

# Some polynomials associated with regular polygons

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**Abstract.** Let  $G_n$  be a regular  $n$ -gon with unit circumradius, and  $m = \lfloor \frac{n}{2} \rfloor$ ,  $\mu = \lfloor \frac{n-1}{2} \rfloor$ . Let the edges and diagonals of  $G_n$  be  $e_{n1} < \dots < e_{nm}$ . We compute the coefficients of the polynomial

$$(x - e_{n1}^2) \cdots (x - e_{n\mu}^2).$$

They appear to form a well-known integer sequence, and we study certain related sequences, too. We also compute the coefficients of the polynomial

$$(x - s_{n1}^2) \cdots (x - s_{nm}^2),$$

where

$$s_{ni} = \cot \frac{(2i-1)\pi}{2n}, \quad i = 1, \dots, m.$$

We interpret  $s_{n1}$  as the sum of all individual edges and diagonals of  $G_n$  divided by  $n$ . We also discuss the interpretation of  $s_{n2}, \dots, s_{nm}$ , and present a conjecture on expressing  $s_{n1}, \dots, s_{nm}$  using  $e_{n1}, \dots, e_{nm}$ .

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# 1 Introduction

Throughout,  $G_n$  is a regular  $n$ -gon with unit circumradius, and

$$m = \left\lfloor \frac{n}{2} \right\rfloor, \quad \mu = \left\lfloor \frac{n-1}{2} \right\rfloor.$$

Long time ago Kepler observed [2] that the squares of the edge and diagonals of  $G_7$  are the zeros of the polynomial  $x^3 - 7x^2 + 14x - 7$ . This raises a general question: Are the squares of (the lengths of) the edge and diagonals of  $G_n$ , excluding the diameter, the zeros of a monic polynomial of degree  $\mu$  with integer coefficients?

Yes, they are. This follows from Savio's and Suruyananarayan's [6] results, which, however, do not give the polynomial explicitly. We will do it in Section 2. A natural further question concerns the edge and diagonals themselves, instead of their squares. They are not zeros of a polynomial described above, but we will in Section 3 see that the squared sum of all individual edges and diagonals is the largest zero of a monic polynomial of degree  $m$  with integer coefficients. We will study geometric interpretation of the square roots of the other zeros in Section 4. In Section 5, we will present a conjecture on expressing these square roots as simple linear combinations of the edge and diagonals. We will in Section 6 notify that the coefficients of the first-mentioned polynomial form an OEIS [4] sequence, and also study OEIS sequences corresponding to certain related polynomials. Finally, we will complete our paper with conclusions and further questions in Section 7.

# 2 Squared chords

Let (the lengths of) the edge and diagonals of  $G_n$  be  $e_{n1} < \dots < e_{nm}$ . Call them (the lengths of) the *chords*. Then

$$e_{nk} = 2 \sin \frac{k\pi}{n}, \quad k = 1, \dots, m.$$

Our problem is to find the coefficients  $a_{mk}$  and  $b_{mk}$  of the polynomials

$$A_m(x) = (x - e_{n+2,1}^2) \cdots (x - e_{n+2,m}^2) = x^m + a_{m,m-1}x^{m-1} + \cdots + a_{m1}x + a_{m0}, \tag{1}$$

where  $n$  is even, and

$$B_m(x) = (x - e_{n1}^2) \cdots (x - e_{nm}^2) = x^m + b_{m,m-1}x^{m-1} + \cdots + b_{m1}x + b_{m0}, \tag{2}$$

where  $n$  is odd. We solve it in two theorems. Mustonen [3] found them experimentally and sketched their proofs.

Let  $\text{tridiag}_m(x, y)$  denote the symmetric tridiagonal  $m \times m$  matrix with all main diagonal entries  $x$  and first super- and subdiagonal entries  $y$ . For  $m \geq 2$ , define

$$\mathbf{A}_m = \text{tridiag}_m(2, 1)$$

and

$$\mathbf{B}_m \text{ is as } \mathbf{A}_m \text{ but the } (m, m) \text{ entry equals } 3.$$

Also define  $\mathbf{A}_1 = (2)$  and  $\mathbf{B}_1 = (3)$ . Denote by  $\text{spec}$  the (multi)set of eigenvalues.

