Acta Univ. Sapientiae, Mathematica, 4, 1 (2012) 36–52

Existence and generalized duality of strong vector equilibrium problems

Adela Capătă
Technical University of Cluj-Napoca
Department of Mathematics Cluj-Napoca, Romania
email: adela.capata@math.utcluj.ro

Abstract. In this paper, with the help of the duality operator and K. Fan's lemma, we present existence results for strong vector equilibrium problems, under pseudomonotonicity assumptions and without any pseudomonotonicity assumptions, respectively. Then, as an application, the main results allow us to state existence theorems for strong vector variational inequality problems.

1 Introduction and mathematical tools

Let A be a nonempty subset of a topological space E, let C be a nontrivial pointed convex cone of a real topological linear space Z, and let $\varphi : A \times A \to Z$ be a given bifunction. In [1], the scalar equilibrium problem was extended to vector-valued bifunctions in the following way:

$$\text{find } \bar{a} \in A \text{ such that } \varphi(\bar{a}, b) \notin -C \setminus \{0\} \text{ for all } b \in A. \tag{VEP}$$

Throughout this paper we deal with (VEP), which is called *the strong vector equilibrium problem.* In the last decade, the study of strong vector equilibrium problems and their particular cases received a special attention from many authors, see, for instance: [1, 3, 7, 8, 12, 14, 15, 16, 24, 26].

Most of the existence results for vector equilibrium problems, are given in the hypothesis of a cone with nonempty interior, but, there are important

2010 Mathematics Subject Classification: 49J52
Key words and phrases: duality operator, C-quasiconvexity, coercivity condition, maximal pseudomonotonicity, vector variational inequality

ordered topological linear spaces whose ordering cones have an empty interior. For example, when $Z := L^p(T, \mu)$, where (T, μ) is a σ-finite measure space and $p \in [1, +\infty[$, the cone

$$C := \{u \in L^p(T, \mu) \mid u(t) \geq 0 \text{ a.e. in } [0, T]\}$$

has an empty interior. Therefore, for optimization problems stated in infinite dimensional spaces, several generalized interior-point conditions were given in order to assure strong duality. To this purpose some generalizations of the classical interior have been introduced (see, for example, [5, 6, 17, 20, 25]).

Another way to overcome this problem is to introduce approximative solutions. H. W. Kuhn and A. W. Tucker [22] observed that some efficient solutions of optimization problems are not satisfactorily characterized by a scalar minimization problem. To eliminate such anomalous efficient points various concepts of proper efficiency for optimization problems have been introduced (see, for instance, [4, 13, 18, 19]).

The aim of this paper is to present new existence results for (VEP) without any solidness assumption for the cone C.

The paper is organized as follows. In Section 2 with the help of the duality operator, we attach to problem (VEP) a generalized dual strong vector equilibrium problem. By introducing a new generalization of the maximal g-pseudomonotonicity due to W. Oettli [24], new existence results for the strong vector equilibrium problem (VEP) are given. To see which convexity notion satisfies assumption (iv) of Theorem 1, in Corollary 1 this assumption is replaced by the one in which we demand the C-quasiconvexity of the vector-function $\varphi(a, \cdot)$ for all $a \in A$.

Then, in Section 3, considering two particular cases of the duality operator, we present existence results for (VEP) under pseudomonotonicity assumptions, respectively, without any pseudomonotonicity assumptions. The results allow us to recover already established results from the literature. For example we recover results from [10] and [24].

Section 4 is devoted to applications. Thus, we give existence results for strong vector variational inequalities under pseudomonotonicity assumptions, and without any pseudomonotonicity assumptions, respectively. In Example 1 is showed that the set of operators which satisfies the assumptions of Theorem 2 is nonempty. Furthermore, from this example we see that there exist maximal pseudomonotone operators which are not strongly pseudomonotone in the sense of Definition 5.

In what follows, by using Ky Fan's lemma we will present new existence results for (VEP) without any solidness assumption on the cone C.

Definition 1 *Let* A *be a nonempty subset of a real topological linear space* E*. A multifunction* $T : A \to 2^A$ *is said to be a KKM-operator if, for every finite subset* $\{a_1, a_2, \dots, a_n\}$ *of* A*, the following inclusion holds:*

$$co\{a_1, a_2, \dots, a_n\} \subseteq \bigcup_{i=1}^{n} T(a_i).$$

In finite dimensional spaces the next lemma was given by B. Knaster, C. Kuratowski, S. Mazurkiewicz in [21], while in infinite dimensional spaces it was established by Ky Fan.

Lemma 1 ([9]) *Let* A *be a nonempty subset of a real Hausdorff topological linear space* E*, and let* $T : A \to 2^A$ *be a KKM-operator satisfying the following conditions:*

(i) $T(a)$ *is closed for all* $a \in A$*;*

(ii) *there is* $\bar{a} \in A$ *such that* $T(\bar{a})$ *is a compact set.*

Then

$$\bigcap_{a \in A} T(a) \neq \emptyset.$$

Now, let us recall the following weakened convexity notion, which can be found, by example, in [11].

