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On a new subclass of bi-univalent functions satisfying subordinate conditions

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Abstract. In the present investigation, we find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the function class $\mathcal{S}_{\Sigma}(\lambda, \phi)$. The results presented in this paper improve or generalize the recent work of Magesh and Yamini [15].

1 Introduction and definitions

Let A denote the class of analytic functions in the unit disk

$$U = \{z \in \mathbb{C} : |z| < 1\}$$

that have the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Further, by S we shall denote the class of all functions in A which are univalent in U.

The Koebe one-quarter theorem [8] states that the image of U under every function f from S contains a disk of radius $\frac{1}{4}$. Thus every such univalent function has an inverse f^{-1} which satisfies

$$f^{-1}(f(z)) = z, \quad (z \in U)$$

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and

$$f\left(f^{-1}(w)\right)=w\ ,\quad \left(|w|<r_0(f)\ ,\ r_0(f)\geq\frac{1}{4}\right),$$

where

$$f^{-1}(w)=w\ -a_2w^2+\left(2a_2^2-a_3\right)w^3-\left(5a_2^3-5a_2a_3+a_4\right)w^4+\cdots.$$

A function $f(z)\in A$ is said to be bi-univalent in U if both $f(z)$ and $f^{-1}(z)$ are univalent in U.

If the functions f and g are analytic in U, then f is said to be subordinate to g, written as

$$f(z)\prec g(z)\,,\qquad (z\in U)$$

if there exists a Schwarz function $w(z)$, analytic in U, with

$$w(0)=0\ \text{ and }\quad |w(z)|<1\qquad (z\in U)$$

such that

$$f(z)=g(w(z))\qquad (z\in U)\,.$$

Let Σ denote the class of bi-univalent functions defined in the unit disk U. For a brief history and interesting examples in the class Σ, (see [20]).

Lewin [14] studied the class of bi-univalent functions, obtaining the bound 1.51 for modulus of the second coefficient $|a_2|$. Subsequently, Brannan and Clunie [5] conjectured that $|a_2|\leq\sqrt{2}$ for $f\in\Sigma$. Netanyahu [16] showed that $\max|a_2|=\frac{4}{3}$ if $f(z)\in\Sigma$.

Brannan and Taha [4] introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses. $S^{\star}(\alpha)$ and $K(\alpha)$ of starlike and convex function of order α $(0<\alpha\leq 1)$ respectively (see [16]). Thus, following Brannan and Taha [4], a function $f(z)\in A$ is the class $S_{\Sigma}^{\star}(\alpha)$ of strongly bi-starlike functions of order α $(0<\alpha\leq 1)$ if each of the following conditions is satisfied:

$$f\in\Sigma,\quad \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right|<\frac{\alpha\pi}{2}\quad (0<\alpha\leq 1,\ z\in U)$$

and

$$\left|\arg\left(\frac{wg'(w)}{g(w)}\right)\right|<\frac{\alpha\pi}{2}\quad (0<\alpha\leq 1,\ w\in U)$$

where g is the extension of f^{-1} to U. The classes $S_{\Sigma}^{\star}(\alpha)$ and $K_{\Sigma}(\alpha)$ of bi-starlike functions of order α and bi-convex functions of order α, corresponding to the function classes $S^{\star}(\alpha)$ and $K(\alpha)$, were also introduced analogously. For each of the function classes $S_{\Sigma}^{\star}(\alpha)$ and $K_{\Sigma}(\alpha)$, they found non-sharp estimates on the initial coefficients. Recently, many authors investigated bounds for various subclasses of bi-univalent functions ([1], [3], [7], [9], [13], [15], [20], [21], [22]).

Not much is known about the bounds on the general coefficient $|a_n|$ for $n \geq 4$. In the literature, the only a few works determining the general coefficient bounds $|a_n|$ for the analytic bi-univalent functions ([2], [6], [10], [11], [12]). The coefficient estimate problem for each of $|a_n|$ ($n \in \mathbb{N}\backslash\{1,2\}$; $\mathbb{N} = \{1,2,3,...\}$) is still an open problem.

In this paper, by using the method [17] different from that used by other authors, we obtain bounds for the coefficients $|a_2|$ and $|a_3|$ for the subclasses of bi-univalent functions considered Magesh and Yamini and get more accurate estimates than that given in [15].