**Lemma 1** For all  $m \geq 1$ ,

$$\begin{aligned} \text{spec } \mathbf{A}_m &= \left\{ 4 \sin^2 \frac{k\pi}{n+2} \mid k = 1, \dots, m \right\} = \{e_{n+2,1}^2, \dots, e_{n+2,m}^2\}, \\ \text{spec } \mathbf{B}_m &= \left\{ 4 \sin^2 \frac{k\pi}{n} \mid k = 1, \dots, m \right\} = \{e_{n1}^2, \dots, e_{nm}^2\}. \end{aligned} \quad (3)$$

**Proof.** See [1, 5, 6]. □

**Theorem 1** In (1),

$$a_{mk} = (-1)^{m-k} \binom{m+1+k}{2k+1}. \quad (4)$$

**Proof.** Denoting

$$P_m(x) = x^m + \sum_{k=0}^{m-1} (-1)^{m-k} \binom{m+1+k}{2k+1} x^k,$$

our claim is that

$$P_m(x) = A_m(x) \quad (5)$$

for all  $m \geq 1$ . Expanding  $\det(x\mathbf{I}_m - \mathbf{A}_m)$  along the last row, we have

$$A_{m+1}(x) = (x-2)A_m(x) - A_{m-1}(x)$$

for all  $m \geq 2$ . Since

$$P_1(x) = x - 2 = A_1(x)$$

and

$$P_2(x) = x^2 - 4x + 3 = A_2(x),$$

the claim (5) follows by showing that

$$P_{m+1}(x) = (x - 2)P_m(x) - P_{m-1}(x) \tag{6}$$

for all  $m \geq 2$ . Mustonen [3] did it by using Mathematica. We will do the computations algebraically in the appendix.  $\square$

The formula (4) yields  $\alpha_{mm} = 1$ , consistently with the coefficient of  $x^m$  in (1). It also allows to define  $\alpha_{00} = 1$ . The polynomial

$$\tilde{A}_{m+1}(x) = (x - 4)A_m(x) = x^{m+1} + \alpha_{m+1,m}x^m + \dots + \alpha_{m+1,1}x + \alpha_{m+1,0} \tag{7}$$

has  $e_{n+2,m+1}^2 = 4$  as the additional zero. By (4),

$$\alpha_{m+1,k} = (-1)^{m-k+1} \left( \binom{m+k}{2k-1} + 4 \binom{m+1+k}{2k+1} \right). \tag{8}$$

(We define  $\binom{n}{k} = 0$  if  $k < 0$ .)

**Theorem 2** *In (2),*

$$b_{mk} = (-1)^{m-k} \frac{2m+1}{m-k} \binom{m+k}{2k+1} = (-1)^{m-k} \left( \binom{m+1+k}{2k+1} + \binom{m+k}{2k+1} \right). \tag{9}$$

**Proof.** The second equation follows from trivial computation. To show the first, denote

$$Q_m(x) = x^m + \sum_{k=0}^{m-1} (-1)^{m-k} \frac{2m+1}{m-k} \binom{m+k}{2k+1} x^k$$

and claim that

$$Q_m(x) = B_m(x) \tag{10}$$

for all  $m \geq 1$ . Expanding  $\det(x\mathbf{I}_m - \mathbf{B}_m)$ , we have

$$B_{m+1}(x) = (x - 3)A_m(x) - A_{m-1}(x)$$

for all  $m \geq 2$ . Since

$$Q_1(x) = x - 3 = B_1(x)$$

and

$$Q_2(x) = x^2 - 5x + 5 = B_2(x),$$

the claim (10) follows by showing that

$$Q_{m+1}(x) = (x - 3)P_m(x) - P_{m-1}(x) \quad (11)$$

for all  $m \geq 2$ . Mustonen [3] did also this by using Mathematica, and we will do the computations algebraically in the appendix.  $\square$

For  $k = m$ , the first expression in (9) is undefined but the second is defined. (We define  $\binom{n}{k} = 0$  if  $n < k$ .) It gives  $b_{mm} = 1$ , the coefficient of  $x^m$  in (2). It also allows to define  $b_{00} = 1$ .

