Acta Univ. Sapientiae, Mathematica, 4, 1 (2012) 65–73

On (κ, μ)-contact metric manifolds with certain curvature restrictions

Satyabrota Kundu
Loreto College,
Department of Mathematics,
Kolkata, India
email: satyacaluniv@gmail.com

Joydeep Sengupta
University of North Bengal
Department of Mathematics,
Raja Ramohanpur,
District Darjeeling, India
email: joydeep1972@yahoo.com

Abstract. In this paper we give a classification of (κ, μ)-contact metric manifolds with certain curvature restrictions.

1 Introduction

In 1995, Blair, Koufogiorgos and Papantoniou [3] introduced a type of contact metric manifolds $M^{(2n+1)}(\phi, \xi, \eta, g)$ whose curvature tensor R satisfies

$$R(X, Y)\xi = \kappa\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)hX - \eta(X)hY\}, \ \forall \ X, Y \in \chi(M).$$

Here, (κ, μ) are real constants and $2h$ denotes the Lie-Derivative in the direction of ξ. In this case we say that the characteristic vector field ξ belongs to the (κ, μ)-nullity distribution and the class of contact metric manifolds satisfying this condition are called (κ, μ)-contact metric manifolds. In case the vector field ξ is Killing, this class of manifolds are called Sasakian manifolds. In 1999, Boeckx [5] proved that a (κ, μ)-contact metric manifolds is either Sasakian or locally ϕ-symmetric. Later in 2000, Boeckx [6] gave a full classification of non-Sasakian (κ, μ)-contact metric manifolds. In 2008, Ghosh [7] proved that all conformally recurrent (κ, μ)-contact metric manifolds are locally isometric either to the unit sphere S^{2n+1} or to $E^{n+1} \times S^n$. In this paper, we study (κ, μ)-contact metric manifolds with different curvature restrictions and classify such manifolds.

2010 Mathematics Subject Classification: 53C15, 53C21, 53C25.
Key words and phrases: contact manifold, hyper-generalized recurrent manifold

2 Preliminaries

Let (M^{2n+1}, g) be an almost contact metric manifold with an almost contact metric structure (ϕ, ξ, η, g). Then we have

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \quad g(X, \xi) = \eta(X); \tag{1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(\phi X, Y) = d\eta(X, Y) = -g(X, \phi Y) \tag{2}$$

for all $X, Y \in \chi(M)$. The operator h satisfies the following results [2], [3], [4]:

$$h\phi = -\phi h, \quad \eta \circ h = 0, \quad g(hX, Y) = g(X, hY), \quad h^2 = (\kappa - 1)\phi^2; \tag{3}$$

$$h\xi = 0, \quad g(X, \phi h Z) = g(\phi h X, Z); \tag{4}$$

$$\nabla_X \xi = -\phi X - \phi h X, \quad (\nabla_X \eta)(Y) = g(X + hX, \phi Y), \tag{5}$$

where ∇ is the Riemannian connection of g. In a (κ, μ)-contact metric manifolds we have the following [3], [4]:

$$R(\xi, X)Y = \kappa\{g(X, Y)\xi - \eta(Y)X\} + \mu\{g(hX, Y)\xi - \eta(Y)hX\}; \tag{6}$$

$$R(X, Y)\xi = \kappa\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)hX - \eta(X)hY\}; \tag{7}$$

$$\begin{aligned} S(X, Y) = {} & \{2(n-1) - n\mu\}g(X, Y) + \{2(n-1) + \mu\}g(hX, Y) \\ & + \{2(1-n) + n(2\kappa + \mu)\}\eta(X)\eta(Y); \end{aligned} \tag{8}$$

$$S(X, \xi) = 2n\kappa\eta(X), \qquad Q\xi = 2n\kappa\xi; \tag{9}$$

$$r = 2n(2n - 2 + \kappa - n\mu); \tag{10}$$

and

$$\begin{aligned} (\nabla_X h)(Y) - (\nabla_Y h)(X) = {} & (1-\kappa)\{2g(X, \phi Y)\xi + \eta(X)\phi Y - \eta(Y)\phi X\} \\ & + (1-\mu)\{\eta(X)\phi h Y - \eta(Y)\phi h X\} \end{aligned} \tag{11}$$

for all $X, Y \in \chi(M)$ where S, r are respectively the Ricci tensor and the scalar curvature of M.

