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On the weighted integral inequalities for convex function

Abstract. In this paper, we establish several weighted inequalities for some differantiable mappings that are connected with the celebrated Hermite-Hadamard-Fejér type and Ostrowski type integral inequalities. The results presented here would provide extensions of those given in earlier works.

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1 Introduction

The following result is known in the literature as Ostrowski's inequality [10]:

Theorem 1 *Let* $f : [a,b] \to \mathbb{R}$ *be a differentiable mapping on* (a,b) *whose derivative* $f' : (a,b) \to \mathbb{R}$ *is bounded on* (a,b)*, i.e.,* $\|f'\|_\infty = \sup_{t\in(a,b)} |f'(t)| < \infty$*. Then, the inequality:*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{(x-\frac{a+b}{2})^2}{(b-a)^2} \right] (b-a) \left\| f' \right\|_\infty \tag{1}$$

holds for all $x \in [a,b]$*. The constant* $\frac{1}{4}$ *is the best possible.*

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Inequality (1) has wide applications in numerical analysis and in the theory of some special means; estimating error bounds for some special means, some mid-point, trapezoid and Simpson rules and quadrature rules, etc. Hence inequality (1) has attracted considerable attention and interest from mathematicans and researchers. Due to this, over the years, the interested reader is also refered to ([1]-[7],[12]-[17]) for integral inequalities in several independent variables. In addition, the current approach of obtaining the bounds, for a particular quadrature rule, have depended on the use of Peano kernel. The general approach in the past has involved the assumption of bounded derivatives of degree greater than one.

If $f : [a, b] \to \mathbb{R}$ is differentiable on $[a, b]$ with the first derivative f' integrable on $[a, b]$, then Montgomery identity holds:

$$f(x) = \frac{1}{b-a}\int_a^b f(t)dt + \int_a^b P(x,t)f'(t)dt, \tag{2}$$

where $P(x, t)$ is the Peano kernel defined by

$$P(x,t) := \begin{cases} \dfrac{t-a}{b-a}, & a \leq t < x \\ \\ \dfrac{t-b}{b-a}, & x \leq t \leq b. \end{cases}$$

Definition 1 *The function* $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$*, is said to be convex if the inequality*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

holds for all $x, y \in [a, b]$ *and* $\lambda \in [0, 1]$*. We say that* f *is concave if* $(-f)$ *is convex.*

The following inequality is well known in the literature as the Hermite-Hadamard integral inequality (see, [11]):

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{3}$$

holds, where $f : I \subset \mathbb{R} \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$.

The most well-known inequalities related to the integral mean of a convex function are the Hermite-Hadamard inequalities or its weighted versions, the so-called Hermite-Hadamard-Fejér inequalities (see, [18]-[22]). In [8], Fejér gave a weighted generalization of the inequality (3) as the following:

Theorem 2 *Let* $f : [a, b] \to \mathbb{R}$*, be a convex function, then the inequality*

$$f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx \le \frac{1}{b-a}\int_a^b f(x)w(x)dx \le \frac{f(a)+f(b)}{2}\int_a^b w(x)dx \quad (4)$$

holds, where $w : [a, b] \to \mathbb{R}$ *is nonnegative, integrable, and symmetric regarding* $x = \frac{a+b}{2}$.

In [18], some inequalities of Hermite-Hadamard-Fejér type for differentiable convex mappings were proved using the following lemma.

