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# Inclusion relations for multiplier transformation

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**Abstract.** Due to widely study of $K$-uniformly typed of functions, we establish here the inclusion relations for $K$-uniformly starlike, $K$-uniformly convex, close to convex and quasi-convex functions under the $D^{\lambda;m}_{\mu,a}$ operator introduced by the authors [1].

## 1 Introduction

Let $U = \{z : z \in C \ \ |z| < 1\}$ be the open unit disk and $A$ denotes the class of functions $f$ normalized by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

which is analytic in the open unit disk $U$ and satisfies the condition $f(0) = f'(0)-1 = 0$. A function $f \in A$ is said to be in $UST(k, \alpha)$, the class of $k$-uniformly starlike functions of order $\alpha$, $0 \le \alpha < 1$ if it satisfies the condition

$$\Re\left(\frac{zf'(z)}{f(z)}\right) - \alpha \ge k\left|\frac{zf'(z)}{f(z)} - 1\right|, \qquad k \ge 0, \ \ 0 \le \alpha < 1.$$

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Similarly, a function $f \in A$ is said to be in $UCV(k, \alpha)$, the class of $k$-uniformly convex functions of order $\alpha$, $0 \leq \alpha < 1$ if it satisfies the condition

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) - \alpha \geq k\left|\frac{zf''(z)}{f'(z)}\right|, \qquad k \geq 0,\ \ 0 \leq \alpha < 1.$$

The classes of uniformly convex and uniformly starlike were introduced by Goodman [3,4] and later generalized by Kanas and Wisniowska ([14],[15]) (see also the work of Kanas and Srivastava [16], Ronning ([7],[8]), Ma and Minda [20] and Gangadharan et al. [2]).

Let $F$ and $G$ be analytic functions in the unit disk $U$. The function $F$ is subordinate to $G$ written $F \prec G$. If $G$ is univalent, then $F(0) = G(0)$ and $F(U) \subset G(U)$.

In general, given two functions $F$ and $G$ which are analytic in $U$, the function $F$ is said to be subordinate to $G$ if there exist a function $w$ analytic in $U$ with

$$w(0) = 0 \text{ and } (\forall z \in U):\ |w(z)| < 1,$$

such that

$$(\forall z \in U):\ F(z) = G(w(z)).$$

For arbitrarily chosen $k \in [0, \infty[$ and $0 \leq \alpha < 1$, let $\Omega_{k,\alpha}$ denote the domain

$$\Omega_{k,\alpha} = \{u + iv,\ \ (u-\alpha)^2 > k^2(u-1)^2 + k^2v^2\}.$$

This characterization enables us to designate precisely the domain $\Omega_{k,\alpha}$ as a convex domain contain in the right half-plane. Moreover, $\Omega_{k,\alpha}$ is an elliptic region for $k > 1$, parabolic for $k = 1$, hyperbolic for $0 < k < 1$ and finally $\Omega_{0,0}$ is the whole right half-plane.

Let $q_{k,\alpha}(z) : U \to \Omega_{k,\alpha}$ denote the conformal mapping of $U$ onto $\Omega_{k,\alpha}$ so that $q_{k,\alpha}(0) = 0$, ${q_{k,\alpha}}'(0) > 0$. The explicit forms of $q_{k,\alpha}(z)$, were obtained in [13] as follows:

$$q_{k,\alpha}(z) = \begin{cases} \frac{1+(1-2\alpha)z}{1-z} & \text{for } k = 0, \\ \frac{1-\alpha}{1-k^2}\cos\left\{\frac{2}{\pi}\arccos(k) i \log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right\} - \frac{k^2-\alpha}{1-k^2} & \text{for } k \in (0,1) \\ 1 + \frac{2(1-\alpha)}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 & \text{for } k = 1, \\ \frac{1-\alpha}{k^2-1}\sin\left\{\frac{\pi}{2K(x)}\int_0^{\frac{u(z)}{\sqrt{x}}}\frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}\right\} + \frac{k^2-\alpha}{k^2-1} & \text{for } k > 1, \end{cases}$$

where $u(z) = \frac{z-\sqrt{x}}{1-\sqrt{x}z}$, $x \in (0,1)$ and $K$ is such $k = \cosh\frac{\pi K'(x)}{4K(x)}$.

