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Mark sequences in bipartite multidigraphs and constructions

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Abstract. A bipartite r-digraph is an orientation of a bipartite multigraph without loops and contains at most r edges between any pair of vertices from distinct parts. In this paper, we obtain necessary and sufficient conditions for a pair of sequences of non-negative integers in non-decreasing order to be a pair of sequences of numbers, called marks (or r-scores), attached to the vertices of a bipartite r-digraph. One of the characterizations is combinatorial and the other is recursive. As an application, these characterizations provide algorithms to construct a bipartite r-digraph with given mark sequences.

1 Introduction

An r-digraph is an orientation of a multigraph without loops and contains at most r edges between any pair of distinct vertices. So, 1-digraph is an oriented graph, and a complete 1-digraph is a tournament. Let D be an r-digraph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$, and let $d^+_{v_i}$ and $d^-_{v_i}$ denote the outdegree and indegree, respectively, of a vertex v_i. Define p_{v_i} (or simply p_i)$= r(n-1) + d^+_{v_i} - d^-_{v_i}$ as the mark (or r-score) of v_i, implying $0 \le p_{v_i} \le 2r(n-1)$. Then the sequence $P = [p_i]_1^n$ in non-decreasing order is called the mark sequence of D.

The following criterion for marks in r-digraphs due to Pirzada et al. [8] is analogous to a result on scores in tournaments given by Landau [6].

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Theorem 1 *A sequence* $P = [p_i]_1^n$ *of non-negative integers in non-decreasing order is the mark sequence of an* r*-digraph if and only*

$$\sum_{i=1}^{t} p_i \geq rt(t-1),$$

for $1 \leq t \leq n$*, with equality when* $t = n$*.*

Many results on marks in digraphs can be seen in [7, 9, 12, 14]. Also results for scores in oriented graphs can be found in [1, 11], while on tournaments we refer to [3, 4, 5]. Also it is important to mention here that the concept of scores has been extended to hypertournaments [15, 16, 17].

A bipartite r-digraph is an orientation of a bipartite multigraph without loops and contains at most r edges between any pair of vertices from distinct parts. So bipartite 1-digraph is an oriented bipartite graph and a complete bipartite 1-digraph is a bipartite tournament. Let $D(X, Y, A)$ be a bipartite r-digraph with vertex sets $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ and arc set A with each arc having one end in X and the other end in Y. For any vertex v_i in $D(X, Y)$, let $d^+_{v_i}$ and $d^-_{v_i}$ be the outdegree and indegree, respectively, of v_i. Define p_{x_i} (or simply p_i) $= rn + d^+_{x_i} - d^-_{x_i}$ and q_{y_j} (or simply q_j)$=$ $rm + d^+_{y_j} - d^-_{y_j}$ as the marks (or r-scores) of x_i in X and y_j in Y respectively. Clearly, $0 \leq p_{x_i} \leq 2rn$ and $0 \leq q_{y_j} \leq 2rm$. Then the sequences $P = [p_i]_1^m$ and $Q = [q_j]_1^n$ in non-decreasing order are called the mark sequences of $D(X, Y, A)$.

A bipartite r-digraph can be interpreted as the result of a competition between two teams in which each player of one team plays with every player of the other team at most r times in which ties (draws) are allowed. A player receives two points for each win, and one point for each tie. With this marking system, player x_i (respectively y_j) receives a total of p_{x_i} (respectively q_{y_j}) points. The sequences P and Q of non-negative integers in non-decreasing order are said to be realizable if there exists a bipartite r-digraph with mark sequences P and Q.

In a bipartite r-digraph $D(X, Y, A)$, if there are a_1 arcs directed from a vertex $x \in X$ to a vertex $y \in Y$ and a_2 arcs directed from vertex y to vertex x, with $0 \leq a_1, a_2 \leq r$ and $0 \leq a_1 + a_2 \leq r$, we denote it by $x(a_1 - a_2)y$. For example, if there are exactly r arcs directed from $x \in X$ to $y \in Y$ and no arc directed from y to x, this is denoted by $x(r-0)y$, and if there is no arc directed from x to y and no arc directed from y to x, this is denoted by $x(0-0)y$.