Lemma 1 *Let* $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* I°*,* $a, b \in I^\circ$ *with* $a < b$*, and* $w : [a, b] \to [0, \infty)$ *be a differentiable mapping. If* $f' \in L[a, b]$*, then the following equality holds:*

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)w(x)dx - \frac{1}{b-a}f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx \\ = (b-a)\int_0^1 k(t)f'(ta+(1-t)b)dt\end{aligned} \quad (5)$$

for each $t \in [0, 1]$*, where*

$$k(t) = \begin{cases} \int_0^t w(as+(1-s)b)ds, & t \in [0, \frac{1}{2}) \\ -\int_t^1 w(as+(1-s)b)ds, & t \in [\frac{1}{2}, 1]. \end{cases}$$

The main result in [18] is as follows:

Theorem 3 *Let* $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* I°*,* $a, b \in I^\circ$ *with* $a < b$*, and* $w : [a, b] \to [0, \infty)$ *be a differentiable mapping and symmetric to* $\frac{a+b}{2}$*. If* $|f'|$ *is convex on* $[a, b]$*, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)w(x)dx - \frac{1}{b-a}f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx\right| \\ &\le \left(\frac{1}{(b-a)^2}\int_{\frac{a+b}{2}}^b w(x)\left[(x-a)^2-(b-x)^2\right]dx\right)\left(\frac{|f'(a)|+|f'(b)|}{2}\right).\end{aligned} \quad (6)$$

In this article, using functions whose derivatives absolute values are convex, we obtained new inequalities of Fejer-Hermite-Hadamard type and Ostrowski type. The results presented here would provide extensions of those given in earlier works.

2 Main results

We will establish some new results connected with the left-hand side of (4) and Ostrowski type inequalities used the following Lemma. Now, we give the following new Lemma for our results:

Lemma 2 *Let* $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* I°, $a, b \in I^\circ$ *with* $a < b$ *and let* $w : [a, b] \to \mathbb{R}$. *If* $f', w \in L[a, b]$, *then, for all* $x \in [a, b]$, *the following equality holds:*

$$\begin{aligned}
&\int_a^x \left(\int_a^t w(s)ds\right)^{\alpha} f'(t)dt - \int_x^b \left(\int_t^b w(s)ds\right)^{\alpha} f'(t)dt \\
&= \left[\left(\int_a^x w(s)ds\right)^{\alpha} + \left(\int_x^b w(s)ds\right)^{\alpha}\right] f(x) \\
&- \alpha\int_a^x \left(\int_a^t w(s)ds\right)^{\alpha-1} w(t)f(t)dt - \alpha\int_x^b \left(\int_t^b w(s)ds\right)^{\alpha-1} w(t)f(t)dt.
\end{aligned} \tag{7}$$

Proof. By integration by parts, we have the following equalities:

$$\begin{aligned}
&\int_a^x \left(\int_a^t w(s)ds\right)^{\alpha} f'(t)dt = \left(\int_a^t w(s)ds\right)^{\alpha} f(t)\Bigg|_a^x - \alpha\int_a^x \left(\int_a^t w(s)ds\right) w(t)f(t)dt \\
&= \left(\int_a^x w(s)ds\right)^{\alpha} f(x) - \alpha\int_a^x \left(\int_a^t w(s)ds\right)^{\alpha-1} w(t)f(t)dt
\end{aligned} \tag{8}$$

and

$$\begin{aligned}
&\int_x^b \left(\int_t^b w(s)ds\right)^{\alpha} f'(t)dt \\
&= \left(\int_t^b w(s)ds\right)^{\alpha} f(t)\Bigg|_x^b + \alpha\int_x^b \left(\int_t^b w(s)ds\right)^{\alpha-1} w(t)f(t)dt \\
&= -\left(\int_x^b w(s)ds\right)^{\alpha} f(x) + \alpha\int_x^b \left(\int_t^b w(s)ds\right)^{\alpha-1} w(t)f(t)dt.
\end{aligned} \tag{9}$$

Subtracting (8) from (9), we obtain (7)

$$\begin{aligned}
&\int\limits_a^x \left(\int\limits_a^t w(s)ds\right)^{\alpha} f'(t)dt - \int\limits_x^b \left(\int\limits_t^b w(s)ds\right)^{\alpha} f'(t)dt \\
= &\left[\left(\int\limits_a^x w(s)ds\right)^{\alpha} + \left(\int\limits_x^b w(s)ds\right)^{\alpha}\right] f(x) \\
&-\alpha\int\limits_a^x \left(\int\limits_a^t w(s)ds\right)^{\alpha-1} w(t)f(t)dt - \alpha\int\limits_x^b \left(\int\limits_t^b w(s)ds\right)^{\alpha-1} w(t)f(t)dt.
\end{aligned}$$