Let $P$ denote the class of Caratheodory functions analytic in $U$ e.g.

$$P = \{p : \text{p analytic in } U,\ p(0) = 1,\ \Re\, p(z) > 0\}.$$

The characterization of the classes $UST(k, \alpha)$ and $UCV(k, \alpha)$, can be expressed in terms of subordination as follows,

$$f \in UST(k, \alpha) \Leftrightarrow p(z) = \frac{zf'(z)}{f(z)} \prec q_{k,\alpha}(z), \quad z \in U,$$

and

$$f \in UCV(k, \alpha) \Leftrightarrow p(z) = \frac{zf''(z)}{f'(z)} + 1 \prec q_{k,\alpha}(z), \quad z \in U.$$

So that

$$\Re\, p(z) > \Re\, q_{k,\alpha}(z) > \frac{k + \alpha}{k + 1}. \tag{1}$$

Define $UCC(k, \alpha, \beta)$ to be the family of functions $f \in A$ such that

$$\frac{zf'(z)}{g(z)} \prec q_{k,\alpha}(z), \quad z \in U,$$

for some $g(z) \in UST(k, \beta)$. On the other hand, let $UQC(k, \alpha, \beta)$ be the family of functions $f \in A$ such that

$$\frac{(zf'(z))'}{g'(z)} \prec q_{k,\alpha}(z), \quad z \in U,$$

for some $g(z) \in UCV(k, \beta)$.

We observe that, $UCC(0, \alpha, \beta)$ is the class of close-to-convex functions of order $\alpha$ and type $\beta$ and $UQC(0, \alpha, \beta)$ is the class of quasi-convex functions of order $\alpha$ and type $\beta$.

We now state the following definition.

**Definition 1** ([1]) *Let the function $f \in A$, then for $\mu, m \in C$, $a \in C/\{-1, -2, \dots\}$, and $\lambda > -1$, we define the following operator:*

$$D^{\lambda,m}_{\mu,a} f(z) = z + \sum_{k=2}^{\infty} \left(\frac{k + a}{1 + a}\right)^m \frac{(\lambda + 1)_{k-1}}{(\mu)_{k-1}} a_k z^k. \tag{2}$$

Here $(x)_k$ is Pochhammer symbol (or the shifted factorial), defined by

$$(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)} \begin{cases} 1, k=0 & \text{and } x \in C\backslash\{0\}; \\ x(x+1)\dots(x+k-1) \text{ if} k \in N & \text{and } x \in C, \end{cases}$$

and $\Gamma(x),\ (x \in C)$ denotes the Gamma function.

It should be noted that the operator $D^{\lambda,m}_{\mu,a}f(z)$ is a generalization of many operators considered earlier. For $m \in Z$, $a \geq 1$, $\mu = 1$ and $\lambda = 0$ the operator $D^{\lambda,m}_{\mu,a}$ were studied by Cho and Srivastava [6], for $m = -1$, $\mu = 1$ and $\lambda = 0$ the operator is the integral operator studied by Owa and Srivastava [17], for any negative real number $m$ and $\mu = 1$, $a = 1$, $\lambda = 0$ the operator $D^{\lambda,m}_{\mu,a}$ is the integral operator studied by Jung et. al [5], for any nonnegative integer number $m$ and $\mu = 1$, $a = 0$, $\lambda = 0$ the operator $D^{\lambda,m}_{\mu,a}$ is the differential operator defined by Salagean [9], for $m = 0$, $\mu = 1$, $\lambda > -1$ the operator $D^{\lambda,m}_{\mu,a}$ is the differential operator defined by Ruscheweyh [19], for $\mu = 1$ and $\lambda > -1$ the operator $D^{\lambda,m}_{\mu,a}$ is the multiplier transformations defined by Al-Shaqsi and Darus [10] and for $D^{\lambda,m}_{\mu,a}$ the operator $D^{\lambda,m}_{\mu,a}$ is the derivative operator given by Al-Shaqsi and Darus [11]. In particular, we note that $D^{0,0}_{1,a} = f(z)$ and $D^{0,1}_{1,0} = zf'(z)$.

