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On the maximal exponent of the prime power divisor of integers

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Abstract. The largest exponent of the prime powers function is investigated on the set of numbers of form one plus squares of primes.

1 Introduction

1.1. Notation. Let, as usual, $\mathcal{P}$, $\mathbb{N}$ be the set of primes, positive integers, respectively. For a prime divisor p of n let $\nu_p(n)$ be defined by $p^{\nu_p(n)} \| n$. Then $n = \prod_{p|n} p^{\nu_p(n)}$. Let

$$H(n) = \max_{p|n} \nu_p(n) \quad \text{and} \quad h(n) = \min_{p|n} \nu_p(n).$$

We denote by $\pi(x)$ the number of primes $p \le x$ and by $\pi(x, k, \ell)$ the number of primes $p \le x, p \equiv \ell \pmod{k}$.

1.2. Preliminaries. A. Niven proved in [7] that

$$\sum_{n \le x} h(n) = x + \frac{\zeta(3/2)}{\zeta(3)}\sqrt{x} + o(\sqrt{x}) \quad (x \to \infty) \tag{1}$$

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and that

$$\frac{1}{x}\sum_{n\le x} H(n) \to B \quad (x\to\infty), \quad \text{where} \quad B = 1+\sum_{k=2}^{\infty}\left(1-\frac{1}{\zeta(k)}\right). \tag{2}$$

W. Schwarz and J. Spilker showed in [8] that

$$\sum_{n\le x} H(n) = \mathcal{M}(H)x + O\left(x^{3/4}\exp(-\gamma\sqrt{\log x})\right) \quad (x\to\infty), \tag{3}$$

$$\sum_{n\le x} \frac{1}{H(n)} = \mathcal{M}\left(\frac{1}{H}\right)x + O(x^{3/4}\exp(-\gamma\sqrt{\log x}) \quad (x\to\infty), \tag{4}$$

where $\gamma > 0$ is a suitable constant, $\mathcal{M}(H) = B$, $\mathcal{M}\left(\frac{1}{H}\right)$ are suitable positive numbers.

D. Suryanayana and Sita Ramachandra Rao [9] proved that the error term in (3) and (4) can be improved to

$$O(\sqrt{x}\exp(-\gamma(\log x)^{3/5}(\log\log x)^{-1/5})).$$

They proved furthermore that

$$\sum_{n\le x} h(n) = c_1x + c_2x^{1/2} + c_3x^{1/3} + c_4x^{1/4} + c_5x^{1/5} + O(x^{1/6}), \tag{5}$$

$$\sum_{n\le x} \frac{1}{h(n)} = d_1x + d_2x^{1/2} + d_3x^{1/3} + d_4x^{1/4} + d_5x^{1/5} + O(x^{1/6}). \tag{6}$$

Gu Tongxing and Cao Huizhong announced in [4] that they can improve the error term in (3) to

$$O(\sqrt{x}\exp(-c(\log x)^{3/5}(\log\log x)^{-1/5})).$$

I. Kátai and M. V. Subbarao [5] investigated the asymptotic of

$$A_x(r) := \natural\{n\in[x,x+Y] \mid H(n)=r\}, \quad Y = x^{\frac{1}{2r+1}}\log x,$$

and

$$B_x(r) := \natural\{p\in\mathcal{P}, p\in[x,x+Y] \mid H(p+1)=r\}, \quad Y = x^{\frac{7}{12}+\epsilon}$$

for fixed $r\ge 1$.

Namely, they proved that

$$A_x(r) = Y\Big(\eta(r+1) - \eta(r)\Big) + O\Big(\frac{Y}{\log x}\Big),\ \eta(s) = \frac{1}{\zeta(s)} - 1 \quad (s = 1, 2, \cdots)$$

and

$$B_x(r) = e(r)\frac{Y}{\log x} + O\Big(\frac{Y}{(\log x)^2}\Big),$$

where

$$e(1) = \prod_{p\in\mathcal{P}}\Big(1 - \frac{1}{p(p-1)}\Big),$$

and for $r \geq 2$

$$e(r) = \prod_{p\in\mathcal{P}}\Big(1 - \frac{1}{(p-1)p^r}\Big) - \prod_{p\in\mathcal{P}}\Big(1 - \frac{1}{(p-1)p^{r-1}}\Big).$$

In [6] we can read some results on (5) assuming the Riemann conjecture.

Our main interest now is to give the asymptotic of the number of those $n \leq x, n \in \mathcal{B}$, for which $H(n) = r$ uniformly as $1 \leq r \leq \kappa(x)$, where $\kappa(x)$ is as large as it is possible. We shall investigate it when $\mathcal{B}$ = set of shifted primes.