This completes the proof. □

Corollary 1 *Under the same assumptions as in Lemma 2, if we put* $\alpha = 1$, *then the following identity holds:*

$$\begin{aligned}
&\left(\int\limits_a^b w(s)ds\right) f(x) - \int\limits_a^b w(t)f(t)dt \\
&= \int\limits_a^x \left(\int\limits_a^t w(s)ds\right) f'(t)dt - \int\limits_x^b \left(\int\limits_t^b w(s)ds\right) f'(t)dt
\end{aligned} \tag{10}$$

Remark 1 *If we take* $w(s) = 1$ *in (10), the idendity (10) reduces to the identity (2).*

Definition 2 *Let* $f \in L_1[a,b]$. *The Riemann-Liouville integrals* $J_{a+}^{\alpha}f$ *and* $J_{b-}^{\alpha}f$ *of order* $\alpha > 0$ *with* $a \geq 0$ *are defined by*

$$J_{a^+}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J_{b^-}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b$$

respectively. Here, $\Gamma(\alpha)$ *is the Gamma function and* $J_{a^+}^{0}f(x) = J_{b^-}^{0}f(x) = f(x)$.

Corollary 2 *Under the same assumptions as in Lemma 2, if we put* $w(s) = 1$, *then the following equality holds:*

$$[(x-a)^{\alpha} + (b-x)^{\alpha}]\, f(x) - \Gamma(\alpha+1) J_{x^-}^{\alpha} f(a) - \Gamma(\alpha+1) J_{x^+}^{\alpha} f(b) \quad (11)$$

$$= \int_a^x (t-a)^{\alpha} f'(t)dt - \int_x^b (b-t)^{\alpha} f'(t)dt.$$

Corollary 3 *Under the same assumptions of Corollary 2 with* $x = \frac{a+b}{2}$, *the idendity (11) becomes to the following identity*

$$f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b-a)^{\alpha}} \left[J_{\left(\frac{a+b}{2}\right)^-}^{\alpha} f(a) + J_{\left(\frac{a+b}{2}\right)^+}^{\alpha} f(b) \right]$$

$$= \frac{1}{2^{1-\alpha}(b-a)^{\alpha}} \left\{ \int_a^{\frac{a+b}{2}} (t-a)^{\alpha} f'(t)dt - \int_{\frac{a+b}{2}}^{b} (b-t)^{\alpha} f'(t)dt \right\}.$$

Now, by using the above lemma, we prove our main theorems:

Theorem 4 *Let* $f : I^{\circ} \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* I°, $a, b \in I^{\circ}$ *with* $a < b$ *and let* $w : [a, b] \to \mathbb{R}$ *be continuous on* $[a, b]$. *If* $|f'|$ *is convex on* $[a, b]$, *then the following inequality holds:*

$$\left| \left[\left(\int_a^x w(s)ds \right)^{\alpha} + \left(\int_x^b w(s)ds \right)^{\alpha} \right] f(x) \right.$$
$$\left. -\alpha \int_a^x \left(\int_a^t w(s)ds \right)^{\alpha-1} w(t)f(t)dt - \alpha \int_x^b \left(\int_t^b w(s)ds \right)^{\alpha-1} w(t)f(t)dt \right|$$
$$\leq \frac{\|w\|_{[a,x],\infty}^{\alpha}}{b-a} \left(\frac{(b-a)(x-a)^{\alpha+1}}{\alpha+1} - \frac{(x-a)^{\alpha+2}}{\alpha+2} \right) |f'(a)|$$
$$+ \frac{\|w\|_{[a,x],\infty}^{\alpha}}{b-a} \frac{(x-a)^{\alpha+2}}{\alpha+2} |f'(b)| + \frac{\|w\|_{[x,b],\infty}^{\alpha}}{b-a} \frac{(b-x)^{\alpha+2}}{\alpha+2} |f'(a)|$$
$$+ \frac{\|w\|_{[a,x],\infty}^{\alpha}}{b-a} \left(\frac{(b-a)(b-x)^{\alpha+1}}{\alpha+1} - \frac{(b-x)^{\alpha+2}}{\alpha+2} \right) |f'(b)|$$