It is readily verified from (2) that

$$z(D^{\lambda,m}_{\mu+1,a}f(z))' = \mu D^{\lambda,m}_{\mu,a}f(z) - (\mu-1)D^{\lambda,m}_{\mu+1,a}f(z) \tag{3}$$

$$z(D^{\lambda,m}_{\mu,a}f(z))' = (\lambda+1)D^{\lambda+1,m}_{\mu,a}f(z) - \lambda D^{\lambda,m}_{\mu,a}f(z) \tag{4}$$

$$z(D^{\lambda,m}_{\mu,a}f(z))' = (a+1)D^{\lambda,m+1}_{\mu,a}f(z) - aD^{\lambda,m}_{\mu,a}f(z). \tag{5}$$

## 2 Main results

The main object of this paper is to study the inclusion properties of the above-mentioned classes under the multiplier transformation $D^{\lambda,m}_{\mu,a}f(z)$.

We shall need the following lemmas to prove our theorems:

**Lemma 1 ([12])** *Let $\sigma, \nu$ be complex numbers. Suppose also that $m(z)$ be convex univalent in $U$ with $m(0) = 1$ and $\Re[\sigma m(z) + \nu] > 0$, $z \in U$. If $u(z)$ is analytic in $U$ with $u(0) = 1$, then*

$$u(z) + \frac{zu'(z)}{\sigma\, u(z) + \nu} \prec m(z) \ \Rightarrow u(z) \prec m(z)\ .$$

**Lemma 2 ([18])** *Let $h(z)$ be the convex in the unit disk $U$ and let $E \geq 0$. suppose $B(z)$ is analytic in $U$ with $\Re B(z) > E$. If $g(z)$ is analytic in $U$ and $g(0) = h(0)$. Then*

$$E\, z^2 g''(z) + B(z) z g'(z) + g(z) \prec h(z) \Rightarrow g(z) \prec h(z).$$

Our first result is the following:

**Theorem 1** *Let $f(z) \in A$.*
*If $D^{\lambda,m}_{\mu,a} f(z) \in UST(k,\alpha)$, and $\Re\mu > \frac{1-\alpha}{1+k}$, then $D^{\lambda,m}_{\mu+1,a} f(z) \in UST(k,\alpha)$.*

**Proof.** Let $p(z) = \frac{z\left(D^{\lambda,m}_{\mu+1,a} f(z)\right)'}{D^{\lambda,m}_{\mu+1,a} f(z)}$. In view of (3), we can write

$$\frac{\mu D^{\lambda,m}_{\mu,a} f(z)}{D^{\lambda,m}_{\mu+1,a} f(z)} = p(z) + \mu - 1.$$

Differentiating the above expression yields

$$\frac{z\left(D^{\lambda,m}_{\mu,a} f(z)\right)'}{D^{\lambda,m}_{\mu,a} f(z)} = p(z) + \frac{zp'(z)}{p(z) + \mu - 1}.$$

From this and argument given in the introduction we may write

$$p(z) + \frac{zp'(z)}{p(z) + \mu - 1} \prec q_{k,\alpha}(z).$$

Therefore, the theorem follows by Lemma 1 and the condition (1) since $q_{k,\alpha}(z)$ is univalent and convex in $U$ and $\Re(q_{k,\alpha}(z)) > \frac{k+\alpha}{k+1}$. □

**Theorem 2** *Let $f(z) \in A$.*
*If $D^{\lambda,m}_{\mu,a} f(z) \in UCV(k,\alpha)$, then $D^{\lambda,m}_{\mu+1,a} f(z) \in UCV(k,\alpha)$.*

**Proof.**

$$\begin{aligned} D^{\lambda,m}_{\mu,a} f(z) \in UCV(k,\alpha) &\Leftrightarrow z\left(D^{\lambda,m}_{\mu,a} f(z)\right)' \in UST(k,\alpha) \\ &\Leftrightarrow D^{m}_{\mu,a}(zf'(z)) \in UST(k,\alpha) \\ &\Leftrightarrow D^{m}_{\mu+1,a}(zf'(z)) \in UST(k,\alpha) \\ &\Leftrightarrow D^{m}_{\mu+1,a} f(z) \in UCV(k,\alpha), \end{aligned}$$

and the proof is complete. □

**Theorem 3** *Let* $f(z) \in A$.
*If* $D_{\mu,a}^{\lambda,m} f(z) \in UCC(k, \alpha, \beta)$, *and* $\Re\mu > \frac{1-\alpha}{1+k}$, *then* $D_{\mu+1,a}^{\lambda,m} f(z) \in UCC(k, \alpha, \beta)$.