The following characterization of mark sequences in bipartite 2-digraphs [13] is analogous to a result on scores in bipartite tournaments due to Beineke and Moon [2].

Theorem 2 *Let* $P = [p_i]_1^m$ *and* $Q = [q_j]_1^n$ *be sequences of non-negative integers in non-decreasing order. Then* P *and* Q *are mark sequences of some bipartite 2-digraph if and only if*

$$\sum_{i=1}^{f} p_i + \sum_{j=1}^{g} q_j \geq 4fg,$$

for $1 \leq f \leq m$ *and* $1 \leq g \leq n$ *with equality when* $f = m$ *and* $g = n$.

Analogous results for scores in oriented bipartite graphs can be found in [10].

An oriented tetra in a bipartite r-digraph is an induced 1-subdigraph with two vertices from each part. Define oriented tetras of the form $x(1-0)y(1-0)x'$ $(1-0)y'(1-0)x$ and $x(1-0)y(1-0)x'(1-0)y'(0-0)x$ to be of α-type and all other oriented tetras to be of β-type. A bipartite r-digraph is said to be of α-type or β-type according as all of its oriented tetras are of α-type or β-type respectively. We assume, without loss of generality, that β-type bipartite r-digraphs have no pair of symmetric arcs because symmetric arcs $x(a-a)y$, where $1 \leq a \leq \frac{r}{2}$, can be transformed to $x(0-0)y$ with the same marks. A transmitter is a vertex with indegree zero.

2 Criteria for realizability and construction algorithms

We start with the following observations.

Lemma 1 *Among all bipartite* r*-digraphs with given mark sequences, those with the fewest arcs are of* β*-type.*

Proof. Let $D(X, Y)$ be a bipartite r-digraph with mark sequences P and Q. Assume $D(X, Y)$ is not of β-type. Then $D(X, Y)$ has an oriented tetra of α-type, that is, $x(1-0)y(1-0)x'(1-0)y'(1-0)x$ or $x(1-0)y(1-0)x'(1-0)y'(0-0)x$ where $x, x' \in X$ and $y, y' \in Y$. Since $x(1-0)y(1-0)x'(1-0)y'(1-0)x$ can be transformed to $x(0-0)y(0-0)x'(0-0)y'(0-0)x$ with the same mark sequences and four arcs fewer, and $x(1-0)y(1-0)x'(1-0)y'(0-0)x$ can be transformed to $x(0-0)y(0-0)x'(0-0)y'(0-1)x$ with the same mark sequences and two arcs fewer, therefore, in both cases we obtain a bipartite r-digraph having same mark sequences P and Q with fewer arcs. Note that if there are symmetric arcs between x and y, that is $x(a-a)y$, where $1 \leq a \leq \frac{r}{2}$, then these can be transformed to $x(0-0)y$ with the same mark sequences and a arcs fewer. Hence the result follows. □

Lemma 2 *Let* $P = [p_i]_1^m$ *and* $Q = [q_j]_1^n$ *be mark sequences of a* β*-type bipartite* r*-digraph. Then either the vertex with mark* p_m*, or the vertex with mark* q_n*, or both can act as transmitters.*

We now have some observations about bipartite r-digraphs, as these will be required in application of Theorem 2.10. If $P = [p_1, p_2, \dots, p_m]$ and $Q = [q_1, q_2, \dots, q_n]$ are mark sequences of a bipartite r-digraph, then $p_i \leq 2rn$ and $q_j \leq 2rm$, where $1 \leq i \leq m$ and $1 \leq j \leq m$.

