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Evolution of $\mathfrak{I}$-functional and ω-entropy functional for the conformal Ricci flow

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Abstract. In this paper we define the $\mathfrak{I}$-functional and the ω-entropy functional for the conformal Ricci flow and see how they evolve according to time.

1 Introduction

In 1982 R. Hamilton introduced Ricci flow as a deformation of Riemannian metric [3], [4]. After him many scientists gave attention on it and in 2003–2004 G. Perelman [1], [2] used it to prove Poincaré conjecture. Meanwhile in 2004 A. E. Fischer introduced the concept of conformal Ricci flow equation which is given by

$$\begin{aligned} &\frac{\partial g}{\partial t} + 2\left(S + \frac{g}{n}\right) = -pg \\ &R(g) = -1. \end{aligned} \tag{1}$$

Here p is a scalar non dynamical field. As conformal Ricci flow equation is analogous to the Navier-Stokes equation of fluid mechanics, the scalar field p is also called conformal pressure field.

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The name conformal Ricci flow was introduced because of the role that conformal geometry plays in constraining the scalar curvature and because these equations are the vector field sum of a conformal flow equation and a Ricci flow equation. For the classical Ricci flow equation and the conformal Ricci flow equation, the volume and scalar curvature behave somewhat oppositely. In classical Ricci flow equation, the volume is preserved, that is $\mathrm{vol}(M, g) = 1$, but for non-static flows the scalar is not preserved, whereas for conformal Ricci flow equation the scalar curvature $R(g)$ is kept constant to -1 and for non-static flows the volume varies. Comparing the classical and conformal Ricci flow equations, we observe that the constraint equation changes from $\mathrm{vol}(M, g) = 1$ for the classical Ricci flow to $R(g) = -1$ for the conformal Ricci flow with the concomitant change of the configuration space from M^1 to M_{-1}. Since M^1 is a codimension-1 submanifold of M whereas M_{-1} is a codimension $C^\infty(M, \mathfrak{R})$ submanifold of M, M_{-1} is a much smaller configuration space than M^1. In the view point of geometry having a smaller configuration space is potentially better.

From the lecture note of P. Topping [5], we have been introduced the concept of the $\mathfrak{I}$-functional and Perelman's ω entropy functional for Ricci flow. In our paper we have defined the $\mathfrak{I}$-functional and ω-entropy functional regarding conformal Ricci flow and have shown how they evolve with respect to time t.

2 The $\mathfrak{I}$-functional for the conformal Ricci flow

Let M be a fixed closed manifold, g is a Riemannian metric and f is a function defined on M to the set of real numbers $\mathfrak{R}$.

Then the $\mathfrak{I}$-functional on pair (g, f) is defined as

$$\mathfrak{I}(g, f) = \int \left(-1 + |\nabla f|^2\right) e^{-f} dV. \tag{2}$$

Now we establish how the $\mathfrak{I}$-functional changes according to time under conformal Ricci flow.

Theorem 1 *In conformal Ricci flow, the rate of change of $\mathfrak{I}$-functional with respect of time is given by*

$$\frac{d}{dt}\mathfrak{I}(g, f) = \int \left[-2\mathrm{Ric}(\nabla f, \nabla f) - \left(\frac{2}{n} + p\right) g(\nabla f, \nabla f) - 2\frac{\partial f}{\partial t}(\Delta f - |\nabla f|^2) + (-1 + |\nabla f|^2)\left(-\frac{\partial f}{\partial t} + \frac{1}{2}\mathrm{tr}\frac{\partial g}{\partial t}\right)\right] e^{-f} dV,$$

where $\mathfrak{I}(g,f) = \int(-1+|\nabla f|^2)e^{-f}dV$.

Proof.

$$\frac{\partial}{\partial t}|\nabla f|^2 = \frac{\partial}{\partial t}g(\nabla f, \nabla f) = \frac{\partial g}{\partial t}(\nabla f, \nabla f) + 2g\left(\nabla\frac{\partial f}{\partial t}, \nabla f\right). \tag{3}$$

So using proposition 2.3.12 of [5] we can write

$$\begin{aligned}\frac{d}{dt}\mathfrak{I}(g,f) = &\int\left[\frac{\partial g}{\partial t}(\nabla f, \nabla f) + 2g\left(\nabla\frac{\partial f}{\partial t}, \nabla f\right)\right]e^{-f}dV\\ &+\int(-1+|\nabla f|^2)\left[-\frac{\partial f}{\partial t}+\frac{1}{2}\mathrm{tr}\frac{\partial g}{\partial t}\right]e^{-f}dV.\end{aligned} \tag{4}$$

Using integration by parts of equation (3), we get

$$\int 2g\left(\nabla\frac{\partial f}{\partial t}, \nabla f\right)e^{-f}dV = -2\int\frac{\partial f}{\partial t}(\Delta f - |\nabla f|^2)e^{-f}dV. \tag{5}$$