1.3. Auxiliary results.

Lemma 1 (Brun-Titchmarsh inequality). *We have*

$$\pi(x, k, \ell) < C\frac{x}{\varphi(k)\log\frac{x}{k}}.$$

Lemma 2 (Siegel-Walfisz theorem). *We have*

$$\pi(x, k, \ell) = \frac{\operatorname{li} x}{\varphi(k)}\Big(1 + O(e^{-c\sqrt{\log x}})\Big)$$

uniformly as $(k, \ell) = 1$, $k \leq (\log x)^A$. *Here* A *is arbitrary,* $c > 0$ *is a fixed constant.*

Lemma 3 ([1]) *Let* q *be an odd prime,* $D = q^n$ $(n = 1, 2, \cdots)$, $\epsilon > 0$ *be an arbitrary small, and* M *be an arbitrary large positive number. Then the asymptotic law*

$$\pi(x, D, \ell) = \frac{\operatorname{li} x}{\varphi(D)}\Big(1 + O\Big((\log x)^{-M}\Big)\Big)$$

holds for $D \leq x^{3/8-\epsilon}$, $(\ell, D) = 1$.

Lemma 4 ([2]) *Let* a *be an integer,* $a \geq 2$*. If* $A > 0$*, then there is a* $B > 0$ *for which*

$$\sum_{\substack{d \leq \frac{x^{1/2}}{q}(\log x)^{-B} \\ (d,q)=1}} \max_{(r,qd)=1} \max_{y \leq x} \left| \pi(y, qd, r) - \frac{\operatorname{li} x}{\varphi(qd)} \right| \ll \frac{x}{\varphi(q)(\log x)^A}, \operatorname{li} x = \int_2^x \frac{du}{\log u}$$

uniformly for moduli $q \leq x^{1/3} \exp(-(\log\log x)^3)$ *that are powers of* a.

While the implicit constant in $\ll$ *may depend upon* a, B *is a function of* A *alone.* $B = A + 6$ *is permissible.*

We shall use a special consequence of this assertion:

Corollary. *Let* a *be an integer,* $a \geq 2$, $D = a^n$ $(n = 1, 2, \cdots)$, $D \leq x^{1/3} \exp(-(\log\log x)^3)$*. Let* $A > 0$ *be an arbitrary constant. Then*

$$\pi(x, D, \ell) = \frac{\operatorname{li} x}{\varphi(D)} \Big(1 + O\Big(\frac{1}{(\log x)^A}\Big)\Big), \quad (\ell, D) = 1$$

Lemma 5 ([3]) *Let* $q = p^r$, p *an odd prime,* $q x^{\frac{3}{5}+\epsilon} \leq h \leq x$*. Then*

$$\pi(x+h, q, \ell) - \pi(x, q, \ell) = (1 + o_x(1)) \frac{h}{\varphi(q) \log x}$$

as $x \to \infty$, $(\ell, q) = 1$.

2 Formulation of the theorems

Let $(0 <)U, V$ be coprime integers, and let Q be the smallest prime for which

$$U(1 + 2m) + V \equiv 0 \pmod{Q}$$

has a solution, that is

$$Q = \begin{cases} 2 & \text{if } \ 2|U+V \\ \text{smallest prime for which } \ (Q2U) = 1, & \text{if } \ 2 \nmid U+V. \end{cases}$$

Let

$$M_{U,V}(x \mid k) = \natural\{p \leq x \mid H(Up + V) = k\}.$$

Theorem 1 *Assume that* $r(x) \to \infty$ *arbitrarily slowly. Then, in the interval* $r(x) < k < (\frac{1}{3} - \epsilon)\frac{\log x}{\log Q}$, *we have*

$$M_{U,V}(x \mid k) = \frac{\mathrm{li} x}{\varphi(Q^k)}\Big(1 - \frac{1}{Q}\Big) \cdot (1 + o_x(1)).$$

Let $P(n) = n^2 + 1$. Then $4 \nmid P(n)$, $3 \nmid P(n)$, $5 \mid P(2)$, $5 \mid P(3)$. For every k there exists $1 \le \ell_k < \frac{5^k}{2}$, such that $P(\ell_k) \equiv 0 \pmod{5^k}$. The congruence $P(n) \equiv 0 \pmod{5^k}$ has exactly two solutions: ℓ_k and $5^k - \ell_k$. It obvious that $(\ell_k, 5) = 1$.