Definition 2 Let E and Z be real topological linear spaces, let A be a nonempty subset of E, and let $C \subseteq Z$ be a convex cone. A function $f : A \to Z$ is said to be C-quasiconvex if A is convex and, for all $a_1, a_2 \in A$ and all $\lambda \in [0, 1]$, we have

$$f(\lambda a_1 + (1 - \lambda)a_2) \leq_C f(a_1)$$

or

$$f(\lambda a_1 + (1 - \lambda)a_2) \leq_C f(a_2).$$

2 Existence results via Ky Fan's lemma

From now on, E is considered to be a real Hausdorff topological linear space, $A \subseteq E$ is a nonempty convex subset, and C is a pointed convex cone of the real topological linear space Z.

With the help of an operator, we attach to problem (VEP) a dual problem. Let $\mathcal{D}$ be an operator from $\mathcal{F}(A,Z) := \{\psi \mid \psi : A \times A \to Z\}$ into itself, which is called the duality operator. In fact, $\mathcal{D}$ is a set of fixed rules applied to problem (VEP). By means of $\mathcal{D}$ we introduce the following *generalized dual strong vector equilibrium problem*:

$$\text{find } \bar{a} \in A \text{ such that } \mathcal{D}(\varphi)(\bar{a}, b) \notin -C \setminus \{0\} \text{ for all } b \in A. \tag{DVEP}$$

The following proposition shows that, under a certain hypothesis, the generalized dual of this dual problem becomes the initial problem. Its proof is straightforward.

Proposition 1 *If*

$$\mathcal{D} \circ \mathcal{D}(\varphi) = \varphi,$$

then the generalized dual problem of (DVEP) *is problem* (VEP)*.*

Let $G : A \times A \to Z$ be defined by

$$G(a, b) := -\mathcal{D}(\varphi)(b, a) \text{ for all } a, b \in A.$$

In this framework, problem (DVEP) can be written as:

$$\text{find } \bar{a} \in A \text{ such that } G(b, \bar{a}) \notin C \setminus \{0\} \text{ for all } b \in A. \tag{GVEP}$$

The next notions are generalizations of the g-monotonicity and maximal g-monotonicity, respectively, introduced by W. Oettli [24] in the scalar case.

Definition 3 The bifunction $\varphi : A \times A \to Z$ is said to be:

(i) G-pseudomonotone if, for all $a, b \in A$,

$$\varphi(a, b) \notin -C \setminus \{0\} \text{ implies } G(b, a) \notin C \setminus \{0\};$$

(ii) maximal G-pseudomonotone if it is G-pseudomonotone and, for all $a, b \in A$,

$$G(x, a) \notin C \setminus \{0\} \text{ for all } x \in]a, b] \text{ implies } \varphi(a, b) \notin -C \setminus \{0\}.$$

Proposition 2 *If* $\varphi : A \times A \to Z$ *is maximal* G*-pseudomonotone, then the sets of solutions of problems* (VEP) *and* (GVEP) *coincide.*

Proof. Let $\bar{a} \in A$ be a solution of problem (VEP), i.e.

$$\varphi(\bar{a}, b) \notin -C \setminus \{0\} \text{ for all } b \in A.$$

By the G-pseudomonotonicity of φ we deduce that $G(b, \bar{a}) \notin C \setminus \{0\}$ for all $b \in A$, which assures that $\bar{a}$ is a solution of problem $(GVEP)$.

For the converse inclusion, suppose that $\bar{a} \in A$ is a solution of problem $(GVEP)$, i.e.

$$G(b, \bar{a}) \notin C \setminus \{0\} \text{ for all } b \in A.$$

Take $b \in A$. Thus, by the convexity of the set A we have $]\bar{a}, b] \subseteq A$. Therefore

$$G(x, \bar{a}) \notin C \setminus \{0\} \text{ for all } x \in]\bar{a}, b].$$

Since φ is maximal G-pseudomonotone, we get $\varphi(\bar{a}, b) \notin -C \setminus \{0\}$. Taking into account that $b \in A$ was arbitrarily chosen, it results that $\bar{a}$ is a solution of problem (VEP).

Now, let us consider the case when the set of solutions of problem $(GVEP)$ is empty, i.e. for any $a \in A$ there exists $b_a \in A$ such that

$$G(b_a, a) \in C \setminus \{0\}.$$

Due to fact that φ is maximal G-pseudomonotone we have

$$\varphi(a, b_a) \in -C \setminus \{0\},$$

which means that the set of solutions of problem (VEP) is the empty set. Thus, we proved that the sets of solutions of problems $(GVEP)$ and (VEP) coincide also for this particular case, and this completes the proof. $\square$

By using the dual formulation $(GVEP)$ of problem (VEP) we obtain the following existence results for solutions of problem (VEP).