Lemma 3 *If* $P = [p_1, p_2, \dots, p_{m-1}, p_m]$ *and* $Q = [0, 0, \dots, 0, 0]$ *with each* $p_i = 2rn$ *are mark sequences of some bipartite* r*-digraph, then* $P' = [p_1, p_2, \dots, p_{m-1}]$ *and* $Q' = [0, 0, \dots, 0]$ *are also mark sequences of some bipartite* r*-digraph.*

Proof. Let P and Q as given above be mark sequences of bipartite r-digraph D with parts $X = \{x_1, x_2, \dots, x_{m-1}, x_m\}$ and $Y = \{y_1, y_2, \dots, y_{n-1}, x_n\}$. Since mark of each x_i is $2rn$, so $x_i(r-0)y_j$ for each x_i and each y_j, $1 \leq i \leq m$ and $1 \leq j \leq n$. Deleting x_m will neither change the marks of the vertices x_i, for all $1 \leq i \leq m-1$ nor will change the marks of the vertices y_j, for all $1 \leq j \leq n$. Hence $P' = [p_1, p_2, \dots, p_{m-1}]$ and $Q' = [0, 0, \dots, 0]$ are the mark sequences of the bipartite r-digraph with parts $\{x_1, x_2, \dots, x_{m-1}\}$ and $\{y_1, y_2, \dots, y_{n-1}, x_n\}$, that is the bipartite r-digraph $D - x_n$. □

Lemma 4 *If* $P = [p_1, p_2, \dots, p_{m-1}, p_m]$ *and* $Q = [0, 0, \dots, 0, q_n]$ *with* $4n - p_m = 3$ *and* $q_n \geq 3$ *are mark sequences of some bipartite* r*-digraph, then* $P' = [p_1, p_2, \dots, p_{m-1}]$ *and* $Q' = [0, 0, \dots, 0, q_n - 3]$ *are also mark sequences of some bipartite* r*-digraph.*

Proof. Let P and Q as given above be mark sequences of bipartite r-digraph D with parts $X = \{x_1, x_2, \dots, x_{m-1}, x_m\}$ and $Y = \{y_1, y_2, \dots, y_{n-1}, x_n\}$. Since $4n - p_m = 3$ and $3 \leq q_n \leq 4m$, therefore in D necessarily $x_m(2-0)y_i$, for all $1 \leq i \leq n-1$. Also $y_n(1-0)x_m$, because if $y_n(0-0)x_m$, or $y_n(0-2)x_m$,or $y_n(0-1)x_m$, then in all these cases $p_{x_m} \geq 4(n-1)+2$, a contradiction to our assumption. Also $y_n(2-0)x_m$ is not possible because in that case $p_{x_m} = 4(n-1) < 4n-3$.

Now delete x_m, obviously this keeps marks of y_1,y_2, $\dots$,y_{n-1} as zeros and reduces mark of y_m by 3, and we obtain a bipartite r-digraph with mark sequences $P' = [p_1, p_2, \dots, p_{m-1}]$ and $Q' = [0, 0, \dots, 0, q_n - 3]$, as required. □

Lemma 5 *If* $P = [p_1, p_2, \dots, p_{m-1}, p_m]$ *and* $Q = [0, 0, \dots, 0, q_n]$ *with* $4n - p_m = 4$ *and* $q_n \geq 4$ *are mark sequences of some bipartite* r*-digraph, then* $P' = [p_1, p_2, \dots, p_{m-1}]$ *and* $Q' = [0, 0, \dots, 0, q_n - 4]$ *are also mark sequences of some bipartite* r*-digraph.*

Proof. Let P and Q as given above be mark sequences of a bipartite r-digraph D with parts $X = \{x_1, x_2, \dots, x_{m-1}, x_m\}$ and $Y = \{y_1, y_2, \dots, y_{n-1}, x_n\}$. Since $4n - p_m = 4$ and $4 \leq q_n \leq 4m$, therefore in D necessarily $x_l(2-0)y_i$, for all $1 \leq i \leq n-1$. Also $y_n(2-0)x_l$, because if $y_n(0-0)x_m$, or $y_n(1-0)x_m$, or $y_n(0-2)x_m$,or $y_n(0-1)x_m$, then in all these cases $p_{x_m} \geq 4(n-1)+1$, a contradiction to our assumption.