**Proof.** Since $D_{\mu,a}^{\lambda,m} f(z) \in UCC(k, \alpha, \beta)$, by definition, we can write

$$\frac{z\left(D_{\mu,a}^{\lambda,m} f(z)\right)'}{K(z)} \prec q_{k,\alpha}(z),$$

for some $K(z) \in UST(k, \beta)$. For $g(z)$ such that $D_{\mu,a}^{\lambda,m} g(z) = K(z)$, we have

$$\frac{z\left(D_{\mu,a}^{\lambda,m} f(z)\right)'}{D_{\mu,a}^{\lambda,m} g(z)} \prec q_{k,\alpha}(z). \tag{6}$$

Letting $r(z) = \frac{z\left(D_{\mu+1,a}^{\lambda,m} f(z)\right)'}{D_{\mu+1,a}^{\lambda,m} g(z)}$ *and* $R(z) = \frac{z\left(D_{\mu+1,a}^{\lambda,m} g(z)\right)'}{D_{\mu+1,a}^{\lambda,m} g(z)}$, we observe that $r$ and $R$ are analytic in $U$ and $r(0) = R(0) = 1$. Now, by Theorem 1, $D_{\mu+1,a}^{\lambda,m} g(z) \in UST(k, \beta)$ and so $\Re(R(z)) > \frac{k+\alpha}{k+1}$,also, note that

$$z\left(D_{\mu+1,a}^{\lambda,m} f(z)\right)' = \left(D_{\mu+1,a}^{\lambda,m} g(z)\right) r(z). \tag{7}$$

Differentiating both sides of (7) yields

$$z\frac{\left(z\left(D_{\mu+1,a}^{\lambda,m} f(z)\right)'\right)'}{D_{\mu+1,a}^{\lambda,m} g(z)} = \frac{z\left(D_{\mu+1,a}^{\lambda,m} g(z)\right)'}{D_{\mu+1,a}^{\lambda,m} g(z)} r(z) + zr'(z) = R(z)r(z) + zr'(z).$$

Now using the identity (3), we obtain

$$\begin{aligned}
\frac{z\left(D_{\mu,a}^{\lambda,m} f(z)\right)'}{D_{\mu,a}^{\lambda,m} g(z)} &= \frac{D_{\mu,a}^{\lambda,m}\left(zf'(z)\right)}{D_{\mu,a}^{\lambda,m} g(z)} \\
&= \frac{z(D_{\mu+1,a}^{\lambda,m} zf'(z))' + (\mu-1) D_{\mu+1,a}^{\lambda,m}\left(zf'(z)\right)}{z(D_{\mu+1,a}^{\lambda,m} g(z))' + (\mu-1) D_{\mu+1,a}^{\lambda,m} g(z)} \\
&= \frac{\frac{z(D_{\mu+1,a}^{\lambda,m} zf'(z))'}{D_{\mu+1,a}^{\lambda,m} g(z)} + (\mu-1)\frac{D_{\mu+1,a}^{\lambda,m}(zf'(z))}{D_{\mu+1,a}^{\lambda,m} g(z)}}{\frac{z(D_{\mu+1,a}^{\lambda,m} g(z))'}{D_{\mu+1,a}^{\lambda,m} g(z)} + (\mu-1)}
\end{aligned}$$

$$= \frac{\frac{z\left(z\left(D^{\lambda,m}_{\mu+1,a}f(z)\right)'\right)'}{D^{\lambda,m}_{\mu+1,a}g(z)} + (\mu-1)\frac{z\left(D^{\lambda,m}_{\mu+1,a}f(z)\right)'}{D^{\lambda,m}_{\mu+1,a}g(z)}}{\frac{z(D^{\lambda,m}_{\mu+1,a}g(z))'}{D^{\lambda,m}_{\mu+1,a}g(z)} + (\mu-1)}$$

$$= \frac{R(z)r(z) + zr'(z) + (\mu-1)r(z)}{R(z) + (\mu-1)} \tag{8}$$

$$= r(z) + \frac{zr'(z)}{R(z)+(\mu-1)}.$$

From (6), (7) and (8), we conclude that

$$r(z) + \frac{zr'(z)}{R(z)+(\mu-1)} \prec Q_{k,\alpha}(z).$$

In order to apply Lemma 2, Let $E = 0$ *and* $B(z) = \frac{1}{R(z)+(\mu-1)}$, we obtain

$$\Re(B(z)) = \frac{1}{|R(z)+(\mu-1)|^2}\Re(R(z)+(\mu-1)) > 0.$$

Then we conclude that $r(z) \prec q_{k,\alpha}(z)$ and so the proof is complete. □