**Corollary 1** *The sum of all individual squared chords of  $G_n$  is  $n^2$ . Their product is  $n^n$ .*

**Proof.** By Theorems 1 and 2 (or by [7, Eqs. (20) and (24)]), we obtain

$$\begin{aligned} e_{2m,1}^2 + \cdots + e_{2m,m-1}^2 &= -a_{m-1,m-2} = 2(m-1), \\ e_{2m+1,1}^2 + \cdots + e_{2m+1,m}^2 &= -b_{m,m-1} = 2m+1, \end{aligned}$$

and

$$\begin{aligned} e_{2m,1}^2 \cdots e_{2m,m-1}^2 &= (-1)^m a_{m-1,0} = m, \\ e_{2m+1,1}^2 \cdots e_{2m+1,m}^2 &= (-1)^m b_{m0} = 2m+1. \end{aligned}$$

Denoting by  $\Sigma_n$  the sum and by  $\Pi_n$  the product of all individual squared chords of  $G_n$ , we therefore have

$$\begin{aligned} \Sigma_{2m} &= 2m \cdot 2(m-1) + m \cdot 4 = (2m)^2, \\ \Sigma_{2m+1} &= (2m+1)(2m+1) = (2m+1)^2, \end{aligned}$$

and

$$\Pi_{2m} = m^{2m} 4^m = (2m)^{2m}, \quad \Pi_{2m+1} = (2m+1)^{2m+1}.$$

$\square$

### 3 Sum of chords

The sum of all individual chords of  $G_n$  is

$$S_n = ns_n,$$

where

$$s_n = e_{n1} + \dots + e_{n,m-1} + \frac{1}{2}e_{nm} = e_{n1} + \dots + e_{n,m-1} + 1$$

if  $n$  is even, and

$$s_n = e_{n1} + \dots + e_{nm}$$

if  $n$  is odd, is the sum of different (lengths of) chords but the diameter is halved.

**Theorem 3** For all  $n \geq 3$ ,

$$s_n = \cot \frac{\pi}{2n}.$$

**Proof.** We have [7, Eq. (21)]

$$\sum_{k=1}^{n-1} \sin \frac{k\pi}{n} = \cot \frac{\pi}{2n}. \tag{12}$$

If  $n$  is even, this implies

$$s_n = \sum_{k=1}^{m-1} 2 \sin \frac{k\pi}{n} + \frac{1}{2} \cdot 2 = \sum_{k=1}^{m-1} \sin \frac{k\pi}{n} + 1 + \sum_{k=m+1}^{2m-1} \sin \frac{k\pi}{n} = \sum_{k=1}^{2m-1} \sin \frac{k\pi}{n} = \cot \frac{\pi}{2n}.$$

If  $n$  is odd, then

$$s_n = \sum_{k=1}^m 2 \sin \frac{k\pi}{n} = \sum_{k=1}^m \sin \frac{k\pi}{n} + \sum_{k=m+1}^{2m} \sin \frac{k\pi}{n} = \sum_{k=1}^{2m} \sin \frac{k\pi}{n} = \cot \frac{\pi}{2n}.$$

□

Is  $s_n$  a zero of a monic polynomial of degree  $m$  with integer coefficients? Yes for  $s_4 = \cot \frac{\pi}{8} = 1 + \sqrt{2}$ ; it is a zero of  $x^2 - 2x - 1$ . On the other hand, it is easy to see that  $s_5 = \cot \frac{\pi}{10} = \sqrt{5 + 2\sqrt{5}}$  is not a zero of such a polynomial. But

$s_5^2 = 5 + 2\sqrt{5}$  is a zero of  $x^2 - 10x + 5$ , and the other zero is  $5 - 2\sqrt{5} = \cot^2 \frac{3\pi}{10}$ . Also  $s_4^2 = 3 + 2\sqrt{2}$  has this property: it is a zero of  $x^2 - 6x + 1$ , and the other zero is  $3 - 2\sqrt{2} = \cot^2 \frac{3\pi}{8}$ .

Generally, denoting

$$s_{ni} = \cot \frac{(2i - 1)\pi}{2n}, \quad i = 1, \dots, m,$$

this motivates us to study for even  $n$  the coefficients of the polynomial

$$U_m(x) = (x - s_{n1}^2) \cdots (x - s_{nm}^2) = x^m + u_{m,m-1}x^{m-1} + \cdots + u_{m1}x + u_{m0}, \quad (13)$$

and for odd  $n$  those of

$$V_m(x) = (x - s_{n1}^2) \cdots (x - s_{nm}^2) = x^m + v_{m,m-1}x^{m-1} + \cdots + v_{m1}x + v_{m0}. \quad (14)$$

We will see that they all are integers. The largest zero is  $s_n^2 = s_{n1}^2$ .