Theorem 1 *Suppose that the bifunctions* $\varphi : A \times A \to Z$ *and* $G : A \times A \to Z$ *satisfy the following conditions:*

(i) $\varphi(a, a) \in C$ *for all* $a \in A$*;*

(ii) φ *is maximal* G*-pseudomonotone;*

(iii) *for each* $b \in A$*, the set* $S(b) := \{a \in A \mid G(b, a) \notin C \setminus \{0\}\}$ *is closed;*

(iv) *for each* $a \in A$*, the set* $W(a) := \{b \in A \mid \varphi(a, b) \in -C \setminus \{0\}\}$ *is convex;*

(v) *there exist a nonempty, compact and convex set* $D \subseteq A$ *as well as an element* $\tilde{b} \in D$ *such that*

$$\varphi(x, \tilde{b}) \in -C \setminus \{0\} \text{ *for all* } x \in A \setminus D.$$

Then problem (VEP) *admits a solution.*

Proof. First, we show that the multifunction $T : A \to 2^A$, defined by

$$T(b) := \mathrm{cl}\{a \in A \mid \varphi(a, b) \notin -C \setminus \{0\}\},$$

is a KKM-operator. In view of assumption (i), it results that the set $T(b)$ is nonempty for each $b \in A$.

By contradiction, we suppose that T is not a KKM-operator, i.e. there exist a finite subset $\{b_1, b_2, \dots, b_n\}$ of A and numbers $\lambda_1, \dots, \lambda_n \geq 0$ with $\lambda_1 + \dots + \lambda_n = 1$ such that

$$\bar{b} := \sum_{i=1}^{n} \lambda_i b_i \notin T(b_j) \text{ for all } j \in \{1, 2, \dots, n\}.$$

This relation gives

$$\varphi(\bar{b}, b_j) \in -C \setminus \{0\} \text{ for all } j \in \{1, \dots, n\}.$$

So, $b_j \in W(\bar{b})$ for all $j \in \{1, \dots, n\}$. But, by assumption (iv), the set $W(\bar{b})$ is convex. Consequently, it follows that $\bar{b} \in W(\bar{b})$. This is a contradiction to assumption (i).

Assumption (v) assures the existence of an element $\tilde{b} \in D$ such that

$$\varphi(x, \tilde{b}) \in -C \setminus \{0\} \text{ for all } x \in A \setminus D.$$

Thus, $T(\tilde{b}) \subseteq D$. Because D is compact and $T(\tilde{b})$ is closed, it follows that $T(\tilde{b})$ is a compact set. The assumptions of Lemma 1 are satisfied and, by this we get the existence of a point $\bar{a} \in D$ such that $\bar{a} \in T(b)$ for all $b \in A$. The G-pseudomonotonicity of φ and the closedness of the set $S(b)$ imply that

$$T(b) \subseteq S(b) \text{ for all } b \in A.$$

Therefore, we obtain $\bar{a} \in S(b)$ for all $b \in A$, i.e. $\bar{a}$ is a solution of problem $(GVEP)$. By Proposition 2, $\bar{a}$ is a solution of problem (VEP). $\square$

Remark 1 It is worth to underline that our result is different from Theorem 1 established in [1]. Assumption (i) of Theorem 1 is stronger than condition (i) of Theorem 1 from [1] (we just have to take $B = A$ and T to be the identity operator in condition (i) of Theorem 1 of [1]), while our coercivity condition is weaker than the compactness assumption for the set A. Further, the part of the maximal G-pseudomonotonicity which assures the inclusion between the set of solutions of the dual problem and the set of solution of the initial problem, is different from the one considered in condition (v) of the Theorem 1 from [1].

Corollary 1 *Suppose that the bifunctions* $\varphi : A \times A \to Z$ *and* $G : A \times A \to Z$ *satisfy the following conditions:*

(i) $\varphi(a, a) \in C$ *for all* $a \in A$*;*

(ii) φ *is maximal* G*-pseudomonotone;*

(iii) *for each* $b \in A$*, the set* $S(b) := \{a \in A \mid G(b, a) \notin C \setminus \{0\}\}$ *is closed;*

(iv) *for each* $a \in A$*, the function* $\varphi(a, \cdot) : A \to Z$ *is* C*-quasiconvex;*

(v) *there exist a nonempty, compact and convex set* $D \subseteq A$ *as well as an element* $\tilde{b} \in D$ *such that*

$$\varphi(x, \tilde{b}) \in -C \setminus \{0\} \text{ for all } x \in A \setminus D.$$

Then problem (VEP) *admits a solution.*

Proof. For the proof of this corollary, we have to show that the assumptions of Theorem 1 are satisfied. It is obvious that the assumptions (i), (ii), (iii) and (v) are satisfied. To verify assumption (iv), fix $a \in A$, and let $b_1, b_2 \in A$ and $\lambda \in [0, 1]$ be such that $b_1, b_2 \in W(a)$, i.e.

$$\varphi(a, b_1) \in -C \setminus \{0\} \text{ and } \varphi(a, b_2) \in -C \setminus \{0\}.$$