For (κ, μ)-contact metric manifolds with $h = 0$, we have $\kappa = 1$, and in this case the manifold reduces to a Sasakian one. The following relations hold in a Sasakian manifold [2]:

$$(i)\ \nabla_X \xi = -\phi X, \qquad (ii)\ (\nabla_X \eta)(Y) = g(X, \phi Y), \tag{12}$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{13}$$

$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X, \tag{14}$$

$$S(X, \xi) = 2n\eta(X), \tag{15}$$

$$S(\phi X, \phi Y) = S(X, Y) - 2n\eta(X)\eta(Y), \tag{16}$$

for all $X, Y \in \chi(M)$. The above formulae will be used in the sequel.

3 On a class of (κ,μ)-contact metric manifolds

A Riemannian manifold (M, g) is called a hyper-generalized recurrent manifold (for details we refer to [8]) if and only if its curvature tensor R satisfies the condition

$$
\begin{aligned}
(\nabla_W R)(X,Y)Z = {} & A(W)R(X,Y)Z \\
& + B(W)\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\}
\end{aligned} \tag{17}
$$

for all $X, Y, Z \in \chi(M)$; where A and B are two non-zero 1-forms metrically equivalent to two vector fields σ and ρ, respectively. Moreover, if the the scalar curvature r is a non-zero constant, then these associated 1-forms are related by

$$A + 4nB = 0. \tag{18}$$

Consequently we have

$$\sigma + 4n\rho = 0. \tag{19}$$

Before proceeding for the main theorems of the paper, we are to state the following lemma [7]:

Lemma 1 *For a (κ,μ)-contact metric space, the relation $\nabla_\xi h = \mu h\phi$ holds.*

We are now going to prove the main theorems of the paper:

By contracting (17) with respect to W, we obtain

$$
\begin{aligned}
(div\, R)(X,Y)Z = {} & g(R(X,Y)Z,\sigma) + \big\{S(Y,Z)g(X,\rho) - S(X,Z)g(Y,\rho) \\
& + g(Y,Z)S(X,\rho) - g(X,Z)S(Y,\rho)\big\}.
\end{aligned} \tag{20}
$$

Using the result

$$(div\, R)(X,Y)Z = (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z),$$

in (20), one obtains

$$
\begin{aligned}
(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) = {} & g(R(X,Y)Z,\sigma) + \big[S(Y,Z)g(X,\rho) - S(X,Z)g(Y,\rho) \\
& + g(Y,Z)S(X,\rho) - g(X,Z)S(Y,\rho)\big].
\end{aligned} \tag{21}
$$

Setting $Z = \xi$, yields on using (9)

$$\begin{aligned} & 2n\kappa\big[g(X+hX,\phi Y) - g(Y+hY,\phi X)\big] \\ + \;& S(Y,\phi X) - S(X,\phi Y) + S(Y,\phi hX) - S(X,\phi hY) \\ = \;& g(R(X,Y)\xi,\sigma) + 2n\kappa[\eta(Y)g(X,\rho) - \eta(X)g(Y,\rho)] \\ - \;& [\eta(Y)S(X,\rho) + \eta(X)S(Y,\rho)]. \end{aligned} \tag{22}$$

Replacing X by ϕX and Y by ϕY and using (1) and (3), we have

$$2\kappa + \mu + n\mu - \mu\kappa = 0. \tag{23}$$

In a (κ,μ)-contact metric manifold, the scalar curvature r is a non-zero constant, therefore using (18) in (17) and thereby contraction over W yields

$$\begin{aligned} (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) = g(R(X,Y)Z,\sigma) - \frac{1}{4n}\big[&S(Y,Z)g(X,\sigma) \\ &- S(X,Z)g(Y,\sigma) + g(Y,Z)S(X,\sigma) \\ &- g(X,Z)S(Y,\sigma)\big]. \end{aligned} \tag{24}$$

Using (8) and (11) on (24), one obtains

$$\begin{aligned} &(3\mu + 2n\kappa - n\mu - \mu^2)\big(\eta(X)g(\phi hY,Z) - \eta(Y)g(\phi hX,Z)\big) \\ &= g(R(X,Y)Z,\sigma) - \frac{1}{4n}\big[S(Y,Z)g(X,\sigma) - S(X,Z)g(Y,\sigma) \\ &\quad + g(Y,Z)S(X,\sigma) - g(X,Z)S(Y,\sigma)\big]. \end{aligned} \tag{25}$$