Now putting (5) in (4), we get

$$\begin{aligned}\frac{d}{dt}\mathfrak{I}(g,f) = &\int\left[\frac{\partial g}{\partial t}(\nabla f, \nabla f) - \frac{\partial f}{\partial t}(\Delta f - |\nabla f|^2)\right.\\ &\left.+(-1+|\nabla f|^2)\left(-\frac{\partial f}{\partial t}+\frac{1}{2}\mathrm{tr}\frac{\partial g}{\partial t}\right)\right]e^{-f}dV.\end{aligned} \tag{6}$$

Using (1) in (6), we get the following result for conformal Ricci flow, as

$$\begin{aligned}\frac{d}{dt}\mathfrak{I}(g,f) = &\int\Big[-2\mathrm{Ric}(\nabla f, \nabla f) - \left(\frac{2}{n}+p\right)g(\nabla f, \nabla f)\\ &-2\frac{\partial f}{\partial t}(\Delta f - |\nabla f|^2) + (-1+|\nabla f|^2)\left(-\frac{\partial f}{\partial t}+\frac{1}{2}\mathrm{tr}\frac{\partial g}{\partial t}\right)\Big]e^{-f}dV.\end{aligned} \tag{7}$$

Hence the proof. □

3 ω-entropy functional for the conformal Ricci flow

Let M be a closed manifold, g is a Riemannian metric on M and f is a smooth function defined from M to the set of real numbers $\mathfrak{R}$. We define ω-entropy functional as

$$\omega(g,f,\tau) = \int\left[\tau\left(-1+|\nabla f|^2\right)+f-n\right]u\,dV, \tag{8}$$

where $\tau > 0$ is a scale parameter and u is defined as $u(t) = e^{-f(t)}; \int_M u dV = 1$.

We would also like to define heat operator acting on the function $f : M \times [0, \tau] \longrightarrow \mathfrak{R}$ by $\diamondsuit := \frac{\partial}{\partial t} - \Delta$ and also, $\diamondsuit^* := -\frac{\partial}{\partial t} - \Delta - 1$, conjugate to $\diamondsuit$.

We choose u, such that $\diamondsuit^* u = 0$.

Now we prove the following theorem.

Theorem 2 *If* g*,* f*,* τ *evolve according to*

$$\frac{\partial g}{\partial t} = -2\mathrm{Ric} - \left(\frac{2}{n} + p\right) g \tag{9}$$

$$\frac{\partial \tau}{\partial t} = -1 \tag{10}$$

$$\frac{\partial f}{\partial t} = -\Delta f + |\nabla f|^2 + 1 + \frac{n}{2\tau} \tag{11}$$

and the function ν *defined as* $\nu = [\tau(2\Delta f - |\nabla f|^2 - 1) + f - n]u$*, then the rate of change of* ω*-entropy functional for conformal Ricci flow is* $\frac{d\omega}{dt} = -\int_M \diamondsuit^* \nu$*, where*

$$\diamondsuit^* \nu = 2u(\Delta f - |\nabla f|^2 - 1) - \frac{un}{2\tau} - \nu - u\tau[4 < \mathrm{Ric}, \mathrm{Hess} f > + \left(\frac{2}{n} + p\right) g(\nabla f, \nabla f) - 2g(\nabla|\nabla f|^2, \nabla f) + 4g(\nabla(\Delta f), \nabla f) + 2|\mathrm{Hess} f|^2].$$

Proof.

$$\diamondsuit^* \nu = \diamondsuit^* \left(\frac{\nu}{u} u\right) = \frac{\nu}{u} \diamondsuit^* u + u \diamondsuit^* \left(\frac{\nu}{u}\right).$$

We have defined previously that $\diamondsuit^* u = 0$, so

$$\diamondsuit^* \nu = u \diamondsuit^* \left(\frac{\nu}{u}\right)$$

$$\diamondsuit^* \nu = u \diamondsuit^* [\tau(2\nabla f - |\nabla f|^2 - 1) + f - n].$$

We shall use the conjugate of heat operator, as defined earlier as $\diamondsuit^* = -\left(\frac{\partial}{\partial t} + \Delta + 1\right)$. Therefore $\diamondsuit^* \nu = -u\left(\frac{\partial}{\partial t} + \Delta + 1\right)[\tau(2\Delta f - |\nabla f|^2 - 1) + f - n] \Rightarrow u^{-1} \diamondsuit^* \nu = -\left(\frac{\partial}{\partial t} + \Delta\right)[\tau(2\Delta f - |\nabla f|^2 - 1)] - \left(\frac{\partial}{\partial t} + \Delta\right) f - [\tau(2\Delta f - |\nabla f|^2 - 1) + f - n]$ using equation (10), we have

$$u^{-1} \diamondsuit^* \nu = (2\Delta f - |\nabla f|^2 - 1) - \tau\left(\frac{\partial}{\partial t} + \Delta\right)(2\Delta f - |\nabla f|^2 - 1) - \frac{\partial f}{\partial t} - \Delta f - \frac{\nu}{u}. \tag{12}$$