Mustonen [3] found the following theorem experimentally and also presented its proof. Yaglom and Yaglom [9, Eqs. (7) and (8)] formulated (16) differently.

**Theorem 4** *In (13),*

$$u_{mk} = (-1)^k \binom{n}{2k}. \quad (15)$$

*In (14),*

$$v_{mk} = (-1)^k \binom{n}{2k + 1}. \quad (16)$$

**Proof.** We have [10]

$$\cot nt = \frac{\sum_{k=0}^m (-1)^k \binom{n}{2k} \cot^{n-2k} t}{\sum_{k=0}^m (-1)^k \binom{n}{2k+1} \cot^{n-2k-1} t}. \quad (17)$$

Denote

$$t_i = \frac{(2i - 1)\pi}{2n}, \quad i = 1, \dots, m.$$

Since  $\cot nt_i = 0$ , (17) yields

$$\sum_{k=0}^m (-1)^k \binom{n}{2k} \cot^{n-2k} t_i = 0. \quad (18)$$

First assume  $n$  even. The polynomial

$$\tilde{U}_m(x) = \sum_{k=0}^m (-1)^{m-k} \binom{n}{2k} x^k$$

is monic and has degree  $m$ . For all  $i = 1, \dots, m$ ,

$$\begin{aligned} \tilde{U}_m(s_{ni}^2) &= \sum_{k=0}^m (-1)^{m-k} \binom{2m}{2k} s_{ni}^{2k} = \sum_{l=0}^m (-1)^l \binom{2m}{2m-2l} s_{ni}^{2m-2l} \\ &= \sum_{l=0}^m (-1)^l \binom{2m}{2l} s_{ni}^{2m-2l} = \sum_{l=0}^m (-1)^l \binom{n}{2l} \cot^{n-2l} t_i = 0 \end{aligned}$$

by (18). Hence

$$\tilde{U}_m(x) = (x - s_{n1}^2) \cdots (x - s_{nm}^2) = U_m(x),$$

and (15) follows.

Second, assume  $n$  odd. The polynomial

$$\tilde{V}_m(x) = \sum_{k=0}^m (-1)^{m-k} \binom{n}{2k+1} x^k$$

is monic and has degree  $m$ . For all  $i = 1, \dots, m$ ,

$$\begin{aligned} \tilde{V}_m(s_{ni}^2) &= \sum_{k=0}^m (-1)^{m-k} \binom{2m+1}{2k+1} s_{ni}^{2k} = \sum_{l=0}^m (-1)^l \binom{2m+1}{2m-2l+1} s_{ni}^{2m-2l} = \\ &= s_{ni}^{-1} \sum_{l=0}^m (-1)^l \binom{2m+1}{2m-2l+1} s_{ni}^{2m+1-2l} = s_{ni}^{-1} \sum_{l=0}^m (-1)^l \binom{2m+1}{2l} s_{ni}^{2m+1-2l} \\ &= s_{ni}^{-1} \sum_{l=0}^m (-1)^l \binom{n}{2l} \cot^{n-2l} t_i = 0, \end{aligned}$$

again by (18). Hence

$$\tilde{V}_m(x) = (x - s_{n1}^2) \cdots (x - s_{nm}^2) = V_m(x),$$

and (16) follows. □



**Corollary 2** *The number  $s_n^2$  is the largest zero of the polynomial*

$$x^m + u_{m,m-1}nx^{m-1} + \dots + u_{m1}n^{m-1}x + u_{m0}n^m$$

*if  $n$  is even, and that of*

$$x^m + v_{m,m-1}nx^{m-1} + \dots + v_{m1}n^{m-1}x + v_{m0}n^m$$

*if  $n$  is odd.*

### 4 Interpreting $s_{n,m-k+1}$ , $k = 1, \dots, \lfloor \frac{n-1}{3} \rfloor$ , $n$ odd

The zeros of  $A_m(x)$  and  $B_m(x)$  describe the squared chords of  $G_{2m+2}$  and  $G_{2m+1}$ , respectively, excluding the diameter. The largest zero of  $U_m(x)$ ,  $s_{2m,1}^2 = s_{2m}^2$ , and that of  $V_m(x)$ ,  $s_{2m+1,1}^2 = s_{2m+1}^2$ , describe the squared sum of chords but halving the diameter. In other words, the sum of all individual chords of  $G_n$  is divided by  $n$  and the result is squared.