Let

$$E(x \mid k) = \natural\{p \le x \mid H(p^2 + 1) = k\}.$$

Theorem 2 *Assume that* $r(x) \to \infty$ *arbitrarily slowly. Then, in the interval* $r(x) < k < (\frac{1}{3} - \epsilon)\frac{\log x}{\log 5}$, *we have*

$$E(x \mid k) = \frac{2}{5^k}\mathrm{li} x(1 + o_x(1)).$$

3 Proof of Theorem 1.

It is obvious that

$$M_{U,V}(x \mid k) \le \sum_q{}^{*}\Big[Q(x, q^k, r_{q,k}) - Q(x, q^{k+1}, r_{q,k+1})\Big],$$

where q runs over all those primes for which $U(1 + 2m) + V \equiv 0 \pmod{q}$ has a solution, $r_{q,k} \equiv VU^{-1} \pmod{q^k}$, $r_{q,k+1} \equiv VU^{-1} \pmod{q^{k+1}}$.

By using Lemma 3 and Lemma 1 we obtain that

$$\begin{aligned} M_{U,V}(x \mid k) \le \frac{\mathrm{li} x}{\varphi(Q^k)}\Big(1 - \frac{1}{Q}\Big) \cdot \Big(1 + O\Big(\frac{1}{(\log x)^M}\Big)\Big) + \\ + C \sum_{\substack{q > Q \\ q \in \mathcal{P}}} \frac{\mathrm{li} x}{\varphi(q^k)} + C \sum_{\substack{Q < q \\ q^k \ge \sqrt{x}}} \frac{x}{q^k}. \end{aligned}$$

It is clear that

$$\sum_{\substack{q > Q \\ q \in \mathcal{P}}} \frac{1}{\varphi(q^k)} = \frac{o_x(1)}{\varphi(Q^k)}$$

and that

$$\sum_{\substack{q^k \geq \sqrt{x} \\ q \in \mathcal{P}}} \frac{1}{\varphi(q^k)} = O\Big(\frac{1}{x^{1/4}}\Big),$$

thus

$$M_{U,V}(x \mid k) \leq \Big(1 + o_x(1)\Big) \frac{\mathrm{li} x}{Q^k}.$$

On the other hand

$$M_{U,V}(x \mid k) \geq \Big[Q(x, Q^k, r_{Q,k}) - Q(x, Q^{k+1}, r_{Q,k+1})\Big] - \sum_{\substack{q > Q \\ q \in \mathcal{P}}} Q(x, Q^k q^k, r_{Qq,k}).$$

The sum on right hand side is less than

$$C \frac{\mathrm{li}\ x}{Q^k} \sum_{(Q<)q} \frac{1}{q^k} + O(x^{3/4}) \leq o_x(1) \frac{\mathrm{li} x}{Q^k}.$$

From Lemma 3 our theorem follows.

4 Proof of Theorem 2

We have

$$E(x \mid k) = S + O(T),$$

where

$$S = \natural\{p \leq x : \quad 5^k \| p^2 + 1\}$$

and

$$T = \sum_{\substack{q \in \mathcal{P} \\ q > 5}} \natural\{p \leq x : \quad q^k \| p^2 + 1\}.$$

Thus, by using Lemma 1 and $k \geq \gamma(x)$,

$$T \leq \sum_{\substack{q \in \mathcal{P} \\ q > 5}} \frac{2C \mathrm{li} x}{\varphi(q^k)} + \sum_{\substack{q^k > x \\ q \in \mathcal{P}}} \frac{x}{q^k} = o_x(1) \frac{\mathrm{li} x}{5^k}.$$

Hence we obtain that

$$E(x \mid k) \leq \frac{2}{5^k} \mathrm{li} x (1 + o_x(1)).$$

On the other hand

$$E(x \mid k) \geq S - \sum_{\substack{q \in \mathcal{P} \\ q > 5}} \natural\{p \leq x : \quad 5^k \cdot q^k \| p^2 + 1\}.$$

By using Lemma 1, the sum on the right can be overestimated by

$$\frac{C \operatorname{li} x}{5^k} \sum_{q > 5} \frac{1}{\varphi(q^k)} + \frac{x}{5^k} \sum_{q^k > \sqrt{x}} \frac{1}{q^k},$$

which is clearly $o_x(1)S$.

This completes the proof of Theorem 2.

5 Further remarks

By using Lemma 5 we can prove short interval version of Theorem 1 and 2.

Theorem 3 *Let* $5^k x^{3/5+\epsilon} \leq h \leq x$, $k \geq g(x)$. *Then*

$$E(x + h \mid k) - E(x) = \frac{h}{5^k} \frac{1}{\log x} (1 + o_x(1)).$$

Theorem 4 *Let Let* U, V *be coprime integers,* $U > 0, U + V$=*odd,* Q *be the smallest prime which is not a divisor of* $2U$. *Let* $k \geq g(x)$, $Q^k x^{3/5+\epsilon} \leq h \leq x$. *Then*

$$M_{U,V}(x + h \mid k) - M_{U,V}(x) = (1 + o_x(1)) \frac{h}{Q^k} \frac{1}{\log x} \quad \text{as} \quad x \to \infty.$$

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