2 Coefficient estimates

In the following, let ϕ be an analytic function with positive real part in U, with $\phi(0) = 1$ and $\phi'(0) > 0$. Also, let $\phi(U)$ be starlike with respect to 1 and symmetric with respect to the real axis. Thus, ϕ has the Taylor series expansion

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots \quad (B_1 > 0). \tag{2}$$

Suppose that $u(z)$ and $v(w)$ are analytic in the unit disk U with $u(0) = v(0) = 0$, $|u(z)| < 1$, $|v(w)| < 1$, and suppose that

$$u(z) = b_1 z + \sum_{n=2}^{\infty} b_n z^n, \quad v(w) = c_1 w + \sum_{n=2}^{\infty} c_n w^n \quad (|z| < 1). \tag{3}$$

It is well known that

$$|b_1| \leq 1, \quad |b_2| \leq 1 - |b_1|^2, \quad |c_1| \leq 1, \quad |c_2| \leq 1 - |c_1|^2. \tag{4}$$

Next, the equations (2) and (3) lead to

$$\phi(u(z)) = 1 + B_1 b_1 z + \left(B_1 b_2 + B_2 b_1^2\right) z^2 + \cdots, \quad |z| < 1 \tag{5}$$

and

$$\phi(v(w)) = 1 + B_1 c_1 w + \left(B_1 c_2 + B_2 c_1^2\right) w^2 + \cdots, \quad |w| < 1. \tag{6}$$

Definition 1 *A function* $f \in \Sigma$ *is said to be* $S_{\Sigma}(\lambda,\phi)$, $0 \le \lambda \le 1$, *if the following subordination hold*

$$\frac{zf'(z)+\left(2\lambda^2-\lambda\right)z^2f''(z)}{4(\lambda-\lambda^2)z+(2\lambda^2-\lambda)zf'(z)+(2\lambda^2-3\lambda+1)f(z)} \prec \phi(z)$$

and

$$\frac{wg'(w)+\left(2\lambda^2-\lambda\right)w^2g''(w)}{4(\lambda-\lambda^2)w+(2\lambda^2-\lambda)wg'(w)+(2\lambda^2-3\lambda+1)g(w)} \prec \phi(w)$$

where $g(w)=f^{-1}(w)$.

Theorem 1 *Let* f *given by (1) be in the class* $S_{\Sigma}(\lambda,\phi)$. *Then*

$$|a_2| \le \frac{B_1\sqrt{B_1}}{\sqrt{\left|(12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1)B_1^2-(1+3\lambda-2\lambda^2)^2B_2\right|+(1+3\lambda-2\lambda^2)^2B_1}} \tag{7}$$

and

$$|a_3| \le \begin{cases} \frac{B_1}{2(2\lambda^2+1)}; & \text{if } B_1 \le \frac{\left(1+3\lambda-2\lambda^2\right)^2}{2(2\lambda^2+1)} \\ \frac{\left|\left(12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1\right)B_1^2-\left(1+3\lambda-2\lambda^2\right)^2B_2\right|B_1+2\left(2\lambda^2+1\right)B_1^3}{2(2\lambda^2+1)\left[\left|(12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1)B_1^2-(1+3\lambda-2\lambda^2)^2B_2\right|+(1+3\lambda-2\lambda^2)^2B_1\right]}; & \\ & \text{if } B_1 > \frac{\left(1+3\lambda-2\lambda^2\right)^2}{2(2\lambda^2+1)}. \end{cases} \tag{8}$$

Proof. Let $f \in S_{\Sigma}(\lambda,\phi)$, $0 \le \lambda \le 1$. Then there are analytic functions $u, v : U \to U$ given by (3) such that

$$\frac{zf'(z)+\left(2\lambda^2-\lambda\right)z^2f''(z)}{4(\lambda-\lambda^2)z+(2\lambda^2-\lambda)zf'(z)+(2\lambda^2-3\lambda+1)f(z)} = \phi(u(z)) \tag{9}$$

and

$$\frac{wg'(w)+\left(2\lambda^2-\lambda\right)w^2g''(w)}{4(\lambda-\lambda^2)w+(2\lambda^2-\lambda)wg'(w)+(2\lambda^2-3\lambda+1)g(w)} = \phi(v(w)) \tag{10}$$

where $g(w)=f^{-1}(w)$. Since

$$\begin{aligned} &\frac{zf'(z)+\left(2\lambda^2-\lambda\right)z^2f''(z)}{4(\lambda-\lambda^2)z+(2\lambda^2-\lambda)zf'(z)+(2\lambda^2-3\lambda+1)f(z)} \\ = \;& 1+\left(1+3\lambda-2\lambda^2\right)a_2z \\ &+\left[\left(12\lambda^4-28\lambda^3+11\lambda^2+2\lambda-1\right)a_2^2+\left(4\lambda^2+2\right)a_3\right]z^2+\cdots \end{aligned}$$