Using a similar argument in Theorem 2, we can prove

**Theorem 4** *Let* $f(z) \in A$.
*If* $D^{\lambda,m}_{\mu,a}f(z) \in UQC(k,\alpha,\beta)$, *then* $D^{\lambda,m}_{\mu+1,a}f(z) \in UQC(k,\alpha,\beta)$.

*Now, we examine the closure property of the above classes of functions under the generalized Bernardi-Libera-Livingston operator* $\Psi_c(f)$ *which is defined by*

$$\Psi_c(z) = \frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt \;\; (c > -1), \quad f(z) \in A. \tag{9}$$

**Theorem 5** *Let* $c > \frac{-(k+\alpha)}{k+1}$.
*If* $D^{\lambda,m}_{\mu+1,a}f(z) \in UST(k,\alpha)$, *then* $D^{\lambda,m}_{\mu+1,a}\Psi_c(f(z)) \in UST(k,\alpha)$, *where* $\Psi_c$ *is the integral operator defined by (9).*

**Proof.** From (3) and (9), we have

$$z(D^{\lambda,m}_{\mu+1,a}\Psi_c f(z))' = (c+1)D^{\lambda,m}_{\mu+1,a}f(z) - cD^{\lambda,m}_{\mu+1,a}\Psi_c f(z). \tag{10}$$

Substituting $p(z) = \frac{z\left(D^{\lambda,m}_{\mu+1,a}\Psi_c f(z)\right)'}{D^{\lambda,m}_{\mu+1,a}\Psi_c f(z)}$ in (10), we can write

$$(c+1)\frac{D^{\lambda,m}_{\mu+1,a}f(z)}{D^{\lambda,m}_{\mu+1,a}\Psi_c f(z)} = p(z) + c \tag{11}$$

Differentiating (11) yields

$$\frac{z\left(D^{\lambda,m}_{\mu+1,a}f(z)\right)'}{D^{\lambda,m}_{\mu+1,a}f(z)} = p(z) + \frac{zp'(z)}{p(z)+c}$$

Applying Lemma 1, it follows that $p(z) \prec q_{k,\alpha}(z)$, that is,

$$\frac{z\left(D^{\lambda,m}_{\mu+1,a}\Psi_c f(z)\right)'}{D^{\lambda,m}_{\mu+1,a}\Psi_c f(z)} \prec q_{k,\alpha}(z),$$

and so

$$D^{\lambda,m}_{\mu+1,a}\Psi_c f(z) \in UST(k,\alpha).$$

□

A similar argument leads to:

**Theorem 6** *Let* $c > \frac{-(k+\alpha)}{k+1}$.
*If* $D^{\lambda,m}_{\mu+1,a}f(z) \in UCV(k,\alpha)$, *then* $D^{\lambda,m}_{\mu+1,a}\Psi_c(f(z)) \in UCV(k,\alpha)$, *where* $\Psi_c$ *is the integral operator defined by (9).*

**Theorem 7** *Let* $c > \frac{-(k+\alpha)}{k+1}$.
*If* $D^{\lambda,m}_{\mu+1,a}f(z) \in UCC(k,\alpha)$, *then* $D^{\lambda,m}_{\mu+1,a}\Psi_c(f(z)) \in UCC(k,\alpha)$.