What about the other zeros?

Let the vertices of  $G_n$  be  $P_0, \dots, P_{n-1}$ , where  $P_k = (\cos \frac{k\pi}{n}, \sin \frac{k\pi}{n})$ . Then  $e_{nk} = P_0P_k = 2 \sin \frac{k\pi}{n}$ ,  $k = 1, \dots, m$ . Since  $P_0P_{n-k} = P_0P_k$ , we define  $e_{n,n-k} = e_{nk}$ ,  $k = 1, \dots, m$ .

Fix  $n$  and denote  $e_k = e_{nk}$  for brevity. Assume that  $3k < n$ ; i.e.,  $k < \frac{n}{3}$ . Then the line segments  $P_0P_{2k}$  and  $P_kP_{n-k}$  intersect; let  $Q_k$  be their intersection point and denote  $x_k = P_0Q_k$ . Because  $\triangle Q_kP_0P_k \sim \triangle Q_kP_{2k}P_{n-k}$ , we have

$$\frac{x_k}{e_{2k} - x_k} = \frac{e_k}{e_{3k}}.$$

Hence

$$\begin{aligned} x_k &= \frac{e_k e_{2k}}{e_k + e_{3k}} = \frac{2 \sin \frac{k\pi}{n} \sin \frac{2k\pi}{n}}{\sin \frac{k\pi}{n} + \sin \frac{3k\pi}{n}} = \\ &= \frac{2 \sin \frac{k\pi}{n} \sin \frac{2k\pi}{n}}{\sin(\frac{2k\pi}{n} - \frac{k\pi}{n}) + \sin(\frac{2k\pi}{n} + \frac{k\pi}{n})} = \frac{\sin \frac{k\pi}{n} \sin \frac{2k\pi}{n}}{\sin \frac{2k\pi}{n} \cos \frac{k\pi}{n}} = \tan \frac{k\pi}{n}. \end{aligned}$$

If  $n$  is odd, then

$$\tan \frac{k\pi}{n} = \cot \left( \frac{\pi}{2} - \frac{k\pi}{2m+1} \right) = \cot \frac{[2(m-k)+1]\pi}{2n} = s_{n,m-k+1}.$$

Thus  $s_{n,m-k+1} = P_0Q_k$ ,  $k = 1, \dots, \lfloor \frac{n-1}{3} \rfloor$ . In other words, the  $\lfloor \frac{n-1}{3} \rfloor$  smallest zeros of  $V_m(x)$  are the squared line segments  $P_0Q_k$ ,  $k = 1, \dots, \lfloor \frac{n-1}{3} \rfloor$ . Mustonen [3] found this experimentally. The largest zero is already interpreted, but the interpretation of the rest of zeros remains open. For some experimental observations, see [3]. Interpretation of the zeros of  $U_m(x)$ , except the largest, remains open, too.

### 5 Expressing $s_{n1}, \dots, s_{nm}$ using $e_{n1}, \dots, e_{nm}$

Mustonen's [3] experiments make conjecture that, given  $n$ , there are numbers  $\lambda_{nk}^{(i)} \in \{0, \pm 1\}$ ,  $i, k = 1, \dots, m$ , such that

$$s_{ni} = \lambda_{n1}^{(i)}e_{n1} + \dots + \lambda_{n,m-1}^{(i)}e_{n,m-1} + \lambda_{nm}^{(i)}e'_{nm}, \quad i = 1, \dots, m,$$

where

$$e'_{nm} = \begin{cases} \frac{1}{2}e_{nm} & \text{if } n \text{ is even,} \\ e_{nm} & \text{if } n \text{ is odd.} \end{cases}$$

In other words,

$$\cot \frac{(2i-1)\pi}{2n} = 2 \left[ \lambda_{n1}^{(i)} \sin \frac{\pi}{n} + \dots + \lambda_{n,m-1}^{(i)} \sin \frac{(m-1)\pi}{n} + \theta_n \lambda_{nm}^{(i)} \sin \frac{m\pi}{n} \right],$$

where

$$\theta_n = \begin{cases} \frac{1}{2} & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

This is true by (12) when  $i = 1$  ( $s_{n1} = s_n$ ,  $\lambda_{n1}^{(1)} = \dots = \lambda_{nm}^{(1)} = 1$ ) but remains generally open.