Now using the equality $\frac{\partial}{\partial t}(2\Delta f - |\nabla f|^2 - 1) = 2\frac{\partial}{\partial t}(\Delta f) - \frac{\partial}{\partial t}|\nabla f|^2$ and the proposition 2.5.6 of [5], we have

$$\frac{\partial}{\partial t}(2\Delta f - |\nabla f|^2 - 1) = 2\Delta \frac{\partial f}{\partial t} + 4 < Ric, Hessf > - \frac{\partial g}{\partial t}(\nabla f, \nabla f) - 2g\left(\frac{\partial}{\partial t}\nabla f, \nabla f\right).$$

Now using the conformal Ricci flow equation (1), we have

$$\begin{aligned}\frac{\partial}{\partial t}(2\Delta f - |\nabla f|^2 - 1) = 2\Delta \frac{\partial f}{\partial t} + & \; 4 < Ric, Hessf > +2Ric(\nabla f, \nabla f) \\ & + \left(\frac{2}{n} + p\right) g(\nabla f, \nabla f) - 2g\left(\frac{\partial}{\partial t}\nabla f, \nabla f\right).\end{aligned} \tag{13}$$

Using (11) in (13), we get

$$\begin{aligned}\frac{\partial}{\partial t}(2\Delta f - |\nabla f|^2 - 1) = 2\Delta & \left(-\Delta f + |\nabla f|^2 + 1 + \frac{n}{2\tau}\right) \\ & + 4 < Ric, Hessf > +2Ric(\nabla f, \nabla f) \\ & + \left(\frac{2}{n} + p\right) g(\nabla f, \nabla f) - 2g\left(\frac{\partial}{\partial t}\nabla f, \nabla f\right).\end{aligned} \tag{14}$$

Now let us compute

$$\Delta(2\Delta f - |\nabla f|^2 - 1) = 2\Delta^2 f - \Delta|\nabla f|^2. \tag{15}$$

Using (14) and (15) in (12) we obtain after a brief calculation

$$\begin{aligned}u^{-1}\diamondsuit^* v = & \; (2\Delta f - |\nabla f|^2 - 1) - \tau\Big[- 2\Delta^2 f + 2\Delta|\nabla f|^2 \\ & + 4 < Ric, Hessf > +2Ric(\nabla f, \nabla f) \\ & + \left(\frac{2}{n} + p\right) g(\nabla f, \nabla f) - 2g\left(\frac{\partial}{\partial t}\nabla f, \nabla f\right) \\ & + 2\Delta^2 f - \Delta|\nabla f|^2\Big] - \frac{\partial f}{\partial t} - \Delta f - \frac{v}{u} \\ = & \; \Delta f - |\nabla f|^2 - 1 - \tau[\Delta|\nabla f|^2 + 4 < Ric, Hessf > +2Ric(\nabla f, \nabla f) \\ & + \left(\frac{2}{n} + p\right) g(\nabla f, \nabla f) - 2g\left(\frac{\partial}{\partial t}\nabla f, \nabla f\right)\Big] - \frac{\partial f}{\partial t} - \frac{v}{u}\end{aligned}$$

$$= \Delta f - |\nabla f|^2 - 1 - \tau\Big[\Delta|\nabla f|^2 + 4 < \mathrm{Ric}, \mathrm{Hess} f > +2\mathrm{Ric}(\nabla f, \nabla f)$$
$$+ \left(\frac{2}{n} + p\right) g(\nabla f, \nabla f) - 2g\left(\frac{\partial}{\partial t}\nabla f, \nabla f\right)\Big] + \Delta f - |\nabla f|^2 - 1 - \frac{n}{2\tau} - \frac{v}{u}$$
$$= 2(\Delta f - |\nabla f|^2 - 1) - \frac{n}{2\tau} - \frac{v}{u} - \tau\Big[\Delta|\nabla f|^2 + 4 < \mathrm{Ric}, \mathrm{Hess} f > +2\mathrm{Ric}(\nabla f, \nabla f)$$
$$+ \left(\frac{2}{n} + p\right) g(\nabla f, \nabla f) - 2g\left(\frac{\partial}{\partial t}\nabla f, \nabla f\right)\Big]$$