and

$$\begin{aligned}
&\frac{wg'(w)+(2\lambda^2-\lambda)\,w^2g''(w)}{4(\lambda-\lambda^2)\,w+(2\lambda^2-\lambda)\,wg'(w)+(2\lambda^2-3\lambda+1)\,g(w)}\\
=\;& 1-\left(1+3\lambda-2\lambda^2\right)a_2w\\
&+\left[\left(12\lambda^4-28\lambda^3+19\lambda^2+2\lambda+3\right)a_2^2-\left(4\lambda^2+2\right)a_3\right]w^2+\cdots,
\end{aligned}$$

it follows from (5), (6), (9) and (10) that

$$\left(1+3\lambda-2\lambda^2\right)a_2=B_1b_1, \tag{11}$$

$$\left(12\lambda^4-28\lambda^3+11\lambda^2+2\lambda-1\right)a_2^2+\left(4\lambda^2+2\right)a_3=B_1b_2+B_2b_1^2, \tag{12}$$

and

$$-\left(1+3\lambda-2\lambda^2\right)a_2=B_1c_1, \tag{13}$$

$$\left(12\lambda^4-28\lambda^3+19\lambda^2+2\lambda+3\right)a_2^2-\left(4\lambda^2+2\right)a_3=B_1c_2+B_2c_1^2. \tag{14}$$

From (11) and (13) we obtain

$$c_1=-b_1. \tag{15}$$

By adding (14) to (12), further computations using (11) to (15) lead to

$$\left[2\left(12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1\right)B_1^2-2\left(1+3\lambda-2\lambda^2\right)^2B_2\right]a_2^2=B_1^3\left(b_2+c_2\right). \tag{16}$$

(15) and (16), together with (4), give that

$$\left|\left(12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1\right)B_1^2-\left(1+3\lambda-2\lambda^2\right)^2B_2\right||a_2|^2\le B_1^3\left(1-|b_1|^2\right). \tag{17}$$

From (11) and (17) we get

$$|a_2|\le\frac{B_1\sqrt{B_1}}{\sqrt{\left|\left(12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1\right)B_1^2-\left(1+3\lambda-2\lambda^2\right)^2B_2\right|+\left(1+3\lambda-2\lambda^2\right)^2B_1}}.$$

Next, in order to find the bound on $|a_3|$, by subtracting (14) from (12), we obtain

$$4\left(2\lambda^2+1\right)a_3-4\left(2\lambda^2+1\right)a_2^2=B_1\left(b_2-c_2\right)+B_2\left(b_1^2-c_1^2\right). \tag{18}$$

Then, in view of (4) and (15) , we have

$$2\left(2\lambda^2+1\right)B_1|a_3| \le \left[2\left(2\lambda^2+1\right)B_1-\left(1+3\lambda-2\lambda^2\right)^2\right]|a_2|^2+B_1^2.$$

Notice that (7), we get

$$|a_3| \le \begin{cases} \frac{B_1}{2(2\lambda^2+1)}; & \text{if } B_1 \le \frac{\left(1+3\lambda-2\lambda^2\right)^2}{2(2\lambda^2+1)} \\ \frac{\left|\left(12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1\right)B_1^2-\left(1+3\lambda-2\lambda^2\right)^2B_2\right|B_1+2\left(2\lambda^2+1\right)B_1^3}{2(2\lambda^2+1)\left[\left|(12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1)B_1^2-(1+3\lambda-2\lambda^2)^2B_2\right|+(1+3\lambda-2\lambda^2)^2B_1\right]}; \\ \qquad \text{if } B_1 > \frac{\left(1+3\lambda-2\lambda^2\right)^2}{2(2\lambda^2+1)}. \end{cases}$$

□

Putting $\lambda = 0$ in Theorem 1, we have the following corollary.

Corollary 1 *Let* f *given by (1) be in the class* $S_\Sigma^*(\phi)$. *Then*

$$|a_2| \le \frac{B_1\sqrt{B_1}}{\sqrt{\left|B_1^2-B_2\right|+B_1}}$$

and

$$|a_3| \le \begin{cases} \frac{B_1}{2}; & \text{if } B_1 \le \frac{1}{2} \\ \frac{\left|B_1^2-B_2\right|B_1+2B_1^3}{2\left[\left|B_1^2-B_2\right|+B_1\right]}; & \text{if } B_1 > \frac{1}{2}. \end{cases}$$

The estimates on the coefficients $|a_2|$ and $|a_3|$ of Corollary 1 are improvement of the estimates obtained in Corollary 2.1 in [19].