By the C-quasiconvexity of $\varphi(a, \cdot)$ there is an index $i_0 \in \{1, 2\}$ with the property

$$\varphi(a, b_{i_0}) \in \varphi(a, tb_1 + (1 - t)b_2) + C.$$

So, there exists $c \in C$ such that

$$\varphi(a, b_{i_0}) = \varphi(a, tb_1 + (1 - t)b_2) + c. \tag{1}$$

Because $d := \varphi(a, b_{i_0}) \in -C \setminus \{0\}$, by (1) we get

$$\varphi(a, tb_1 + (1-t)b_2) = -c + d \in -C \setminus \{0\}.$$

Thus, the set $W(a)$ is convex. □

Remark 2 Assumption (iv) in Theorem 1 does not imply assumption (iv) of Corollary 1. Indeed, let $E = Z$, let $C \subsetneq Z$ be a pointed convex cone such that the ordering defined by C is not total on A, and let $\varphi : A \times A \to Z$ be defined by

$$\varphi(a, b) := b \text{ for all } a, b \in A.$$

In order to verify assumption (iv) of Theorem 1, fix $a \in A$ and take $b_1, b_2 \in W(a)$. Thus $b_1, b_2 \in -C \setminus \{0\}$. Because $-C \setminus \{0\}$ is convex, we have

$$\lambda b_1 + (1-\lambda)b_2 \in -C \setminus \{0\} \text{ for every } \lambda \in [0, 1].$$

So, $W(a)$ is a convex set.

Now, let $b_1, b_2 \in A$ and $\lambda \in [0, 1]$. Suppose that $\varphi(a, \cdot) : A \to Z$ is C-quasiconvex. Thus, we obtain

$$b_1 \in b_2 + C \text{ or } b_2 \in b_1 + C.$$

Since b_1 and b_2 were arbitrarily chosen and the ordering induced by C on A is not total, it follows that the function $\varphi(a, \cdot)$ is not C-quasiconvex.

3 Particular cases of the generalized dual problem

In what follows we consider two particular cases of the operator $\mathcal{D}$. Firstly we define $\mathcal{D} : \mathcal{F}(A, Z) \to \mathcal{F}(A, Z)$ by

$$\mathcal{D}(\psi)(a, b) := -\psi(b, a) \text{ for all } a, b \in A. \tag{2}$$

So, the generalized dual strong vector equilibrium problem becomes:

$$\text{find } \bar{a} \in A \text{ such that } \varphi(b, \bar{a}) \notin C \setminus \{0\} \text{ for all } b \in A. \tag{DVEP1}$$

Under pseudomonotonicity assumptions we will give an existence result for the strong vector equilibrium problem (VEP). For this, we recall some monotonicity notions, used in the past for vector-valued bifunctions. Taking into consideration that the vector-valued bifunction $G : A \times A \to Z$, associated with the operator $\mathcal{D} : \mathcal{F}(A, Z) \to \mathcal{F}(A, Z)$ defined by (2), coincides with φ, Definition 3 yields the following definition.

Definition 4 The bifunction $\varphi : A \times A \to Z$ is said to be:

(i) pseudomonotone if, for all $a, b \in A$,

$$\varphi(a, b) \notin -C \setminus \{0\} \text{ implies } \varphi(b, a) \notin C \setminus \{0\};$$

(ii) maximal pseudomonotone if it is pseudomonotone and, for all $a, b \in A$,

$$\varphi(x, a) \notin C \setminus \{0\} \text{ for all } x \in]a, b] \text{ implies } \varphi(a, b) \notin -C \setminus \{0\}.$$

Proposition 3 *If $\varphi : A \times A \to Z$ is maximal pseudomonotone, then the sets of solutions of problems* (VEP) *and* $(\mathsf{DVEP1})$ *coincide.*

Proof. Take

$$G(b, a) := \varphi(b, a) \text{ for all } a, b \in A$$

in Proposition 2. □

Theorem 1 provides the next existence result of solutions of (VEP) under a pseudomonotonicity assumption.

Corollary 2 *Suppose that the bifunction $\varphi : A \times A \to Z$ satisfies the following conditions:*

(i) $\varphi(a, a) \in C$ *for all* $a \in A$*;*

(ii) φ *is maximal pseudomonotone;*

(iii) *for each* $b \in A$*, the set* $S(b) := \{a \in A \mid \varphi(b, a) \notin C \setminus \{0\}\}$ *is closed;*

(iv) *for each* $a \in A$*, the set* $W(a) := \{b \in A \mid \varphi(a, b) \in -C \setminus \{0\}\}$ *is convex;*

(v) *there exist a nonempty, compact and convex set* $D \subseteq A$ *as well as an element* $\tilde{b} \in D$ *such that*

$$\varphi(x, \tilde{b}) \in -C \setminus \{0\} \text{ for all } x \in A \setminus D.$$

Then problem (VEP) *admits a solution.*

Now, if we define $\mathcal{D} : \mathcal{F}(A, Z) \to \mathcal{F}(A, Z)$ by $\mathcal{D}(\psi) := \psi$, we obtain an existence result for problem (VEP) without pseudomonotonicity assumptions. It is easy to verify that the assumption of φ to be maximal G-pseudomonotone is fulfilled.