$$\leq \frac{\|w\|_{[a,b],\infty}^{\alpha}}{b-a}\left\{\left(\frac{(b-a)(x-a)^{\alpha+1}}{\alpha+1}+\frac{(b-x)^{\alpha+2}-(x-a)^{\alpha+2}}{\alpha+2}\right)|f^{'}(a)|\right.$$
$$\left.+\left(\frac{(b-a)(b-x)^{\alpha+1}}{\alpha+1}+\frac{(x-a)^{\alpha+2}-(b-x)^{\alpha+2}}{\alpha+2}\right)|f^{'}(b)|\right\}$$

where $\alpha > 0$ *and* $\|w\|_{[a,b],\infty} = \sup_{t\in[a,b]} |w(t)|$.

Proof. We take absolute value of both sizes of (7), we find that

$$\left|\left[\left(\int_a^x w(s)ds\right)^{\alpha}+\left(\int_x^b w(s)ds\right)^{\alpha}\right]f(x)\right.$$
$$\left.-\alpha\int_a^x\left(\int_a^t w(s)ds\right)^{\alpha-1}w(t)f(t)dt-\alpha\int_x^b\left(\int_t^b w(s)ds\right)^{\alpha-1}w(t)f(t)dt\right|$$
$$\leq \int_a^x\left(\left|\int_a^t w(s)ds\right|\right)^{\alpha}\left|f^{'}(t)\right|dt+\int_x^b\left(\left|\int_t^b w(s)ds\right|\right)^{\alpha}|f^{'}(t)|dt$$
$$\leq \|w\|_{[a,x],\infty}^{\alpha}\int_a^x (t-a)^{\alpha}\left|f^{'}(t)\right|dt+\|w\|_{[x,b],\infty}\int_x^b (b-t)^{\alpha}|f^{'}(t)|dt$$
$$= \|w\|_{[a,x],\infty}^{\alpha}\int_a^x (t-a)^{\alpha}\left|f^{'}(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b)\right|dt$$
$$+\|w\|_{[x,b],\infty}^{\alpha}\int_x^b (b-t)^{\alpha}\left|f^{'}(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b)\right|dt$$

Since $|f^{'}|$ is convex on $[a.b]$, it follows that

$$\left|\left[\left(\int_a^x w(s)ds\right)^{\alpha}+\left(\int_x^b w(s)ds\right)^{\alpha}\right]f(x)\right.$$
$$\left.-\alpha\int_a^x\left(\int_a^t w(s)ds\right)^{\alpha-1}w(t)f(t)dt-\alpha\int_x^b\left(\int_t^b w(s)ds\right)^{\alpha-1}w(t)f(t)dt\right|$$