Putting $X = \xi$ and setting $Z = \sigma$ in the above equality, we have by (6)

$$(3\mu + 2n\kappa - n\mu - \mu^2)g(\phi hY,\sigma) = 0. \tag{26}$$

Two cases arise from above

$$\text{(i)}\;\; 3\mu + 2n\kappa - n\mu - \mu^2 = 0, \tag{27}$$

$$\text{(ii)}\;\; \phi h\sigma = 0. \tag{28}$$

From (23) we have

$$-\kappa(\mu - 2) + (n+1)\mu = 0.$$

$$\text{or,}\quad \kappa = (n+1)\frac{\mu}{\mu - 2}. \tag{29}$$

Putting this value of κ in (23) we have

$$\mu(\mu - n - 3)(\mu + 2n - 2) = 0. \tag{30}$$

From (29) and (30) we get the following set of corresponding values of μ and κ:

μ	κ
0	0
$n+3$	$n+3$
$2-2n$	$n-\frac{1}{n}$

Since, $\kappa < 1$ and $n > 1$, therefore only the case $\kappa = 0 = \mu$ is admissible and other possibilities will be ignored. For $\kappa = 0 = \mu$, from (11) we have, $R(X, Y)\xi = 0$, for all X, Y. Therefore by [1], a (κ, μ)-contact metric manifold (M^{2n+1}, g) admitting such a structure is locally isometric to either (i) the unit sphere $S^{2n+1}(1)$ or (ii) to the product space $E^{n+1} \times S^n(4)$.

Next let us consider the case (ii). From (28) we have the following:

$$\begin{aligned}
&\phi g\sigma = 0\\
\Rightarrow\ &\phi^2 h\sigma = -h\sigma + \eta(h\sigma)\xi,\\
\Rightarrow\ &h\sigma = 0, \text{ by } (3)\\
\Rightarrow\ &h^2\sigma = (\kappa - 1)\phi^2\sigma = 0.
\end{aligned}$$

Since, $\kappa < 1$, it follows that $\phi^2\sigma = 0$ and consequently $\sigma = \eta(\sigma)\xi$ i.e. for all vector field W on M, $A(W) = \eta(\sigma)\eta(W)$. Applying (18), we find $B(W) = \eta(\rho)\eta(W)$. Hence putting the values of A and B in (17), one obtains

$$\begin{aligned}
&(\nabla_W R)(X, Y)Z\\
= \;&\eta(\sigma)\eta(W)R(X, Y)Z\\
- \;&\eta(\rho)\eta(W)\{S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY\}.
\end{aligned} \tag{31}$$

Placing $\phi^2 W$ in lieu of W and thereby contracting over W in the resulting equation, we find

$$(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z) = g\big((\nabla_\xi R)(X, Y)Z, \xi\big). \tag{32}$$

Replacing Y by ξ, the above equation reduces to

$$(\nabla_X S)(\xi, Z) - (\nabla_\xi S)(X, Z) = g\big((\nabla_\xi R)(X, Y)Z, \xi\big). \tag{33}$$

We have,

$$\begin{aligned}(\nabla_X S)(\xi, Z) &= 2n\kappa(\nabla_X \eta)(Z) + S(\phi X, Z) + S(\phi h X, Z) \\ &= (2n\kappa + n\mu + \mu)g(hX, \phi Z). \qquad (34)\end{aligned}$$

Again, from (8) we have

$$\begin{aligned}(\nabla_\xi S)(X, Z) &= (2n - 2 + \mu)g((\nabla_\xi h)(X), Z) \\ &= \mu\{2(n-1) + \mu\}g(h\phi X, Z). \qquad (35)\end{aligned}$$

Moreover, applying covariant differentiation with respect to the vector field ξ, we obtain

$$\begin{aligned}(\nabla_\xi R)(X, \xi)Z &= -\mu g((\nabla_\xi h)(X), Z)\xi + \mu(\nabla_\xi \eta)(Z)hX \\ &= -\mu g(\mu(h\phi)(X), Z)\xi + \mu\eta(Z)\mu(h\phi)(X), \text{ by Lemma 3.1} \\ &= -\mu^2 g(h\phi X, Z). \qquad (36)\end{aligned}$$