Corollary 2 *If let*

$$\phi(z) = \left(\frac{1+z}{1-z}\right)^\alpha = 1+2\alpha z+2\alpha^2z^2+\ldots \qquad (0<\alpha\le 1),$$

then inequalities (7) and (8) become

$$|a_2| \le \frac{2\alpha}{\sqrt{|20\lambda^4-44\lambda^3+25\lambda^2-2\lambda+1|\,\alpha+(1+3\lambda-2\lambda^2)^2}} \tag{19}$$

and

$$|a_3| \le \begin{cases} \frac{\alpha}{2\lambda^2+1}; & \text{if } \ 0<\alpha\le \frac{\left(1+3\lambda-2\lambda^2\right)^2}{4(2\lambda^2+1)} \\ \frac{\left[\left|20\lambda^4-44\lambda^3+25\lambda^2-2\lambda+1\right|+4\left(2\lambda^2+1\right)\right]\alpha^2}{(2\lambda^2+1)\left[\left|20\lambda^4-44\lambda^3+25\lambda^2-2\lambda+1\right|\alpha+(1+3\lambda-2\lambda^2)^2\right]}; & \text{if } \ \frac{\left(1+3\lambda-2\lambda^2\right)^2}{4(2\lambda^2+1)}<\alpha\le 1. \end{cases} \tag{20}$$

The bounds on $|a_2|$ and $|a_3|$ given by (19) and (20) are more accurate than that given in Theorem 2.1 in [15].

We note that for $\lambda=0$, the class $\mathcal{S}_\Sigma(\lambda,\phi)$ reduces to the class of strongly bi-starlike functions of order α $(0<\alpha\le 1)$ and denoted by $S^\star_\Sigma(\alpha)$.
Putting $\lambda=0$ in Corollary 2, we have the following corollary.

Corollary 3 *Let* f *given by (1) be in the class* $S^*_\Sigma(\alpha)$, $(0<\alpha\le 1)$. *Then*

$$|a_2| \le \frac{2\alpha}{\sqrt{\alpha+1}} \tag{21}$$

and

$$|a_3| \le \begin{cases} \alpha; & \text{if } \ 0<\alpha\le \frac{1}{4} \\ \frac{5\alpha^2}{\alpha+1}; & \text{if } \ \frac{1}{4}<\alpha\le 1. \end{cases} \tag{22}$$

The bounds on $|a_3|$ given by (22) is more accurate than that given by Remark 2.2 in [17] and Theorem 2.1 in [4].

Remark 1 *The bounds on $|a_3|$ given by (22) is more accurate than that given in Corollary 2.3 in [18].*

Corollary 4 *If let*

$$\phi(z) = \frac{1+(1-2\alpha)z}{1-z} = 1+2(1-\alpha)z+2(1-\alpha)z^2+\cdots \qquad (0<\alpha\le 1),$$

then inequalities (7) and (8) become

$$|a_2| \le \frac{2(1-\alpha)}{\sqrt{\left|2(1-\alpha)(12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1)-(1+3\lambda-2\lambda^2)^2\right|+(1+3\lambda-2\lambda^2)^2}} \tag{23}$$

and

$$|a_3| \leq \begin{cases} \frac{1-\alpha}{2\lambda^2+1}; & \text{if } \frac{4(2\lambda^2+1)-(1+3\lambda-2\lambda^2)^2}{4(2\lambda^2+1)} \leq \alpha < 1 \\ \frac{\left[\left|2(1-\alpha)(12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1)-(1+3\lambda-2\lambda^2)^2\right|+4(1-\alpha)(2\lambda^2+1)\right](1-\alpha)}{(2\lambda^2+1)\left[\left|2(1-\alpha)(12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1)-(1+3\lambda-2\lambda^2)^2\right|+(1+3\lambda-2\lambda^2)^2\right]}; & \\ & \text{if } 0 \leq \alpha < \frac{4(2\lambda^2+1)-(1+3\lambda-2\lambda^2)^2}{4(2\lambda^2+1)}. \end{cases} \tag{24}$$

The bounds on $|a_2|$ and $|a_3|$ given by (23) and (24) are more accurate than that given in Theorem 3.1 in [15].

Putting $\lambda = 0$ in Corollary 4, we have the following corollary.

Corollary 5 *Let f given by (1) be in the class $S^{\star}_{\Sigma}(\alpha)$, $(0 \leq \alpha < 1)$. Then*

$$|a_2| \leq \frac{2(1-\alpha)}{\sqrt{1+|1-2\alpha|}} \tag{25}$$

and

$$|a_3| \leq \begin{cases} 1-\alpha; & \text{if } \frac{3}{4} \leq \alpha < 1 \\ \frac{(1-\alpha)|1-2\alpha|+4(1-\alpha)^2}{1+|1-2\alpha|}; & \text{if } 0 \leq \alpha < \frac{3}{4}. \end{cases} \tag{26}$$

The bounds on $|a_3|$ given by (26) is more accurate than that given by Remark 2.2 in [17] and Theorem 3.1 in [4].

Remark 2 *The bounds on $|a_3|$ given by (26) is more accurate than that given in Corollary 3.3 in [18].*

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