Now delete x_m, obviously this keeps marks of y_1,y_2, $\dots$,y_{n-1} as zeros and reduces mark of y_n by 4, and we obtain a bipartite r-digraph with mark sequences $P' = [p_1, p_2, \dots, p_{m-1}]$ and $Q' = [0, 0, \dots, 0, q_n - 4]$, as required. □

Lemma 6 *If* $P = [p_1, p_2, \dots, p_{m-1}, p_m]$ *and* $Q = [0, 0, \dots, 0, q_n]$ *with* $4n - p_m = 4$ *and* $q_n \geq 3$ *are mark sequences of some bipartite* r*-digraph, then* $P' = [p_1, p_2, \dots, p_{m-1}]$ *and* $Q' = [0, 0, \dots, 0, q_n - 3]$ *are also mark sequences of some bipartite* r*-digraph.*

Proof. The proof follows by using the same argument as in Lemma 5. □

Lemma 7 *If* $P = [p_1, p_2, \dots, p_{m-1}, p_m]$ *and* $Q = [0, 0, \dots, 0, 1, 3]$ *with* $4n - p_m = 4$*, are mark sequences of some bipartite* r*-digraph, then* $P' = [p_1, p_2, \dots, p_{m-1}]$ *and* $Q' = [0, 0, \dots, 0, 0, 0]$ *are also mark sequences of some bipartite* r*-digraph.*

Lemma 8 *If* $P = [p_1, p_2, \dots, p_{m-1}, p_m]$ *and* $Q = [0, 0, \dots, 0, 1, 1, 2]$ *with* $4n - p_m = 4$*, are mark sequences of some bipartite* r*-digraph, then* $P' = [p_1, p_2, \dots, p_{m-1}]$ *and* $Q' = [0, 0, \dots, 0, 0, 0]$ *are also mark sequences of some bipartite* r*-digraph.*

Lemma 9 *If* $P = [p_1, p_2, \dots, p_{m-1}, p_m]$ *and* $Q = [0, 0, \dots, 0, 1, 1, 1, 1]$ *with* $4n - p_m = 4$*, are mark sequences of some bipartite* r*-digraph, then* $P' = [p_1, p_2, \dots, p_{m-1}]$ *and* $Q' = [0, 0, \dots, 0, 0, 0]$ *are also mark sequences of some bipartite* r*-digraph.*

Remarks. We note that the sequences of non-negative integers $[p_1]$ and $[q_1, q_2, \dots, q_n]$, with $p_1 + q_1 + q_2 + \cdots + q_n = 2rn$, are always mark sequences of some bipartite r-digraph. We observe that the bipartite r-digraph

$D(X,Y)$, with vertex sets $X = \{x_1\}$ and $Y = \{y_1, y_2, \dots, y_n\}$, where for q_i even, say $2t$, we have $x_1((r-t)-t)y_i$ and for q_i odd, say $2t+1$, we have $x_1((r-t-1)-t)y_i$, has mark sequences $[p_1]$ and $[q_1, q_2, \dots, q_n]$. Also the sequences $[0]$ and $[2r, 2r, \dots, 2r]$ are mark sequences of some bipartite r-digraph.

The next result provides a useful recursive test whether or not a pair of sequences is realizable.

Theorem 3 *Let $P = [p_i]_1^m$ and $Q = [q_j]_1^n$ be sequences of non-negative integers in non-decreasing order with $p_m \geq q_n$ and $rn \leq p_m \leq 2rn$.*
(A) *If $q_n \leq 2r(m-1)+1$, let P' be obtained from P by deleting one entry p_m, and Q' be obtained as follows.*

For $[2r-(i-1)]n \geq p_m \geq (2r-i)n$, $1 \leq i \leq r$, reducing $[2r-(i-1)]n - p_m$ largest entries of Q by i each, and reducing $p_m - (2r-i)n$ next largest entries by $i-1$ each.
(B) *In case $q_n > 2r(m-1)+1$, say $q_n = 2r(m-1)+1+h$, where $1 \leq h \leq r-1$, then let P' be obtained from P by deleting one entry p_m, and Q' be obtained from Q by reducing the entry q_n by $h+1$.*

Then P and Q are the mark sequences of some bipartite r-digraph if and only if P' and Q' (arranged in non-decreasing order) are the mark sequences of some bipartite r-digraph.

