

# Convergence of three-step iterations for Ciric-quasi contractive operator in $\text{CAT}(0)$ spaces

Gurucharan S. Saluja

Department of Mathematics,  
Govt. Nagarjuna P.G. College of Science,  
Raipur - 492001 (C.G.), India.  
email: saluja1963@gmail.com

**Abstract.** In this paper, we study three-step iteration process for Ciric-quasi contractive operator and establish strong convergence theorems for above mentioned operator and schemes in the setting of  $\text{CAT}(0)$  spaces. Our result extends and generalizes some previous work from the existing literature (see, e.g., [4, 30] and some others).

## 1 Introduction

A metric space  $X$  is a  $\text{CAT}(0)$  space if it is geodesically connected and if every geodesic triangle in  $X$  is at least as 'thin' as its comparison triangle in the Euclidean plane. It is well known that any complete, simply connected Riemannian manifold having non-positive sectional curvature is a  $\text{CAT}(0)$  space. For a thorough discussion of these spaces and of the fundamental role they play in geometry, we refer the reader to Bridson and Haefliger [8].

Fixed point theory in  $\text{CAT}(0)$  spaces was first studied by Kirk (see [22, 23]). He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete  $\text{CAT}(0)$  space always has a fixed

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**2010 Mathematics Subject Classification:** 54H25, 54E40.

**Key words and phrases:** Ciric-quasi contractive operator, three-step iteration scheme, fixed point, strong convergence,  $\text{CAT}(0)$  space.

point. Since, then the fixed point theory for various mappings and iteration schemes in a  $CAT(0)$  space has been rapidly developed and a lot of papers appeared (see, [3, 11, 13, 14, 20, 21, 24, 25, 27, 31, 32]). It is worth mentioning that the results in  $CAT(0)$  spaces can be applied to any  $CAT(k)$  space with  $k \leq 0$  since any  $CAT(k)$  space is a  $CAT(k')$  space for every  $k' \geq k$  (see, e.g., [8]).

Let  $(X, d)$  be a metric space. A *geodesic path* joining  $x \in X$  to  $y \in X$  (or, more briefly, a geodesic from  $x$  to  $y$ ) is a map  $c$  from a closed interval  $[0, l] \subset \mathbb{R}$  to  $X$  such that  $c(0) = x$ ,  $c(l) = y$ , and let  $d(c(t), c(t')) = |t - t'|$  for all  $t, t' \in [0, l]$ . In particular,  $c$  is an isometry, and  $d(x, y) = l$ . The image  $\alpha$  of  $c$  is called a geodesic (or metric) *segment* joining  $x$  and  $y$ . We say  $X$  is (i) a *geodesic space* if any two points of  $X$  are joined by a geodesic and (ii) *uniquely geodesic* if there is exactly one geodesic joining  $x$  and  $y$  for each  $x, y \in X$ , which we will denote by  $[x, y]$ , called the segment joining  $x$  to  $y$ .

A *geodesic triangle*  $\triangle(x_1, x_2, x_3)$  in a geodesic metric space  $(X, d)$  consists of three points in  $X$  (the vertices of  $\triangle$ ) and a geodesic segment between each pair of vertices (the *edges* of  $\triangle$ ). A *comparison triangle* for geodesic triangle  $\triangle(x_1, x_2, x_3)$  in  $(X, d)$  is a triangle  $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x}_1, \overline{x}_2, \overline{x}_3)$  in  $\mathbb{R}^2$  such that  $d_{\mathbb{R}^2}(\overline{x}_i, \overline{x}_j) = d(x_i, x_j)$  for  $i, j \in \{1, 2, 3\}$ . Such a triangle always exists (see [8]).

### **CAT(0) space**

A geodesic metric space is said to be a  $CAT(0)$  space if all geodesic triangles of appropriate size satisfy the following  $CAT(0)$  comparison axiom.