$$\leq \|w\|_{[a,x],\infty}^{\alpha} \int_a^x (t-a)^{\alpha} \left[\frac{b-t}{b-a}\left|f'(a)\right| + \frac{t-a}{b-a}|f'(b)|\right] dt$$
$$+ \|w\|_{[x,b],\infty}^{\alpha} \int_x^b (b-t)^{\alpha} \left[\frac{b-t}{b-a}|f'(a)| + \frac{t-a}{b-a}|f'(b)|\right] dt$$
$$= \frac{\|w\|_{[a,x],\infty}^{\alpha}}{b-a}\left\{\left(\frac{(b-a)(x-a)^{\alpha+1}}{\alpha+1} - \frac{(x-a)^{\alpha+2}}{\alpha+2}\right)|f'(a)| + \frac{(x-a)^{\alpha+2}}{\alpha+2}|f'(b)|\right\}$$
$$+ \frac{\|w\|_{[x,b],\infty}^{\alpha}}{b-a}\left\{\frac{(b-x)^{\alpha+2}}{\alpha+2}\left|f'(a)\right| + \left(\frac{(b-a)(b-x)^{\alpha+1}}{\alpha+1} - \frac{(b-x)^{\alpha+2}}{\alpha+2}\right)|f'(b)|\right\}$$
$$\leq \frac{\|w\|_{[a,b],\infty}^{\alpha}}{b-a}\left\{\left(\frac{(b-a)(x-a)^{\alpha+1}}{\alpha+1} + \frac{(b-x)^{\alpha+2}-(x-a)^{\alpha+2}}{\alpha+2}\right)|f'(a)|\right.$$
$$\left. + \left(\frac{(b-a)(b-x)^{\alpha+1}}{\alpha+1} + \frac{(x-a)^{\alpha+2}-(b-x)^{\alpha+2}}{\alpha+2}\right)|f'(b)|\right\}.$$

Hence, the proof of theorem is completed. □

Corollary 4 *Under the same assumptions as in Theorem 4, if we take* $w(s) = 1$*, then the following inequality holds:*

$$\begin{aligned} &\left|\left[(x-a)^{\alpha} + (b-x)^{\alpha}\right] f(x) - \Gamma(\alpha+1)\left[J_{x^-}^{\alpha} f(a) + J_{x^+}^{\alpha} f(b)\right]\right| \\ &\leq \frac{1}{b-a}\left\{\left(\frac{(b-a)(x-a)^{\alpha+1}}{\alpha+1} + \frac{(b-x)^{\alpha+2}-(x-a)^{\alpha+2}}{\alpha+2}\right)|f'(a)|\right. \\ &\left. + \left(\frac{(b-a)(b-x)^{\alpha+1}}{\alpha+1} + \frac{(x-a)^{\alpha+2}-(b-x)^{\alpha+2}}{\alpha+2}\right)|f'(b)|\right\}. \end{aligned} \tag{12}$$

Remark 2 *If we take* $x = \frac{a+b}{2}$ *in (12), we get*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{\left(\frac{a+b}{2}\right)^-}^{\alpha} f(a) + J_{\left(\frac{a+b}{2}\right)^+}^{\alpha} f(b)\right]\right|$$
$$\leq \frac{(b-a)}{4(\alpha+1)}\left(\left|f'(a)\right| + \left|f'(b)\right|\right)$$

which is proved by Sarikaya and Yildirim in [19].

Corollary 5 *Under the same assumptions as in Theorem 4, if we take* $\alpha = 1$*, then the following inequality holds:*

$$\begin{aligned}
&\left| \left(\int_a^b w(s)ds \right) f(x) - \int_a^b w(t)f(t)dt \right| \\
\leq\ & \frac{\|w\|_{[a,b],\infty}}{b-a} \left\{ \left(\frac{(b-a)(x-a)^2}{2} + \frac{(b-x)^3 - (x-a)^3}{3} \right) |f'(a)| \right. \\
&\left. + \left(\frac{(b-a)(b-x)^2}{2} + \frac{(x-a)^3 - (b-x)^3}{3} \right) |f'(b)| \right\}.
\end{aligned}$$

Corollary 6 *Under the same assumptions of Corollary 5 with* $x = \frac{a+b}{2}$*, we get*

$$\begin{aligned}
&\left| \left(\int_a^b w(s)ds \right) f\left(\frac{a+b}{2} \right) - \int_a^b w(t)f(t)dt \right| \\
&\leq \frac{(b-a)^2 \|w\|_{[a,b],\infty}}{4} \left(\frac{\left|f'(a)\right| + \left|f'(b)\right|}{2} \right).
\end{aligned} \tag{13}$$

Remark 3 *If we take* $w(s) = 1$ *in (13), we have*

$$\left| f\left(\frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \frac{(b-a)}{4} \left(\frac{\left|f'(a)\right| + \left|f'(b)\right|}{2} \right)$$

which is proved by Kırmacı in [9].