Proof. Let P' and Q' be the mark sequences of some bipartite r-digraph $D'(X', Y')$. First suppose Q' is obtained from Q as in A. Construct a bipartite r-digraph $D(X,Y)$ as follows. Let $X = X' \cup x$, $Y = Y'$ with $X' \cap x = \phi$. Let $x((r-i)-0)y$ for those vertices y of Y' whose marks are reduced by i in going from P to P' and Q to Q', and $x(r-0)y$ for those vertices y of Y' whose marks are not reduced in going from P to P' and Q to Q'. Then $D(X,Y)$ is the bipartite r-digraph with mark sequences P and Q. Now, if Q' is obtained from Q as in B, then construct a bipartite r-digraph $D(X,Y)$ as follows. Let $X = X' \cup x$, $Y = Y'$ with $X' \cap x = \phi$. Let $x((r-h-1)-0)y$ for that vertex y of Y' whose marks are reduced by h in going from P and Q to P' and Q'. Then $D(X,Y)$ is the bipartite r-digraph with mark sequences P and Q.

Conversely, suppose P and Q be the mark sequences of a bipartite r-digraph $D(X,Y)$. Without loss of generality, we choose $D(X,Y)$ to be of β-type. Then by Lemma 2.2, any of the vertex $x \in X$ or $y \in Y$ with mark p_m or q_n respectively can be a transmitter. Let the vertex $x \in X$ with mark p_m be a transmitter. Clearly, $p_m \geq rn$ and because if $p_m < rn$, then by deleting p_m we have to reduce more than n entries from Q, which is absurd.
(A) Now $q_n \leq 2r(m-1)+1$ because if $q_n > 2r(m-1)+1$, then on reduction

$q'_n = q_n - 1 > 2r(m-1) + 1 - 1 = 2r(m-1)$, which is impossible.

Let $[2r-(i-1)]n \geq p_m \geq (2r-i)n$, $1 \leq i \leq r$, let V be the set of $[2r-(i-1)]n - p_m$ vertices of largest marks in Y, and let W be the set of $p_m - (2r-i)n$ vertices of next largest marks in Y and let $Z = Y - \{V, W\}$. Construct $D(X, Y)$ such that $x((r-i)-0)v$ for all $v \in V$, $x((r-i-1)-0)w$ for all $w \in W$ and $x(r-0)z$ for all $z \in Z$. Clearly, $D(X, Y) - x$ realizes P' and Q' (arranged in non-decreasing order).

(B) Now in D, let $q_n > 2r(m-1)+1$, say $q_n = 2r(m-1)+1+h$, where $1 \leq h \leq r-1$. This means $y_m(r-0)x_i$, for all $1 \leq i \leq m-1$. Since x_m is a transmitter, so there cannot be an arc from y_n to x_m. Therefore $x_m((r-h-1)-0)y_n$, since y_n needs $h+1$ more marks. Now delete x_m, it will decrease the mark of y_n by $h+1$, and the resulting bipartite r-digraph will have mark sequences P' and Q' as desired. □

Theorem 2.10 provides an algorithm of checking whether or not the sequences P and Q of non-negative integers in non-decreasing order are mark sequences, and for constructing a corresponding bipartite r-digraph. Let $P = [p_1, p_2, \ldots, p_m]$ and $Q = [q_1, q_2, \ldots, q_n]$, where $p_m \geq q_n$, $rn \leq p_m \leq 2rn$ and $q_n \leq 2r(m-1)+1$, be the mark sequences of a bipartite r-digraph with parts $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ respectively. Deleting p_m and performing A of Theorem 2.10 if $[2r-(i-1)]n \geq p_m \geq (2r-i)n$, $1 \leq i \leq r$, we get $Q' = [q'_1, q'_2, \ldots, q'_n]$. If the marks of the vertices y_j were decreased by i in this process, then the construction yielded $x_m((r-i)-0)y_j$, if these were decreased by $i-1$, then the construction yielded $x_m((r-i-1)-0)y_j$. If we perform B of Theorem 2.10, the mark of y_n was decreased by $h+1$, the construction yielded $x_m((r-h-1)-0)y_n$. For vertices y_j whose marks remained unchanged, the construction yielded $x_m(r-0)y_j$. Note that if the condition $p_m \geq rn$ does not hold, then we delete q_n for which the conditions get satisfied and the same argument is used for defining arcs. If this procedure is applied recursively, then it tests whether or not P and Q are the mark sequences, and if P and Q are the mark sequences, then a bipartite r-digraph with mark sequences P and Q is constructed.