For example, let  $n = 15$ . Denoting  $s_k = s_{15,k}$  and  $e_k = e_{15,k}$  for brevity, we have [3, p. 17]

$$\begin{aligned} s_1 &= e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 \\ s_2 &= e_3 + e_6 \\ s_3 &= e_5 \\ s_4 &= e_1 - e_2 + e_3 - e_4 + e_5 - e_6 + e_7 \\ s_5 &= -e_3 + e_6 \\ s_6 &= e_1 - e_2 + e_3 + e_4 - e_5 + e_6 - e_7 \\ s_7 &= e_1 + e_2 - e_3 - e_4 + e_5 + e_6 - e_7. \end{aligned}$$

We study the zero coefficients in general. If and only if  $d = \gcd(n, 2i-1) > 1$ , then  $G_n$  "inherits" the chord

$$s_{ni} = \cot \frac{(2i-1)\pi}{2n}$$

from  $G_d$ . Then the chords of  $G_d$  are enough to express  $s_{ni}$ , and the coefficients of the remaining chords are zero. Indeed, in our example,

$$s_2 = s_{15,2} = \cot \frac{3\pi}{30} = \cot \frac{\pi}{10} = 2 \left( \sin \frac{\pi}{5} + \sin \frac{2\pi}{5} \right),$$

$$s_3 = s_{15,3} = \cot \frac{5\pi}{30} = \cot \frac{\pi}{6} = 2 \sin \frac{\pi}{3},$$

$$s_5 = s_{15,5} = \cot \frac{9\pi}{30} = \cot \frac{3\pi}{10} = 2 \left( -\sin \frac{\pi}{5} + \sin \frac{2\pi}{5} \right),$$

showing that  $s_3$  is "inherited" from  $G_3$ , and  $s_2$  and  $s_5$  from  $G_5$ .

So we conjecture additionally that if and only if  $n$  is a prime or a power of 2, then each  $\lambda_{nk}^{(i)} \in \{\pm 1\}$ . Mustonen [3] gives also other experimental results and conjectures about the structure of the three-dimensional array  $(\lambda_{nk}^{(i)})$ , and presents an efficient algorithm to compute these numbers.

## 6 Connections with OEIS sequences

The (lexicographically ordered) sequence  $(a_{mk})$  is A053122 in OEIS. Its first six terms are  $a_{00} = 1, a_{10} = -2, a_{11} = 1, a_{20} = 3, a_{21} = -4, a_{22} = 1$ .

The OEIS sequence A132460 consists of the numbers

$$t_{n0} = 1, \quad n = 0, 1, 2, \dots,$$

$$t_{nk} = (-1)^k \left( \binom{n-k}{k} + \binom{n-k-1}{k-1} \right), \quad n = 2, 3, \dots, \quad k = 1, \dots, m.$$

The first six terms of its subsequence corresponding to odd values of  $n$  are  $t_{10} = 1 = b_{00}, t_{30} = 1 = b_{11}, t_{31} = -3 = b_{10}, t_{50} = 1 = b_{22}, t_{51} = -5 = b_{21}, t_{52} = 5 = b_{20}$ . In general,  $b_{mk} = t_{2m+1, m-k}$ .

Also the characteristic polynomials of certain other tridiagonal matrices have connections with OEIS sequences. We study two of them.