In this case, the generalized dual problem of problem (VEP) is exactly:

$$\text{find } \bar{a} \in A \text{ such that } \varphi(\bar{a}, b) \notin -C \setminus \{0\} \text{ for all } b \in A. \qquad (\mathsf{VEP})$$

Corollary 3 *Suppose that the bifunction* $\varphi : A \times A \to Z$ *satisfies the following conditions:*

(i) $\varphi(a, a) \in C$ *for all* $a \in A$*;*

(ii) *for each* $b \in A$*, the set* $S(b) := \{a \in A \mid \varphi(a, b) \notin -C \setminus \{0\}\}$ *is closed;*

(iii) *for each* $a \in A$*, the set* $W(a) := \{b \in A \mid \varphi(a, b) \in -C \setminus \{0\}\}$ *is convex;*

(iv) *there exist a nonempty, compact and convex set* $D \subseteq A$ *as well as an element* $\tilde{b} \in D$ *such that*

$$\varphi(x, \tilde{b}) \in -C \setminus \{0\} \text{ for all } x \in A \setminus D.$$

Then problem (VEP) *admits a solution.*

Theorem 1 and Corollary 3 allow us to reobtain Lemma 1 and Theorem 2 established by W. Oettli [24], which are existence results for scalar equilibrium problems. Indeed, in what follows assume that $Z := \mathbb{R}$ and $C := \mathbb{R}_+$.

Corollary 4 ([24]) *Let the bifunctions* $\varphi : A \times A \to \mathbb{R}$ *and* $G : A \times A \to \mathbb{R}$ *satisfy the following conditions:*

(i) $\varphi(a, a) \geq 0$ *for all* $a \in A$*;*

(ii) φ *is maximal* G*-pseudomonotone;*

(iii) *for each* $b \in A$*, the set* $S(b) := \{a \in A \mid G(b, a) \leq 0\}$ *is closed;*

(iv) *for each* $a \in A$*, the set* $W(a) := \{b \in A \mid \varphi(a, b) < 0\}$ *is convex;*

(v) *there exist a nonempty, compact and convex set* $D \subseteq A$ *as well as an element* $\tilde{b} \in D$ *such that*

$$\varphi(x, \tilde{b}) < 0 \text{ for all } x \in A \setminus D.$$

Then the scalar equilibrium problem admits a solution.

Corollary 5 ([24]) *Suppose that the bifunction* $\varphi : A \times A \to \mathbb{R}$ *satisfies the following conditions:*

(i) $\varphi(a, a) \geq 0$ *for all* $a \in A$*;*

(ii) *for each* $b \in A$*, the set* $S(b) := \{a \in A \mid \varphi(a, b) \geq 0\}$ *is closed;*

(iii) *for each* $a \in A$*, the set* $W(a) := \{b \in A \mid \varphi(a, b) < 0\}$ *is convex;*

(iv) *there exist a nonempty, compact and convex set* $D \subseteq A$ *as well as an element* $\tilde{b} \in D$ *such that*

$$\varphi(x, \tilde{b}) < 0 \textit{ for all } x \in A \setminus D.$$

Then the scalar equilibrium problem admits a solution.

Corollary 5 is a slight generalization of an existence result established in [10] and recovered by [23]. In the sequel we deduce Fan's result.

Corollary 6 ([10]) *Let* A *be a compact set, and let* $\varphi : A \times A \to \mathbb{R}$ *satisfy the following conditions:*

(i) $\varphi(a, a) \geq 0$ *for all* $a \in A$*;*

(ii) $\varphi(\cdot, b) : A \to \mathbb{R}$ *is upper semicontinuous for all* $b \in A$*;*

(iii) $\varphi(a, \cdot) : A \to \mathbb{R}$ *is quasiconvex for all* $a \in A$*.*

Then the scalar equilibrium problem admits a solution.

Proof. Because A is compact and $\varphi(\cdot, b) : A \to \mathbb{R}$ is upper semicontinuous on A for all $b \in A$, the assumptions (ii) and (iv) of Corollary 5 are satisfied.