We illustrate this reduction and the resulting construction with the following examples.

Example 1. Consider the sequences of non-negative integers $P = [14, 14, 15]$ and $Q = [6, 6, 8, 9]$. We check whether or not P and Q are mark sequences of some bipartite 3-digraph.

1. $P = [14, 14, 15]$, $Q = [6, 6, 8, 9]$.

We delete 15. Clearly $[2r-(i-1)]n = [2.3-(3-1)]4 = 16 \geq 15 \geq (2r-i)n =$

$(2.3-3)4 = 12$. So reduce $[2r-(i-1)]n-p_m = [2.3-(3-1]4-15 = 16-15 = 1$ largest entry of Q by $i = 3$ and $p_m-(2r-i)n = 15-(2.3-3)4 = 15-12 = 3$ next largest entries of Q by $i-1 = 3-1 = 2$ each, we get $P_1 = [14, 14]$, $Q_1 = [4, 4, 6, 6]$, and arcs are defined as $x_3(0-0)y_4$, $x_3(1-0)y_3$, $x_3(1-0)y_2$, $x_3(1-0)y_1$.

2. $P_1 = [14, 14]$, $Q_1 = [4, 4, 6, 6]$.

We delete 14. Here $[2r-(i-1)]n = [2.3-(3-1)]4 = 16 \geq 14 \geq (2r-i)n = (2.3-3)4 = 12$. Reduce $[2r-(i-1)]n-p_m = [2.3-(3-1]4-14 = 16-14 = 2$ largest entries of Q_1 by $i = 3$ and $p_m-(2r-i)n = 14-(2.3-3)4 = 14-12 = 2$ next largest entries of Q_1 by $i-1 = 3-1 = 2$ each, we get $P_2 = [14]$, $Q_2 = [2, 2, 3, 3]$, and arcs are defined as $x_2(0-0)y_4$, $x_2(0-0)y_3$, $x_2(1-0)y_2$, $x_2(1-0)y_1$.

3. $P_2 = [14]$, $Q_2 = [2, 2, 3, 3]$.

We delete 14. Here $[2r-(i-1)]n = [2.3-(3-1)]4 = 16 \geq 14 \geq (2r-i)n = (2.3-3)4 = 12$. Reduce $[2r-(i-1)]n-p_m = [2.3-(3-1]4-14 = 16-14 = 2$ largest entries of Q_2 by $i = 3$ and $p_m-(2r-i)n = 14-(2.3-3)4 = 14-12 = 2$ next largest entries of Q_2 by $i-1 = 3-1 = 2$ each, we get $P_3 = \phi$, $Q_3 = [0, 0, 0, 0]$, and arcs are defined as $x_1(0-0)y_4$, $x_1(0-0)y_3$, $x_1(1-0)y_2$, $x_1(1-0)y_1$.

The resulting bipartite 3-digraph has mark sequences $P = [14, 14, 15]$ and $Q = [6, 6, 8, 9]$ with vertex sets $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3, y_4\}$ and arcs as $x_3(0-0)y_4$, $x_3(1-0)y_3$, $x_3(1-0)y_2$, $x_3(1-0)y_1$, $x_2(0-0)y_4$, $x_2(0-0)y_3$, $x_2(1-0)y_2$, $x_2(1-0)y_1$, $x_1(0-0)y_4$, $x_1(0-0)y_3$, $x_1(1-0)y_2$, $x_1(1-0)y_1$.

Example 2. Consider the two sequences of non-negative integers given by $P = [13, 16, 22, 24]$ and $Q = [5, 6, 10]$. We check whether or not P and Q are mark sequences of some bipartite 4-digraph.