$$u^{-1}\diamondsuit^* v = 2(\Delta f - |\nabla f|^2 - 1) - \frac{n}{2\tau} - \Big[\tau(2\Delta f - |\nabla f|^2 - 1) + f - n] - \tau[\Delta|\nabla f|^2$$
$$+ 4 < \mathrm{Ric}, \mathrm{Hess} f > +2\mathrm{Ric}(\nabla f, \nabla f)$$
$$+ \left(\frac{2}{n} + p\right) g(\nabla f, \nabla f) - 2g\left(\frac{\partial}{\partial t}\nabla f, \nabla f\right)\Big]$$

$$u^{-1}\diamondsuit^* v = 2(\Delta f - |\nabla f|^2 - 1) - \frac{n}{2\tau} - f + n - \tau\Big[2\Delta f - |\nabla f|^2 - 1 + \Delta|\nabla f|^2$$
$$+ 4 < \mathrm{Ric}, \mathrm{Hess} f > +2\mathrm{Ric}(\nabla f, \nabla f) \tag{16}$$
$$+ \left(\frac{2}{n} + p\right) g(\nabla f, \nabla f) - 2g\left(\nabla\frac{\partial f}{\partial t}, \nabla f\right)\Big]$$

using (11), we get

$$u^{-1}\diamondsuit^* v = 2\left(\Delta f - |\nabla f|^2 - 1\right) - \frac{n}{2\tau} - f + n - \tau\Big[2\Delta f - |\nabla f|^2 - 1 + \Delta|\nabla f|^2$$
$$+ 4 < \mathrm{Ric}, \mathrm{Hess} f > +2\mathrm{Ric}(\nabla f, \nabla f) + \left(\frac{2}{n} + p\right) g(\nabla f, \nabla f) \tag{17}$$
$$- 2g\left(\nabla\left(-\Delta f + |\nabla f|^2 + \frac{n}{2\tau} + 1\right), \nabla f\right)\Big].$$

We can rewrite (17) in the following way

$$u^{-1}\diamondsuit^* v = 2(\Delta f - |\nabla f|^2 - 1) - \frac{n}{2\tau} - f + n - \tau[2\Delta f - |\nabla f|^2 - 1$$
$$+ 4 < \mathrm{Ric}, \mathrm{Hess} f > + \left(\frac{2}{n} + p\right) g(\nabla f, \nabla f) \tag{18}$$
$$- 2g(\nabla|\nabla f|^2, \nabla f) + 4g(\nabla(\Delta f), \nabla f)]$$
$$+ \tau[-\Delta|\nabla f|^2 - 2\mathrm{Ric}(\nabla f, \nabla f) + 2g(\nabla(\Delta f), \nabla f)]$$

and using Bochner formula in (18) and simplifying, we get

$$u^{-1}\diamondsuit^* v = 2(\Delta f - |\nabla f|^2 - 1) - \frac{n}{2\tau} - f + n - \tau[2\Delta f - |\nabla f|^2 - 1 \\ + 4 < Ric, Hessf > +(\frac{2}{n} + p)g(\nabla f, \nabla f) - 2g(\nabla|\nabla f|^2, \nabla f) \\ + 4g(\nabla(\Delta f), \nabla f)] - 2\tau|Hessf|^2$$

$$u^{-1}\diamondsuit^* v = 2(\Delta f - |\nabla f|^2 - 1) - \frac{n}{2\tau} - [\tau(2\Delta f - |\nabla f|^2 - 1) + f - n] \\ - \tau[4 < Ric, Hessf > + \left(\frac{2}{n} + p\right) g(\nabla f, \nabla f) - 2g(\nabla|\nabla f|^2, \nabla f) \\ + 4g(\nabla(\Delta f), \nabla f)] - 2\tau|Hessf|^2$$

i.e.

$$u^{-1}\diamondsuit^* v = 2(\Delta f - |\nabla f|^2 - 1) - \frac{n}{2\tau} - \frac{v}{u} - \tau[4 < Ric, Hessf > \\ + \left(\frac{2}{n} +\right) g(\nabla f, \nabla f) - 2g(\nabla|\nabla f|^2, \nabla f) \\ + 4g(\nabla(\Delta f), \nabla f)] - 2\tau|Hessf|^2. \tag{19}$$

So finally we have

$$\diamondsuit^* v = 2u(\Delta f - |\nabla f|^2 - 1) - \frac{un}{2\tau} - v - u\tau[4 < Ric, Hessf > \\ + \left(\frac{2}{n} + p\right) g(\nabla f, \nabla f) - 2g(\nabla|\nabla f|^2, \nabla f) \\ + 4g(\nabla(\Delta f), \nabla f) + 2|Hessf|^2]. \tag{20}$$

Now using remark 8.2.7 of [5], we get

$$\frac{d\omega}{dt} = -\int_M \diamondsuit^* v.$$

So the evolution of ω with respect to time can be found by this integration. □

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