Let  $\text{tridiag}(\mathbf{a}, \mathbf{b}, \mathbf{c})$  denote the tridiagonal matrix with main diagonal, sub-diagonal and superdiagonal entries those of vectors  $\mathbf{a}, \mathbf{b}$  and  $\mathbf{c}$ , respectively, and denote  $x^{(k)} = x, \dots, x, k$  copies. For  $m \geq 3$ , define

$$\mathbf{C}_m = \text{tridiag}((2^{(m)}), ((-1)^{(m-2)}, -2), (-2, (-1)^{(m-2)}))$$

and

$$\mathbf{C}_2 = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \quad \mathbf{C}_1 = (2).$$

For  $m \geq 1$ , consider the polynomial

$$C_m(x) = \det(x\mathbf{I}_m - \mathbf{C}_m) = x^m + c_{m,m-1}x^{m-1} + \cdots + c_{m1}x + c_{m0}$$

and define  $C_0(x) = 1$ ,  $c_{00} = c_{mm} = 1$ . The sequence A140882 consists of the numbers  $(-1)^m c_{mk}$ . Since  $C_0(x) = 1$ ,  $C_1(x) = x - 2$ ,  $C_2(x) = x^2 - 4x$ ,  $C_3(x) = x^3 - 6x^2 + 8x$ , its first ten terms are 1, 2, -1, 0, -4, 1, 0, -8, 6, -1, as listed in [4].

We have  $x\tilde{A}_1(x) = x^2 - 4x = C_2(x)$  and  $x\tilde{A}_2(x) = x^3 - 6x^2 + 8x = C_3(x)$ , and generally

$$C_{m+1}(x) = x\tilde{A}_m(x) \tag{19}$$

for all  $m \geq 1$ . This can be proved similarly to the proofs of Theorems 1 and 2. By (8), a formula for A140882 is then obtained. By (19), (7) and (3),

$$\text{spec } \mathbf{C}_m = \text{spec } \mathbf{A}_{m-2} \cup \{0, 4\} = \left\{ 4 \sin^2 \frac{k\pi}{2m-2} \mid k = 0, \dots, m-1 \right\}$$

for  $m \geq 3$ .

Finally, the sequence A136672 motivates us to study the polynomial

$$F_{m+1}(x) = (x - 2)A_m(x) = x^{m+1} + f_{m+1,m}x^m + \cdots + f_{m+1,1}x + f_{m+1,0} \tag{20}$$

and its connections with the matrix  $\mathbf{D}_m$ , defined by

$$\mathbf{D}_m = \text{tridiag}((2^{(m)}), ((-1)^{(m-2)}, 0), ((-1)^{(m-1)}))$$

if  $m \geq 3$ , and

$$\mathbf{D}_2 = \begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix}, \quad \mathbf{D}_1 = (2).$$

By Theorem 1,

$$f_{m+1,k} = (-1)^{m-k+1} \left( \binom{m+k}{2k-1} + 2 \binom{m+1+k}{2k+1} \right). \tag{21}$$

For  $m \geq 1$ , consider the polynomial

$$D_m(x) = \det(x\mathbf{I}_m - \mathbf{D}_m) = x^m + d_{m,m-1}x^{m-1} + \cdots + d_{m1}x + d_{m0}$$

and define  $D_0(x) = 1$ ,  $d_{00} = d_{mm} = 1$ . The sequence A136672 consists of the numbers  $(-1)^m d_{mk}$ . We have  $D_0(x) = 1$ ,  $D_1(x) = x - 2$ ,  $D_2(x) = x^2 - 4x + 4$ ,  $D_3(x) = x^3 - 6x^2 + 11x - 6$ . So its first ten terms are 1, 2, -1, 4, -4, 1, 6, -11, 6, -1, as listed in [4].

Since  $F_1(x) = x - 2 = D_1(x)$ ,  $F_2(x) = x^2 - 4x + 4 = D_2(x)$ , and  $F_3(x) = x^3 - 6x^2 + 11x - 6 = D_3(x)$ , it seems that

$$D_m(x) = F_m(x) \tag{22}$$

for all  $m \geq 1$ . This can be proved similarly to the previous proofs. By (21), a formula for A136672 follows. By (22), (20) and (3),

$$\text{spec } \mathbf{D}_m = \text{spec } \mathbf{A}_{m-1} \cup \{2\} = \left\{ 4 \sin^2 \frac{k\pi}{2m} \mid k = 1, \dots, m-1 \right\} \cup \{2\}$$

for  $m \geq 2$ .

## 7 Conclusions and further questions

The squared chords of  $G_n$ , excluding the diameter, are the zeros of a monic polynomial of degree  $\mu$  with integer coefficients. Including the diameter, the degree is  $m$ .