1. $P = [13, 16, 22, 24]$ and $Q = [5, 6, 10]$.

We delete 24. Here $[2r-(i-1)]n = [2.4-(1-1)]3 = 24$, so reduce $[2r-(i-1)]n-p_m = [2.4-(1-1]3-24 = 24-24 = 0$ largest entries of Q by $i = 1$, and obviously we reduce $p_m-(2r-i)n = 24-(2.4-1)3 = 24-21 = 3$ next largest entries of Q by $i-1 = 1-1 = 0$ each, we get $P_1 = [13, 16, 22]$ and $Q_1 = [5, 6, 10]$, and arcs are $x_4(4-0)y_3$, $x_4(4-0)y_2$, $x_4(4-0)y_1$.

2. $P_1 = [13, 16, 22]$ and $Q_1 = [5, 6, 10]$.

We delete 22. Here $[2r-(i-1)]n = [2.4-(1-1)]3 = 24 \geq 22 \geq (2r-i)n = (2.4-1)3 = 21$. Reduce $[2r-(i-1)]n-p_m = [2.4-(1-1]3-22 = 24-22 = 2$ largest entries of Q_1 by $i = 1$ and $p_m-(2r-i)n = 22-(2.4-1)3 = 22-21 = 1$ next largest entries of Q_1 by $i-1 = 1-1 = 0$ each, we get $P_2 = [13, 16]$, $Q_2 = [5, 5, 9]$, and arcs are defined as $x_3(3-0)y_3$, $x_3(3-0)y_2$, $x_3(4-0)y_1$.

3. $P_2 = [13, 16]$, $Q_2 = [5, 5, 9]$.

We delete 16. Here $[2r-(i-1)]n = [2.4-(3-1)]3 = 18 \geq 16 \geq (2r-i)n = (2.4-3)3 = 15$. Reduce $[2r-(i-1)]n-p_m = [2.4-(3-1]3-16 = 18-16 = 2$ largest entries of Q_2 by $i = 3$ and $p_m-(2r-i)n = 16-(2.4-3)3 = 16-15 = 1$ next largest entry of Q_2 by $i-1 = 3-1 = 2$, we get $P_3 = [13]$, $Q_3 = [3,2,6]$, and arcs are defined as $x_2(3-0)y_3$, $x_2(3-0)y_2$, $x_2(2-0)y_1$.

4. $P_3 = [13]$, $Q_3 = [3,2,6]$.

Here $13+3+2+6 = 24$ which is same as $2rn = 2.4.3 = 24$. Thus by the argument as discussed in the remarks, P_3 and Q_3 are mark sequences of some bipartite 4-digraph. Here arcs are $x_1(1-3)y_3$, $x_1(3-1)y_2$, $x_1(2-1)y_1$.

The resulting bipartite 4-digraph with mark sequences $P = [13,16,22,24]$ and $Q = [5,6,10]$ has vertex sets $X = \{x_1,x_2,x_3,x_4\}$ and $Y = \{y_1,y_2,y_3\}$ and arcs as $x_4(4-0)y_3$, $x_4(4-0)y_2$, $x_4(4-0)y_1$, $x_3(3-0)y_3$, $x_3(3-0)y_2$, $x_3(4-0)y_1$, $x_2(3-0)y_3$, $x_2(3-0)y_2$, $x_2(2-0)y_1$, $x_1(1-3)y_3$, $x_1(3-1)y_2$, $x_1(2-1)y_1$.

Now we give a combinatorial criterion for determining whether the sequences of non-negative integers are realizable as marks. This is analogous to Landau's theorem [6] on tournament scores and similar to the result by Beineke and Moon [2] on bipartite tournament scores.

Theorem 4 *Let $P = [p_i]_1^m$ and $Q = [q_j]_1^n$ be the sequences of non-negative integers in non-decreasing order. Then P and Q are the mark sequences of some bipartite r-digraph if and only if*

$$\sum_{i=1}^{f} p_i + \sum_{j=1}^{g} q_j \geq 2rfg, \tag{1}$$

for $1 \leq f \leq m$ and $1 \leq g \leq n$, with equality when $f = m$ and $g = n$.