**Proof.** By definition, there exists a function $K(z) \in UST(k,\beta)$ and for $g(z)$ such that $D^{\lambda,m}_{\mu+1,a}g(z) = K(z)$, we have

$$\frac{z\left(D^{\lambda,m}_{\mu+1,a}f(z)\right)'}{D^{\lambda,m}_{\mu+1,a}g(z)} \prec Q_{k,\alpha}(z). \tag{12}$$

Now from (10) we have

$$\begin{aligned}\frac{z\left(D^{\lambda,m}_{\mu+1,a}f(z)\right)'}{D^{\lambda,m}_{\mu+1,a}g(z)} &= \frac{D^{\lambda,m}_{\mu+1,a}\left(zf'(z)\right)}{D^{\lambda,m}_{\mu+1,a}g(z)}\\ &= \frac{z(D^{\lambda,m}_{\mu+1,a}\Psi_c zf'(z))' + cD^{\lambda,m}_{\mu+1,a}\Psi_c zf'(z)}{z(D^{\lambda,m}_{\mu+1,a}\Psi_c g(z))' + cD^{\lambda,m}_{\mu+1,a}\Psi_c g(z)}\\ &= \frac{z\left(z\left(D^{\lambda,m}_{\mu+1,a}\Psi_c f(z)\right)'\right)' + cz\left(D^{\lambda,m}_{\mu+1,a}\Psi_c f(z)\right)'}{z(D^{\lambda,m}_{\mu+1,a}\Psi_c g(z))' + cD^{\lambda,m}_{\mu+1,a}\Psi_c g(z)} \qquad (13)\\ &= \frac{\frac{z\left(z\left(D^{\lambda,m}_{\mu+1,a}\Psi_c f(z)\right)'\right)'}{D^{\lambda,m}_{\mu+1,a}\Psi_c g(z)} + c\frac{z\left(D^{\lambda,m}_{\mu+1,a}\Psi_c f(z)\right)'}{D^{\lambda,m}_{\mu+1,a}\Psi_c g(z)}}{\frac{z(D^{\lambda,m}_{\mu+1,a}\Psi_c g(z))'}{D^{\lambda,m}_{\mu+1,a}\Psi_c g(z)} + c}.\end{aligned}$$

Since $D^{\lambda,m}_{\mu+1,a}g(z) \in UST(k,\beta)$, by Theorem 6, we have $D^{\lambda,m}_{\mu+1,a}\Psi_c(g(z)) \in UST(k,\beta)$. Letting $r(z) = \frac{z\left(D^{\lambda,m}_{\mu+1,a}\Psi_c f(z)\right)'}{D^{\lambda,m}_{\mu+1,a}\Psi_c g(z)}$ *and* $R(z) = \frac{z\left(D^{\lambda,m}_{\mu+1,a}\Psi_c g(z)\right)'}{D^{\lambda,m}_{\mu+1,a}\Psi_c g(z)}$, we observe that $\Re\{R(z)\} > \frac{k+\beta}{k+1}$. Also, note that

$$z\left(D^{\lambda,m}_{\mu+1,a}\Psi_c f(z)\right)' = (D^{\lambda,m}_{\mu+1,a}\Psi_c g(z))r(z) \qquad (14)$$

Differentiating both sides of (14) yields

$$z\frac{\left(z\left(D^{\lambda,m}_{\mu+1,a}\Psi_c f(z)\right)'\right)'}{D^{\lambda,m}_{\mu+1,a}\Psi_c g(z)} = z\frac{\left(D^{\lambda,m}_{\mu+1,a}\Psi_c g(z)\right)'}{D^{\lambda,m}_{\mu+1,a}\Psi_c g(z)}r(z)+zr'(z) = R(z)r(z)+zr'(z). \qquad (15)$$

Therefore from (13) and (10), we obtain

$$\frac{z\left(D^{\lambda,m}_{\mu+1,a}f(z)\right)'}{D^{\lambda,m}_{\mu+1,a}g(z)} = \frac{R(z)r(z)+zr'(z)+cr(z)}{R(z)+c} = r(z) + \frac{zr'(z)}{R(z)+c}.$$

From (12), (14) and (15), we conclude that

$$r(z) + \frac{zr'(z)}{R(z)+c} \prec q_{k,\alpha}(z).$$

In order to apply Lemma 2, Let $E = 0$ *and* $B(z) = \frac{1}{R(z)+c}$, we note that

$$\Re\{B(z)\} > 0 \text{ if } c > -\frac{k+\beta}{k+1}.$$

Then we conclude that $r(z) \prec Q_{k,\alpha}(z)$ and so the proof is complete. A similar argument yields. □

**Theorem 8** *Let* $c > \frac{-(k+\alpha)}{k+1}$.
*If* $D^{\lambda,m}_{\mu+1,a}f(z) \in UQC(k,\alpha,\beta)$, *then* $D^{\lambda,m}_{\mu+1,a}\Psi_c(f(z)) \in UQC(k,\alpha,\beta)$.