It remains to show the convexity of the set $W(a)$ for each $a \in A$. So, let $b_1, b_2 \in A$ and $\lambda \in [0, 1]$. By the quasiconvexity of $\varphi(a, \cdot) : A \to \mathbb{R}$ and the inequality

$$\max\{\varphi(a, b_1), \varphi(a, b_2)\} < 0$$

we deduce that

$$\varphi(a, \lambda b_1 + (1 - \lambda) b_2) < 0.$$

Thus, assumption (iii) of Corollary 5 is also satisfied and the proof is completed. □

4 Applications to strong vector variational inequalities

Strong vector variational inequality problems are particular cases of the strong vector equilibrium problem (VEP). Let A be a nonempty convex subset of a real topological linear space E, and let $F : A \to L(E, Z)$ be a mapping, where $L(E, Z)$ denotes the set of all continuous linear functions from E to a real Hausdorff topological linear space Z. Further, let $C \subseteq Z$ be a nontrivial pointed convex cone. Using these notations, in this section we will study the following variational inequalities:

$$\text{find } \bar{a} \in A \text{ such that } \langle F(b), b - \bar{a} \rangle \notin -C \setminus \{0\} \text{ for all } b \in A; \qquad (MVI)$$

$$\text{find } \bar{a} \in A \text{ such that } \langle F(\bar{a}), b - \bar{a} \rangle \notin -C \setminus \{0\} \text{ for all } b \in A. \qquad (SVI)$$

As in the previous section, for all $a, b \in A$, $\langle F(b), b - a \rangle$ denotes the value of the function $F(b)$ at the point $b-a$. Problem (MVI) is called *the strong Minty vector variational inequality*, while (SVI) is called *the strong Stampacchia vector variational inequality*.

Using the generalized duality theory presented in the main section we deduce that the strong Stampacchia vector variational inequality (SVI) admits as a generalized dual the strong Minty vector variational inequality (MVI). We notice that the vice-versa also holds, i.e. the generalized dual problem of (MVI) is (SVI).

In [12] there is presented an existence result for (SVI) under the following monotonicity property. The mapping $F : A \to L(E, Z)$ is said to be strongly pseudomonotone if, for all $a, b \in A$,

$$\langle F(a), b - a \rangle \notin -C \setminus \{0\} \text{ implies } \langle F(b), b - a \rangle \in C.$$

In what follows we work with the notion of pseudomonotonicity, which is weaker than the above one. To see this, we will give an example.

Definition 5 ([12]) The mapping $F : A \to L(E, Z)$ is said to be pseudomonotone if, for all $a, b \in A$,

$$\langle F(a), b - a \rangle \notin -C \setminus \{0\} \text{ implies } \langle F(b), b - a \rangle \notin -C \setminus \{0\}.$$

Example 1 Let $E := \mathbb{R}^2$, $A := [0, 1] \times [0, 1]$, $Z := \mathbb{R}^2$, $C := \mathbb{R}^2_+$, and define $F : A \to L(\mathbb{R}^2, \mathbb{R}^2)$ by

$$\langle F(a), x \rangle := (x_1 + x_2)(a_1 - 2, a_2 + 2) \text{ for all } a := (a_1, a_2) \in A,\ x := (x_1, x_2) \in \mathbb{R}^2.$$

Let $a := (a_1, a_2)$ and $b := (b_1, b_2)$ be points from A. Since $a_1 - 2 < 0$ and $a_2 + 2 > 0$, it follows from

$$\langle F(a), b - a \rangle = (b_1 + b_2 - a_1 - a_2)(a_1 - 2, a_2 + 2)$$

that $\langle F(a), b-a\rangle \notin -\mathbb{R}^2_+ \setminus \{0\}$. Similarly, taking into consideration that $b_1 - 2 < 0$ and $b_2 + 2 > 0$, we obtain from

$$\langle F(b), b-a\rangle = (b_1 + b_2 - a_1 - a_2)(b_1 - 2, b_2 + 2) \tag{3}$$

that $\langle F(b), b-a\rangle \notin -\mathbb{R}^2_+ \setminus \{0\}$. Consequently, F is pseudomonotone. On the other hand, when $b_1 + b_2 - a_1 - a_2 \neq 0$, then (3) implies that

$$\langle F(b), b-a\rangle \notin \mathbb{R}^2_+.$$

Thus F is not strongly pseudomonotone.

The following notion is a particular case of Definition 3.

Definition 6 The mapping $F : A \to L(E, Z)$ is said to be maximal pseudomonotone if the following conditions are satisfied:

(i) F is pseudomonotone;

(ii) for all $a, b \in A$ the following implication holds: if $\langle F(x), a-x\rangle \notin C \setminus \{0\}$ for all $x \in]a, b]$, then $\langle F(a), a-b\rangle \notin C \setminus \{0\}$.

The next statement follows by Proposition 3.

Proposition 4 *If F is maximal pseudomonotone, then the solution sets of problems (SVI) and (MVI) coincide.*

Using Corollary 2, we obtain the following existence result for (SVI).

Theorem 2 *Suppose that the following conditions are satisfied:*

(i) *F is maximal pseudomonotone;*

(ii) *the set $S(b) := \{a \in A \mid \langle F(b), b-a\rangle \notin -C \setminus \{0\}\}$ is closed for all $b \in A$;*

(iii) *there exist a nonempty, compact and convex set $D \subseteq A$ as well as an element $\tilde{b} \in D$ such that*

$$\langle F(x), \tilde{b} - x\rangle \in -C \setminus \{0\} \text{ for all } x \in A \setminus D.$$

Then the problem (SVI) admits a solution.