Corollary 7 *Under the same assumptions as in Theorem 4, if we put* $|f'(a)| = |f'(b)|$ *in (10), then the following inequality holds:*

$$\begin{aligned}
&\left| \left[\left(\int_a^x w(s)ds \right)^{\alpha} + \left(\int_x^b w(s)ds \right)^{\alpha} \right] f(x) \right. \\
&\left. -\alpha \int_a^x \left(\int_a^t w(s)ds \right)^{\alpha-1} w(t)f(t)dt - \alpha \int_x^b \left(\int_t^b w(s)ds \right)^{\alpha-1} w(t)f(t)dt \right|
\end{aligned}$$

$$\leq \frac{\left|f'(a)\right| \|w\|^{\alpha}_{[a,x],\infty}}{\alpha+1}(x-a)^{\alpha+1} + \frac{\left|f'(a)\right| \|w\|^{\alpha}_{[x,b],\infty}}{\alpha+1}(b-x)^{\alpha+1}$$

$$\leq \frac{\left|f'(a)\right| \|w\|^{\alpha}_{[a,b],\infty}}{\alpha+1}\left[(x-a)^{\alpha+1} + (b-x)^{\alpha+1}\right]$$

Theorem 5 *Let* $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* I°*,* $a, b \in I^\circ$ *with* $a < b$ *and let* $w : [a, b] \to \mathbb{R}$ *be continuous on* $[a, b]$*. If* $|f'|^q$ *is convex on* $[a, b]$*,* $q > 1$*, then the following inequality holds:*

$$\left|\left[\left(\int_a^x w(s)ds\right)^{\alpha} + \left(\int_x^b w(s)ds\right)^{\alpha}\right] f(x) - \alpha\int_a^x \left(\int_a^t w(s)ds\right)^{\alpha-1} w(t)f(t)dt\right.$$

$$\left.-\alpha\int_x^b \left(\int_t^b w(s)ds\right)^{\alpha-1} w(t)f(t)dt\right| \leq \frac{\|w\|^{\alpha}_{[a,x],\infty}}{(b-a)^{\frac{1}{q}}}\left(\frac{(x-a)^{\alpha p+1}}{\alpha p+1}\right)^{\frac{1}{p}}$$

$$\left(\frac{(b-a)^2-(b-x)^2}{2}|f'(a)|^q + \frac{(x-a)^2}{2}|f'(b)|^q\right)^{\frac{1}{q}} + \frac{\|w\|^{\alpha}_{[x,b],\infty}}{(b-a)^{\frac{1}{q}}}$$

$$\left(\frac{(b-x)^{\alpha p+1}}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{(b-x)^2}{2}|f'(a)|^q + \frac{(b-a)^2-(x-a)^2}{2}|f'(b)|^q\right)^{\frac{1}{q}} \tag{14}$$

$$\leq \frac{\|w\|^{\alpha}_{[a,b],\infty}}{(b-a)^{\frac{1}{q}}}\left\{\left(\frac{(x-a)^{\alpha p+1}}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{(b-a)^2-(b-x)^2}{2}|f'(a)|^q\right.\right.$$

$$\left.+\frac{(x-a)^2}{2}|f'(b)\||^q\right)^{\frac{1}{q}} + \left(\frac{(b-x)^{\alpha p+1}}{\alpha p+1}\right)^{\frac{1}{p}}$$

$$\left.\left(\frac{(b-x)^2}{2}|f'(a)|^q + \frac{(b-a)^2-(x-a)^2}{2}|f'(b)|^q\right)^{\frac{1}{q}}\right\}$$

where $\alpha > 0$*,* $\frac{1}{p} + \frac{1}{q} = 1$*, and* $\|w\|_{[a,b],\infty} = \sup_{t\in[a,b]} |w(t)|$.