Similarly by using (4) and (5) we obtain the following results. Since the proof of the results is similar to the proof of Theorems 1-8, it will be omitted.

**Theorem 9** *Let* $f(z) \in A$.
*If* $D^{\lambda+1,m}_{\mu,a}f(z) \in UST(k,\alpha)$, *then* $D^{\lambda,m}_{\mu,a}f(z) \in UST(k,\alpha)$.

**Theorem 10** *Let* $f(z) \in A$.
*If* $D^{\lambda+1,m}_{\mu,a}f(z) \in UCV(k,\alpha)$, *then* $D^{\lambda,m}_{\mu,a}f(z) \in UCV(k,\alpha)$.

**Theorem 11** *Let* $f(z) \in A$.
*If* $D^{\lambda+1,m}_{\mu,a}f(z) \in UCC(k,\alpha,\,\beta)$, *then* $D^{\lambda,m}_{\mu,a}f(z) \in UCC(k,\alpha,\,\beta)$.

**Theorem 12** *Let* $f(z) \in A$.
*If* $D^{\lambda+1,m}_{\mu,a}f(z) \in UQC(k,\alpha,\,\beta)$, *then* $D^{\lambda,m}_{\mu,a}f(z) \in UQC(k,\alpha,\,\beta)$.

**Theorem 13** *Let* $f(z) \in A$.
*If* $D^{\lambda,m+1}_{\mu,a}f(z) \in UST(k,\alpha)$, *and* $\Re\{a\}\,\frac{-(k+\alpha)}{k+1}$, *then* $D^{\lambda,m}_{\mu,a}f(z) \in UST(k,\alpha)$.

**Theorem 14** *Let* $f(z) \in A$.
*If* $D^{\lambda,m+1}_{\mu,a}f(z) \in UCV(k,\alpha)$, *then* $D^{\lambda,m}_{\mu,a}f(z) \in UCV(k,\alpha)$.

**Theorem 15** *Let* $f(z) \in A$.
*If* $D^{\lambda,m+1}_{\mu,a}f(z) \in UCC(k,\alpha,\,\beta)$, *and* $\Re\{a\} > \frac{-(k+\alpha)}{k+1}$, *then* $D^{\lambda,m}_{\mu,a}f(z) \in UCC(k,\alpha,\,\beta)$.

**Theorem 16** *Let* $f(z) \in A$.
*If* $D^{\lambda,m+1}_{\mu,a}f(z) \in UQC(k,\alpha,\,\beta)$, *then* $D^{\lambda,m}_{\mu,a}f(z) \in UQC(k,\alpha,\,\beta)$.

**Theorem 17** *Let* $c > \frac{-(k+\alpha)}{k+1}$.
*If* $D^{\lambda,m}_{\mu,a}f(z) \in UST(k,\alpha)$, *then* $D^{\lambda,m}_{\mu,a}\Psi_c(f(z)) \in UST(k,\alpha)$.

**Theorem 18** *Let* $c > \frac{-(k+\alpha)}{k+1}$.
*If* $D^{\lambda,m}_{\mu,a} f(z) \in UCV(k,\alpha)$, *then* $D^{\lambda,m}_{\mu,a} \Psi_c(f(z)) \in UCV(k,\alpha)$.

**Theorem 19** *Let* $c > \frac{-(k+\alpha)}{k+1}$.
*If* $D^{\lambda,m}_{\mu,a} f(z) \in UCC(k,\alpha)$, *then* $D^{\lambda,m}_{\mu,a} \Psi_c(f(z)) \in UCC(k,\alpha)$.

**Theorem 20** *Let* $c > \frac{-(k+\alpha)}{k+1}$.
*If* $D^{\lambda,m}_{\mu,a} f(z) \in UQC(k,\alpha,\beta)$, *then* $D^{\lambda,m}_{\mu,a} \Psi_c(f(z)) \in UQC(k,\alpha,\beta)$.

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