Combining the results (33), (34), (35) and (36) we finally obtain

$$(2n\kappa + n\mu + \mu)g(hX, \phi Z) - \mu\{2(n-1) + \mu\}g(h\phi X, Z) = \mu^2 g(h\phi X, Z).$$

$$\text{i.e., } (2n\kappa + n\mu + \mu)g(h\phi X, Z) = 0.$$

$$\text{i.e., } (2n\kappa + n\mu + \mu) = 0, \text{ since } g(h\phi X, Z) \not\equiv 0\ . \qquad (37)$$

From (37) we get $\kappa = \frac{n-3}{2n}\mu$. Putting this value of κ in (23) we find

$$\mu\{\mu(n-3) - 2(n-1)(n+3)\} = 0. \qquad (38)$$

So, either $\mu = 0$ or $\mu = \frac{2(n-1)(n+3)}{n-3}$. Hence we obtain the following set of values for κ and μ:

μ	κ
0	0
$\frac{2(n-1)(n+3)}{n-3}$	$\frac{(n-1)(n+3)}{n}$, unless $n = 3$

In case $n = 3$ from (37) we find $\kappa = 0$. Hence from (23) we find $\mu = 0$, whenever $n = 3$.

By similar argument, as explained earlier, we are to consider $\kappa = 0 = \mu$. Hence the same result follows for case (ii). Thus we can state:

Theorem 1 *A hyper-generalized recurrent* (κ, μ)*-contact metric manifold* (M^{2n+1}, g) *is locally isometric to either (i) the unit sphere* $S^{2n+1}(1)$ *or (ii) to the product space* $E^{n+1} \times S^n(4)$.

Recalling Theorem (2.1) (viii) of [8], a hyper-generalized recurrent (κ,μ)-contact metric manifold is generalized 2-Ricci recurrent. Hence we can state as follows:

Corollary 1 *A generalized 2-Ricci recurrent (κ,μ)-contact metric manifold is locally isometric to either (i) the unit sphere $S^{2n+1}(1)$ or (ii) to the product space $E^{n+1} \times S^n(4)$.*

Again by virtue of Theorem (2.1) (v) of [8], a hyper-generalized recurrent (κ,μ)-contact metric manifold is generalized conharmonically recurrent. Thus we have the following:

Corollary 2 *A generalized conharmonically recurrent (κ,μ)-contact metric manifold is locally isometric to either (i) the unit sphere $S^{2n+1}(1)$ or (ii) to the product space $E^{n+1} \times S^n(4)$.*

Again, in a (κ,μ)-contact metric manifold, if $\kappa = 1$, then it reduces to a Sasakian manifold. Now we are going to find the consequences of the above theorem for $\kappa = 1$ i.e. for the case of Sasakian manifolds.

Taking $X = \xi$ in (21) gives

$$\begin{aligned} & (\nabla_\xi S)(Y,Z) - (\nabla_Y S)(\xi, Z) \\ = \; & g(R(\xi,Y)Z,\sigma) \\ + \; & \big[S(Y,Z)\eta(\rho) - 2n\eta(Z)g(Y,\rho) + 2ng(Y,Z)\eta(\rho) - \eta(Z)S(Y,\rho)\big]. \end{aligned}$$

Since, for a Sasakian manifold ξ is a Killing vector field, therefore $\pounds_\xi S = 0$ and hence $\nabla_\xi S = 0$. Thereby from the above we obtain

$$\begin{aligned} & -S(\phi Y,Z) + 2ng(\phi Y,Z) \\ = \; & g(R(\xi,Y)Z,\sigma) \\ + \; & \big[S(Y,Z)\eta(\rho) - 2n\eta(Z)g(Y,\rho) + 2ng(Y,Z)\eta(\rho) - \eta(Z)S(Y,\rho)\big]. \end{aligned} \tag{39}$$

Replacing Y and Z by ϕY and ϕZ respectively and using (1) and (2) yields

$$\begin{aligned} & S(Y,\phi Z) - 2ng(Y,\phi Z) \\ = \; & \eta(\sigma)\{g(Y,Z) - \eta(Y)\eta(Z)\} \\ + \; & \eta(\rho)\big\{S(Y,Z) + 2ng(Y,Z) - 4n\eta(Y)\eta(Z)\big\}. \end{aligned} \tag{40}$$

Again replacing ϕY for Y in (40), we obtain

$$S(Y,Z) - 2ng(Y,Z) = \eta(\sigma)g(\phi Y,Z) + \eta(\rho)\big\{S(\phi Y,Z) + 2ng(\phi Y,Z)\big\}. \tag{41}$$