Proof. Let $\varphi : A \times A \to Z$ be defined by $\varphi(a, b) := \langle F(a), b - a\rangle$. We show that φ satisfies the assumptions of Corollary 2. Indeed, the assumptions (ii), (iii) and (v) are satisfied by the hypothesis (i), (ii) and (iii), respectively. It remains to verify the assumptions (i) and (iv) of Corollary 2. Let $a \in A$. Since $F(a) \in L(E, Z)$, it follows that $\varphi(a, a) = 0$ and that the set

$$W(a) := \{b \in A \mid \langle F(a), b - a\rangle \in -C \setminus \{0\}\}$$

is convex. So, all the assumptions of Corollary 2 are fulfilled. Hence, there exists $\bar{a} \in A$ which is a solution for (SVI). □

Example 2 To show that there exists mappings which satisfy the assumptions of Theorem 2, let $E := \mathbb{R}^2$, $A := [0, 1] \times [0, 1]$, $Z := \mathbb{R}^2$, $C := \mathbb{R}^2_+$, and define $F : [0, 1] \times [0, 1] \to L(\mathbb{R}^2, \mathbb{R}^2)$ by

$$\langle F(a), x\rangle := (x_1 + x_2)(a_1 + 1, a_2 + 1) \tag{4}$$

for all $a := (a_1, a_2) \in A$ and all $x := (x_1, x_2) \in \mathbb{R}^2$.

Since

$$(a_1 + 1, a_2 + 1) \in \mathbb{R}^2_+ \setminus \{0\} \text{ for each } a := (a_1, a_2) \in A,$$

it results from (4) that

$$\forall\, a \in A\ :\ \{x \in \mathbb{R}^2 \mid \langle F(a), x\rangle \notin \mathbb{R}^2_+ \setminus \{0\}\} = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1{+}_2 \le 0\}. \tag{5}$$

This inequality implies that

$$\forall\, a, b \in A\ :\ \{x \in \mathbb{R}^2 \mid \langle F(a), x\rangle \notin \mathbb{R}^2_+ \setminus \{0\}\} = \{x \in \mathbb{R}^2 \mid \langle F(b), x\rangle \notin \mathbb{R}^2_+ \setminus \{0\}\}. \tag{6}$$

Let $a := (a_1, a_2)$ and $b := (b_1, b_2)$ be points from A. Suppose that

$$\langle F(a), b - a\rangle \notin -\mathbb{R}^2_+ \setminus \{0\}.$$

Then we have $\langle F(a), a - b\rangle \notin \mathbb{R}^2_+ \setminus \{0\}$. By virtue of (6) we obtain

$$\langle F(b), a - b\rangle \notin \mathbb{R}^2_+ \setminus \{0\}, \text{ whence } \langle F(b), b - a\rangle \notin -\mathbb{R}^2_+ \setminus \{0\}.$$

Thus F is a pseudomonotone mapping

Next suppose that

$$\langle F(x), a - x\rangle \notin \mathbb{R}^2_+ \setminus \{0\} \text{ for all } x \in]a, b].$$

In particular, we have

$$\langle F(b), a-b\rangle \notin \mathbb{R}^2_+ \setminus \{0\}.$$

By virtue of (6) we get $\langle F(a), a-b\rangle \notin \mathbb{R}^2_+ \setminus \{0\}$. Hence F is a maximal pseudomonotone mapping. In other words, condition (i) in Theorem 2 is satisfied.

From (5) it follows that

$$S(b) = \{a \in A \mid \langle F(b), a-b\rangle \notin \mathbb{R}^2_+ \setminus \{0\}\} = \{(a_1, a_2) \in A \mid a_1 + a_2 \leq b_1 + b_2\}$$

for each $b := (b_1, b_2) \in A$. Consequently, condition (ii) in Theorem 2 is also satisfied.

Finally, it is obvious that condition (iii) in Theorem 2 is satisfied for $D := A$.

By Corollary 3 we obtain an existence result for the strong Stampacchia vector variational inequality without monotonicity assumptions. This new existence result is a slight generalization of Theorem 2.1 from [12].

Theorem 3 *Suppose that the following conditions are satisfied:*

(i) *for all* $b \in A$ *the set* $S(b) := \{a \in A \mid \langle F(a), b-a\rangle \notin -C \setminus \{0\}\}$ *is closed;*

(ii) *there exist a nonempty, compact and convex set* $D \subseteq A$ *as well as an element* $\tilde{b} \in D$ *such that*

$$\langle F(x), \tilde{b} - x\rangle \in -C \setminus \{0\} \text{ for all } x \in A \setminus D.$$

Then problem (SVI) *admits a solution.*

Proof. Define the bifunction $\varphi : A \times A \to Z$ by

$$\varphi(a, b) := \langle F(a), b-a\rangle \text{ for all } a, b \in A.$$

Let $a \in A$. Since $F(a) \in L(E, Z)$, it follows that $\varphi(a, a) = 0$ and that the set

$$W(a) := \{b \in A \mid \langle F(a), b-a\rangle \in -C \setminus \{0\}\}$$

is convex. By virtue of this observation, all the assumptions of Corollary 3 are satisfied. So, the strong Stampacchia vector variational inequality admits a solution. □