The squared sum of all individual chords is the largest zero of a monic polynomial of degree  $m$  with integer coefficients. An equivalent fact is that the squared sum of all different (lengths of) chords but the diameter is halved, is a zero of such a polynomial. The zeros of this polynomial seem to be linear combinations of the chords with all coefficients 0 or  $\pm 1$ .

Lemma 1, stating that  $e_{n1}^2, \dots, e_{n\mu}^2$  are the eigenvalues of a tridiagonal matrix with integer entries, follows from certain properties of the Chebychev polynomials. So squared chords have interesting connections with these topics. But what about  $s_{n1}^2, \dots, s_{nm}^2$ ? Are also they the eigenvalues of such a tridiagonal matrix? This question remains open.

The coefficients of the polynomial  $(x - e_{n1}^2) \cdots (x - e_{n\mu}^2)$  form an OEIS sequence, and so do also those of certain related polynomials. What about the coefficients of  $(x - s_{n1}^2) \cdots (x - s_{nm}^2)$ ? Do also they form such a sequence? This question remains open, too.

## Appendix: Proofs of (6) and (11)

*Proof of (6)*

$$\begin{aligned}
 & (x-2)P_m(x) - P_{m-1}(x) \\
 &= (x-2) \sum_{k=0}^m (-1)^{m-k} \binom{m+1+k}{2k+1} x^k - \sum_{k=0}^{m-1} (-1)^{m-1-k} \binom{m+k}{2k+1} x^k \\
 & - x^{m+1} + \sum_{k=0}^{m-1} (-1)^{m-k} \binom{m+1+k}{2k+1} x^{k+1} - 2 \sum_{k=0}^m (-1)^{m-k} \binom{m+1+k}{2k+1} x^k \\
 & \quad - \sum_{k=0}^{m-1} (-1)^{m-1-k} \binom{m+k}{2k+1} x^k \\
 &= x^{m+1} + \sum_{k=1}^m (-1)^{m+1-k} \binom{m+k}{2k-1} x^k + 2 \sum_{k=0}^m (-1)^{m+1-k} \binom{m+1+k}{2k+1} x^k \\
 & \quad - \sum_{k=0}^{m-1} (-1)^{m+1-k} \binom{m+k}{2k+1} x^k \\
 &= x^{m+1} - \left( \binom{2m}{2m-1} + 2 \binom{2m+1}{2m+1} \right) x^m \\
 & \quad + \sum_{k=1}^{m-1} (-1)^{m+1-k} \left( \binom{m+k}{2k-1} + 2 \binom{m+1+k}{2k+1} - \binom{m+k}{2k+1} \right) x^k \\
 & \quad + (-1)^{m+1} \left( 2 \binom{m+1}{1} - \binom{m}{1} \right) \\
 &= x^{m+1} - (2m+2)x^m + \sum_{k=1}^{m-1} (-1)^{m+1-k} \binom{m+2+k}{2k+1} x^k + (-1)^{m+1} (m+2) \\
 &= \sum_{k=0}^{m+1} (-1)^{m+1-k} \binom{m+1+1+k}{2k+1} x^k = P_{m+1}(x).
 \end{aligned}$$

*Proof of (11)*

$$\begin{aligned}
 & (x-3)P_m(x) - P_{m-1}(x) \\
 &= \dots = x^{m+1} - \left( \binom{2m}{2m-1} + 3 \binom{2m+1}{2m+1} \right) x^m \\
 & \quad + \sum_{k=1}^{m-1} (-1)^{m+1-k} \left( \binom{m+k}{2k-1} + 3 \binom{m+1+k}{2k+1} - \binom{m+k}{2k+1} \right) x^k \\
 & \quad + (-1)^{m+1} \left( 3 \binom{m+1}{1} - \binom{m}{1} \right) \\
 &= x^{m+1} - (2m+3)x^m + \sum_{k=1}^{m-1} (-1)^{m+1-k} \frac{2m+3}{m-k+1} \binom{m+1+k}{2k+1} x^k \\
 & \quad + (-1)^{m+1}(2m+3) \\
 &= x^{m+1} + \sum_{k=0}^m (-1)^{m+1-k} \frac{2(m+1)+1}{m+1-k} \binom{m+1+k}{2k+1} x^k = Q_{m+1}(x).
 \end{aligned}$$

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