$\left(\frac{x^3\omega'(x)}{x\omega^2(x) - x\omega'(x) - 2\omega(x)}\right)' = \frac{2(\alpha-1)(\alpha-2)(x^2+(1-\alpha)x)}{((\alpha-1-x)^2 + 2x + 1 - \alpha)^2}.$$

This last expression is clearly positive if $0 < \alpha \leq 1$ and $x > 0$, and thus, by using part (a) of Theorem 2 we conclude that Mills ratio m is reciprocally convex on $(0,\infty)$ for all $0 < \alpha \leq 1$.

Similarly, since

$$\frac{x^2\omega'(x) - x\omega(x) + 2}{x\omega^2(x) - x\omega'(x) - 2\omega(x)} = \frac{x^2 + 2(2-\alpha)x}{x^2 + 2(2-\alpha)x + (\alpha-1)(\alpha-2)},$$

we get

$$\left(\frac{x^2\omega'(x)-x\omega(x)+2}{x\omega^2(x)-x\omega'(x)-2\omega(x)}\right)'=\frac{2(\alpha-1)(\alpha-2)(x+2-\alpha)}{(x^2+2(2-\alpha)x+(\alpha-1)(\alpha-2))^2}$$

and this is negative if $1\leq\alpha\leq 2$ and $x>0$. Consequently, by using part (b) of Theorem 2 we get that the Mills ratio of the gamma distribution is indeed reciprocally concave for $1\leq\alpha\leq 2$. Here we used that if x tends to ∞, then the expressions $f(x)/\omega(x)$ and $xf(x)/(1-x\omega(x))$ tend to 0.

Finally, we note that the convexity (concavity) of $x\mapsto m(1/x)$ can be verified also by using the integral representation of Mills ratio of the gamma distribution. More precisely, if we rewrite $m(x)$ as

$$m(x)=\int_0^\infty\left(1+\frac{u}{x}\right)^{\alpha-1}e^{-u}du,$$

then

$$x^2m'(x)=-\int_0^\infty(\alpha-1)\left(1+\frac{u}{x}\right)^{\alpha-2}ue^{-u}du$$

and

$$[x^2m'(x)]'=\int_0^\infty(\alpha-1)(\alpha-2)\left(1+\frac{u}{x}\right)^{\alpha-3}\frac{u^2}{x^2}e^{-u}du.$$

This shows that $x\mapsto x^2m'(x)$ is decreasing on $(0,\infty)$ if $1\leq\alpha\leq 2$ and increasing on $(0,\infty)$ if $0<\alpha\leq 1$ or $\alpha\geq 2$. Summarizing, the Mills ratio of the gamma distribution is reciprocally convex on $(0,\infty)$ if $0<\alpha\leq 1$ and reciprocally concave on $(0,\infty)$ if $1\leq\alpha\leq 2$. When $\alpha>2$ the functions $x\mapsto m(x)$ and $x\mapsto m(1/x)$ are convex on $(0,\infty)$, thus in this case m is nor reciprocally convex and neither reciprocally concave on its support.

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