Corollary 7 ([12]) *Let* E *be a real Banach space, let A be a compact subset, let* Z *be a real Banach space ordered by a nonempty pointed solid convex cone C. Suppose that* $F : A \to L(E, Z)$ *is a mapping such that for every* $b \in A$ *the set*

$$\{a \in A \mid \langle F(a), b-a\rangle \in -C \setminus \{0\}\}$$

is open in A. Then problem (SVI) *is solvable.*

References

[1] Q. H. Ansari, W. Oettli, D. Schläger, A generalization of vector equilibria, *Math. Methods Oper. Res.*, **46** (1997), 147–152.

[2] H. P. Benson, An improved definition of proper efficiency for vector maximization with respect to cones, *J. Math. Anal. Appl.*, **71** (1979), 232–241.

[3] G. Bigi, A. Capătă, G. Kassay, Existence results for strong vector equilibrium problems and their applications, *Optimization*, **61** (2012), 567–583.

[4] J. M. Borwein, Proper efficient points for maximizations with respect to cones, *SIAM J. Control Optim.*, **15** (1977), 57–63.

[5] J. M. Borwein, V. Jeyakumar, A. S. Lewis, H. Wolkowicz, *Constrained approximation via convex programming*, preprint, University of Waterloo, 1998.

[6] J. M. Borwein, A. S. Lewis, Partially finite convex programming, part I: Quasi relative interiors and duality theory, *Math. Program.*, **57** (1992), 15–48.

[7] A. Capătă, Existence results for proper efficient solutions of vector equilibrium problems and applications, *J. Global Optim.*, **51** (2011), 657–675.

[8] G. Y. Chen, S. H. Hou, Existence of solutions for vector variational inequalities, in: F. Giannessi (ed.), *Vector Variational Inequalities and Vector Equilibria*, Kluwer Academic Publishers, Dordrecht 2000, 73–86.

[9] K. Fan, A generalization of Tychonoff's fixed point theorem, *Math. Ann.*, **142** (1961), 305–310.

[10] K. Fan, A minimax inequality and applications, in: O. Shisha (ed.), *Inequalities III*, Academic Press, New York, **3** (1972), 103–113.

[11] Y. P. Fang, N. J. Huang, Vector equilibrium type problems with $(S)_+$-conditions, *Optimization*, **53** (2004), 269–279.

[12] Y. P. Fang, N. J. Huang, Strong vector variational inequalities in Banach spaces, *Appl. Math. Lett.*, **19** (2006), 362–368.

[13] A. M. Geoffrion, Proper efficiency and the theory of vector maximization, *J. Math. Anal. Appl.*, **22** (1968), 613–630.

[14] F. Giannessi, Theorems of alternative, quadratic programs and complementarity problems, in: R. W. Cottle, F. Giannessi and J. L. Lions (eds.), *Variational Inequalities and Complementarity Problems*, Wiley, Chichester, 1980, 151–186.

[15] X. H. Gong, Strong vector equilibrium problems, *J. Global Optim.*, **36** (2006), 339–349.

[16] X. H. Gong, The strong minimax theorem and strong saddle points of vector-valued functions, *Nonlinear Anal.*, **68** (2008), 2228–2241.

[17] M. S. Gowda, M. Teboulle, A comparison of constraint qualifications in infinite-dimensional convex programming, *SIAM J. Control Optim.*, **28** (1990), 925–935.

[18] R. Hartley, On cone-efficiency, cone-convexity and cone-compactness, *SIAM J. Appl. Math.*, **34** (1978), 211–222.

[19] M. I. Henig, Proper efficiency with respect to cones, *J. Optim. Theory Appl.*, **36** (1982), 387–407.

[20] R. B. Holmes, *Geometric functional analysis*, Springer, Berlin, 1975.

[21] B. Knaster, C. Kuratowski, S. Mazurkiewicz, Ein beweis des fixpunktsatzes für n-dimensionale simplexe, *Fundam. Math.*, **11** (1935), 767–776.

[22] H. W. Kuhn, A. W. Tucker, Nonlinear programming, in: J. Neyman (ed.), *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, University of California Press, Berkeley California, 1951, 481–492.

[23] L. J. Lin, Z. T. Yu, G. Kassay, Existence of equilibria for multivalued mappings and its applications to vectorial equilibria, *J. Optim. Theory Appl.*, **114** (2002), 189–208.

[24] W. Oettli, A remark on vector-valued equilibria and generalized monotonicity, *Acta Math. Vietnam.*, **22** (1997), 213–221.

[25] R. T. Rockafellar, *Conjugate duality and optimization*, Society for Industrial and Applied Mathematics, Philadelphia, 1974.

[26] N. X. Tan, P. N. Tinh, On the existence of equilibrium points of vector functions, *Numer. Funct. Anal. Optim.*, **19** (1998), 141–156.

Received: June 9, 2011