Since, S and g are symmetric, the left hand side of (41) is symmetric with respect to Y and Z. Hence we have

$$S(Y,Z) = 2ng(Y,Z). \tag{42}$$

Thus a hyper-generalized recurrent Sasakian manifold is an Einstein manifold with non-vanishing scalar curvature $r = 2n(2n+1)$. Using (42) in (17) and thereafter by (18), we acquire

$$(\nabla_W R)(X,Y)Z = -4nB(W)\big\{R(X,Y)Z - g(Y,Z)X + g(X,Z)Y\big\}. \tag{43}$$

On cyclic transposition of the last equation twice over X, Y, W and thereafter summing up these resulting equations we get by virtue of the second Bianchi identity,

$$\begin{aligned} & B(W)\big\{R(X,Y)Z - g(Y,Z)X + g(X,Z)Y\big\} \\ + \; & B(Y)\big\{R(W,X)Z - g(X,Z)W + g(W,Z)X\big\} \\ + \; & B(X)\big\{R(Y,W)Z - g(W,Z)Y + g(Y,Z)W\big\} = 0. \end{aligned} \tag{44}$$

On contraction with respect to W and using (42), we obtain

$$R(X,Y)\rho = B(Y)X - B(X)Y. \tag{45}$$

In a similar fashion, we can also find

$$R(Z,\rho)X = B(X)Z - g(X,Z)\rho. \tag{46}$$

Assigning $W = \rho$ in (44) and utilizing (45) and (46) one determines

$$g(\rho,\rho)\{R(X,Y)Z - g(Y,Z)X + g(X,Z)Y\} = 0.$$

Since $\rho \neq 0$, one must have for arbitrary vector fields X, Y and Z on M

$$R(X,Y)Z = g(Y,Z)X - g(X,Z)Y. \tag{47}$$

This implies the space under consideration 1 is of constant curvature 1 and hence locally isometric to the unit sphere. This gives the following theorem:

Theorem 2 *A hyper-generalized recurrent Sasakian manifold* (M^{2n+1}, g) *is of constant curvature 1 and hence locally isometric to a unit sphere* $S^{2n+1}(1)$.

Also by virtue of Theorem (2.1) (v) of [8], a hyper-generalized recurrent Sasakian manifold is generalized conharmonically recurrent. Hence we state the following:

Corollary 3 *A generalized conharmonically recurrent Sasakian manifold is of constant curvature 1 and hence locally isometric to a unit sphere* $S^{2n+1}(1)$.

Retrieving the Theorem (2.1) (viii) of [8], a hyper-generalized recurrent Sasakian manifold is generalized 2-Ricci recurrent. Thus one obtains,

Corollary 4 *A generalized 2-Ricci recurrent Sasakian manifold is of constant curvature 1 and hence locally isometric to a unit sphere* $S^{2n+1}(1)$.

Acknowledgement

The first author was financially supported by University Grants Commission (India). The authors are also grateful to Prof. M. Tarafdar (Department of Pure Mathematics, University of Calcutta, India) and Dr. Amalendu Ghosh (Krishnagar Govt. College, India) for their kind help.

References

[1] D. E. Blair, Two remarks on contact metric structure, *Tohoku Math. J.*, **29** (1977), 319–324.

[2] D. E. Blair, Riemannian geometry of contact and symplectic manifolds, *Progress in Math*, 2nd ed., Birkhäuser, Boston 2010.

[3] D. E. Blair, T. Koufogiorgos, B. J. Papantoniou, Contact metric manifolds satisfying a nullity condition, *Israel J. Math.* **91** (1995), 189–214.

[4] D. E. Blair, H. Chen, A classification of 3-dimensional contact metric manifold with $Q\phi = \phi Q$, *Bull. Inst. Math. Acad. Sinica*, **20** (1992), 379–383.

[5] E. Boeckx, A class of locally ϕ-symmetric contact metric spaces, *Arch. Math.*, **72** (1999), 466–472.

[6] E. Boeckx, A full classification of (κ, μ)-spaces, *Illinois J. Math.*, **44** (2000), 212–219.

[7] A. Ghosh, Conformally recurrent (κ, μ)-contact manifolds, *Note di Matematica*, **28** (2008), 207–212.

[8] A. A. Shaikh, A. Patra, On a generalized class of recurrent manifolds, *Archivum Mathematicum*, Tomus **46** (2010), 71–78.

Received: January 2, 2012