Proof. We take absolute value of (7). Using Holder's inequality, we find that

$$\left|\left[\left(\int_a^x w(s)ds\right)^{\alpha}+\left(\int_x^b w(s)ds\right)^{\alpha}\right]f(x)\right.$$

$$\left.-\alpha\int_a^x\left(\int_a^t w(s)ds\right)^{\alpha-1}w(t)f(t)dt-\alpha\int_x^b\left(\int_t^b w(s)ds\right)^{\alpha-1}w(t)f(t)dt\right|$$

$$\leq \int_a^x\left|\int_a^t w(s)ds\right|^{\alpha}|f'(t)|dt+\int_x^b\left|\int_t^b w(s)ds\right|^{\alpha}|f'(t)|dt$$

$$\leq \int_a^x\left(\left|\int_a^t w(s)ds\right|^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_a^x|f'(t)|^q dt\right)^{\frac{1}{q}}$$

$$+\int_x^b\left(\left|\int_t^b w(s)ds\right|^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_x^b|f'(t)|^q dt\right)^{\frac{1}{q}}$$

$$\leq \|w\|_{[a,x],\infty}^{\alpha}\left(\int_a^x|t-a|^{\alpha p}\,dt\right)^{\frac{1}{p}}\left(\int_a^x|f'(t)|^q dt\right)^{\frac{1}{q}}$$

$$+\|w\|_{[x,b],\infty}^{\alpha}\left(\int_x^b|b-t|^{\alpha p}\,dt\right)^{\frac{1}{p}}\left(\int_x^b|f'(t)|^q dt\right)^{\frac{1}{q}}$$

Since $\left|f'(t)\right|^q$ is convex on $[a,b]$

$$\left|f'(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b)\right|^q\leq\frac{b-t}{b-a}\left|f'(a)\right|^q+\frac{t-a}{b-a}\left|f'(b)\right|^q \tag{15}$$

From (15), it follows that

$$\left|\left[\left(\int_a^x w(s)ds\right)^{\alpha}+\left(\int_x^b w(s)ds\right)^{\alpha}\right]f(x)\right.$$

$$\left.-\alpha\int_a^x\left(\int_a^t w(s)ds\right)^{\alpha-1}w(t)f(t)dt-\alpha\int_x^b\left(\int_t^b w(s)ds\right)^{\alpha-1}w(t)f(t)dt\right|$$

$$\leq \frac{\|w\|_{[a,x],\infty}^{\alpha}}{(b-a)^{\frac{1}{q}}}\left(\frac{(x-a)^{\alpha p+1}}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{(b-a)^2-(b-x)^2}{2}|f'(a)|^q\right.$$

$$\left.+\frac{(x-a)^2}{2}\left|f'(b)\right|^q\right)^{\frac{1}{q}}+\frac{\|w\|_{[x,b],\infty}^{\alpha}}{(b-a)^{\frac{1}{q}}}\left(\frac{(b-x)^{\alpha p+1}}{\alpha p+1}\right)^{\frac{1}{p}}$$

$$\left(\frac{(b-x)^2}{2}|f'(a)|^q+\frac{(b-a)^2-(x-a)^2}{2}|f'(b)|^q\right)^{\frac{1}{q}}\leq\frac{\|w\|_{[a,b],\infty}^{\alpha}}{(b-a)^{\frac{1}{q}}}$$

$$\left\{\left(\frac{(x-a)^{\alpha p+1}}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{(b-a)^2-(b-x)^2}{2}|f'(a)|^q+\frac{(x-a)^2}{2}|f'(b)|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{(b-x)^{\alpha p+1}}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{(b-x)^2}{2}|f'(a)|^q+\frac{(b-a)^2-(x-a)^2}{2}|f'(b)|^q\right)^{\frac{1}{q}}\right\}$$