Proof. The necessity of the condition follows from the fact that the sub-bipartite r-digraph induced by f vertices from the first part and g vertices from the second part has a sum of marks $2rfg$.

For sufficiency, assume that $P = [p_i]_1^m$ and $Q = [q_j]_1^n$ are the sequences of non-negative integers in non-decreasing order satisfying conditions (2.1) but are not mark sequences of any bipartite r-digraph. Let these sequences be chosen in such a way that m and n are the smallest possible and p_1 is the least with that choice of m and n. We consider the following two cases.

Case (a). Suppose the equality in (2.1) holds for some $f \leq m$ and $g \leq n$, so that

$$\sum_{i=1}^{f} p_i + \sum_{j=1}^{g} q_j = 2rfg.$$

By the minimality of m and n, $P_1 = [p_i]_1^f$ and $Q_1 = [q_j]_1^g$ are the mark sequences of some bipartite r-digraph $D_1(X_1, Y_1)$. Let $P_2 = [p_{f+1} - 2rg, p_{f+2} - 2rg, \dots, p_m - 2rg]$ and $Q_2 = [q_{g+1} - 2rf, q_{g+2} - 2rf, \dots, q_n - 2rf]$.
Consider the sum

$$\begin{aligned}\sum_{i=1}^{s}(p_{f+i} - 2rg) + \sum_{j=1}^{t}(q_{g+j} - 2rf) &= \sum_{i=1}^{f+s} p_i + \sum_{j=1}^{g+t} q_j - \left(\sum_{i=1}^{f} p_i + \sum_{j=1}^{g} q_j\right) \\ &\quad - 2rsg - 2rtf \\ &\geq 2r(f+s)(g+t) - 2rfg - 2rsg - 2rtf \\ &= 2r(fg + ft + sg + st - fg - sg - tf) \\ &= 2rst,\end{aligned}$$

for $1 \leq s \leq m - f$ and $1 \leq t \leq n - g$, with equality when $s = m - f$ and $t = n - g$. Thus, by the minimality of m and n, the sequences P_2 and Q_2 form the mark sequences of some bipartite r-digraph $D_2(X_2, Y_2)$. Now construct a new bipartite r-digraph $D(X, Y)$ as follows.

Let $X = X_1 \cup X_2$, $Y = Y_1 \cup Y_2$ with $X_1 \cap X_2 = \phi$, $Y_1 \cap Y_2 = \phi$. Let $x_2(r-0)y_1$ and $y_2(r-0)x_1$ for all $x_i \in X_i, y_i \in Y_i$, where $1 \leq i \leq 2$, so that we get the bipartite r-digraph $D(X, Y)$ with mark sequences P and Q, which is a contradiction.

Case (b). Suppose the strict inequality holds in (2.1) for some $f \neq m$ and $g \neq n$. Also, assume that $p_1 > 0$. Let $P_1 = [p_1 - 1, p_2, \dots, p_{m-1}, p_m + 1]$ and $Q_1 = [q_1, q_2, \dots, q_n]$. Clearly, P_1 and Q_1 satisfy the conditions (2.1). Thus, by the minimality of p_1, the sequences P_1 and Q_1 are the mark sequences of some bipartite r-digraph $D_1(X_1, Y_1)$. Let $p_{x_1} = p_1 - 1$ and $p_{x_m} = p_m + 1$. Since $p_{x_m} > p_1 + 1$, therefore there exists a vertex $y \in Y_1$ such that $x_m(1-0)y(1-0)x_1$, or $x_m(0-0)y(1-0)x_1$, or $x_m(1-0)y(0-0)x_1$, or $x_m(0-0)y(0-0)x_1$, is an induced sub-bipartite 1-digraph in $D_1(X_1, Y_1)$, and if these are changed to $x_m(0-0)y(0-0)x_1$, or $x_m(0-1)y(0-0)x_1$, or $x_m(0-0)y(0-1)x_1$, or $x_m(0-1)y(0-1)x_1$ respectively, the result is a bipartite r-digraph with mark sequences P and Q, which is a contradiction. Hence the result follows. □

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