which completes the proof. □

Corollary 8 *Under the same assumptions as in Theorem 4, if we put* $w(s)=1$*, then the following inequality holds:*

$$\left|\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]f(x)-\Gamma(\alpha+1)\left[J_{x^-}^{\alpha}f(a)+J_{x^+}^{\alpha}f(b)\right]\right|\leq\frac{1}{(b-a)^{\frac{1}{q}}}$$

$$\left\{\left(\frac{(x-a)^{\alpha p+1}}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{(b-a)^2-(b-x)^2}{2}|f'(a)|^q+\frac{(x-a)^2}{2}|f'(b)|^q\right)^{\frac{1}{q}}\right. \tag{16}$$

$$\left.+\left(\frac{(b-x)^{\alpha p+1}}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{(b-x)^2}{2}|f'(a)|^q+\frac{(b-a)^2-(x-a)^2}{2}|f'(b)|^q\right)^{\frac{1}{q}}\right\}.$$

Remark 4 *If we take* $x=\frac{a+b}{2}$ *in (16), we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{\left(\frac{a+b}{2}\right)^-}^{\alpha}f(a)+J_{\left(\frac{a+b}{2}\right)^+}^{\alpha}f(b)\right]\right|$$

$$\leq\frac{(b-a)}{4(\alpha p+1)^{\frac{1}{p}}}\left[\left(\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right]$$

which is proved by Sarikaya and Yildirim in [19].

Corollary 9 *Let the conditions of Theorem 5 hold. If we take* $\alpha = 1$ *in (14), then the following inequality holds:*

$$\left| \left(\int_a^b w(s)ds \right) f(x) - \int_a^b w(t)f(t)dt \right| \leq \frac{\|w\|_{[a,b],\infty}}{(b-a)^{\frac{1}{q}}}$$
$$\left\{ \left(\frac{(x-a)^{p+1}}{p+1} \right)^{\frac{1}{p}} \left(\frac{(b-a)^2-(b-x)^2}{2} |f^{'}(a)|^q + \frac{(x-a)^2}{2} |f^{'}(b)|^q \right)^{\frac{1}{q}} \right.$$
$$\left. + \left(\frac{(b-x)^{p+1}}{p+1} \right)^{\frac{1}{p}} \left(\frac{(b-x)^2}{2} \left|f^{'}(a)\right|^q + \frac{(b-a)^2-(x-a)^2}{2} |f^{'}(b)|^q \right)^{\frac{1}{q}} \right\}$$

Corollary 10 *Under the same assumptions of Corollary 9 with* $x = \frac{a+b}{2}$, *we get*

$$\left| \left(\int_a^b w(s)ds \right) f\left(\frac{a+b}{2} \right) - \int_a^b w(t)f(t)dt \right| \leq \frac{(b-a)^2 \|w\|_{[a,b],\infty}}{2^{2+\frac{1}{q}} (p+1)^{\frac{1}{p}}}$$
$$\left\{ \left(\frac{3|f^{'}(a)|^q + |f^{'}(b)|^q}{2} \right)^{\frac{1}{q}} + \left(\frac{|f^{'}(a)|^q + 3|f^{'}(b)|^q}{2} \right)^{\frac{1}{q}} \right\}. \tag{17}$$

Remark 5 *If we take* $w(s) = 1$ *in (17), we have*

$$\left| f\left(\frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(t)dt \right|$$
$$\leq \frac{(b-a)}{2^{2+\frac{1}{q}} (p+1)^{\frac{1}{p}}} \left\{ \left(\frac{3|f^{'}(a)|^q + |f^{'}(b)|^q}{2} \right)^{\frac{1}{q}} + \left(\frac{|f^{'}(a)|^q + 3|f^{'}(b)|^q}{2} \right)^{\frac{1}{q}} \right\}$$

which is proved by Kırmacı in [9].

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