Let  $\triangle$  be a geodesic triangle in  $X$ , and let  $\overline{\triangle} \subset \mathbb{R}^2$  be a comparison triangle for  $\triangle$ . Then  $\triangle$  is said to satisfy the  $CAT(0)$  inequality if for all  $x, y \in \triangle$  and all comparison points  $\overline{x}, \overline{y} \in \overline{\triangle}$ ,

$$d(x, y) \leq d_{\mathbb{R}^2}(\overline{x}, \overline{y}). \quad (1)$$

Complete  $CAT(0)$  spaces are often called *Hadamard spaces* (see [19]). If  $x, y_1, y_2$  are points of a  $CAT(0)$  space and  $y_0$  is the mid point of the segment  $[y_1, y_2]$  which we will denote by  $(y_1 \oplus y_2)/2$ , then the  $CAT(0)$  inequality implies

$$d^2\left(x, \frac{y_1 \oplus y_2}{2}\right) \leq \frac{1}{2} d^2(x, y_1) + \frac{1}{2} d^2(x, y_2) - \frac{1}{4} d^2(y_1, y_2). \quad (2)$$

The inequality (2) is the (CN) inequality of Bruhat and Tits [9].

Let us recall that a geodesic metric space is a  $CAT(0)$  space if and only if it satisfies the (CN) inequality (see [[8], p.163]). Moreover, if  $X$  is a  $CAT(0)$

metric space and  $x, y \in X$ , then for any  $\alpha \in [0, 1]$ , there exists a unique point  $\alpha x \oplus (1 - \alpha)y \in [x, y]$  such that

$$d(z, \alpha x \oplus (1 - \alpha)y) \leq \alpha d(z, x) + (1 - \alpha)d(z, y), \quad (3)$$

for any  $z \in X$  and  $[x, y] = \{\alpha x \oplus (1 - \alpha)y : \alpha \in [0, 1]\}$ .

A subset  $C$  of a  $CAT(0)$  space  $X$  is convex if for any  $x, y \in C$ , we have  $[x, y] \subset C$ .

**Algorithm 1.** The sequence  $\{x_n\}$  defined by  $x_1 \in C$  and

$$x_{n+1} = a_n Tx_n + (1 - a_n)x_n, \quad n \geq 1, \quad (4)$$

where  $\{a_n\}_{n=1}^{\infty}$  is a sequence in  $(0, 1)$  is called a Mann iterative sequence (see [26]).

**Algorithm 2.** The sequence  $\{x_n\}$  defined by  $x_1 \in C$  and

$$\begin{aligned} y_n &= b_n Tx_n + (1 - b_n)x_n, \\ x_{n+1} &= a_n Ty_n + (1 - a_n)x_n, \quad n \geq 1, \end{aligned} \quad (5)$$

where  $\{a_n\}_{n=1}^{\infty}$  and  $\{b_n\}_{n=1}^{\infty}$  are appropriate sequences in  $[0, 1]$  is called an Ishikawa iterative sequence (see [17]).

**Algorithm 3.** The sequence  $\{x_n\}$  defined by  $x_1 \in C$  and

$$\begin{aligned} z_n &= c_n Tx_n + (1 - c_n)x_n, \\ y_n &= b_n Tz_n + (1 - b_n)x_n, \\ x_{n+1} &= a_n Ty_n + (1 - a_n)x_n, \quad n \geq 1, \end{aligned} \quad (6)$$

where  $\{a_n\}_{n=1}^{\infty}$ ,  $\{b_n\}_{n=1}^{\infty}$ ,  $\{c_n\}_{n=1}^{\infty}$  are appropriate sequences in  $(0, 1)$  is called Noor iterative sequence (see [28]).

**Algorithm 4.** The sequence  $\{x_n\}$  defined by  $x_1 \in C$  and

$$\begin{aligned} y_n &= b_n Tx_n + (1 - b_n)x_n, \\ x_{n+1} &= a_n Ty_n + (1 - a_n)Tx_n, \quad n \geq 1, \end{aligned} \quad (7)$$

where  $\{a_n\}_{n=1}^{\infty}$  and  $\{b_n\}_{n=1}^{\infty}$  are appropriate sequences in  $(0, 1)$  is called S-iterative sequence (see [2]).

Recently, Abbas and Nazir [1] introduced the following iterative process:

**Algorithm 5.** The sequence  $\{x_n\}$  defined by  $x_1 \in C$  and

$$\begin{aligned} x_{n+1} &= (1 - a_n)Ty_n + a_nTz_n, \\ y_n &= (1 - b_n)Tx_n + b_nTz_n, \\ z_n &= (1 - c_n)x_n + c_nTx_n, \quad n \geq 1 \end{aligned} \quad (8)$$

where  $\{a_n\}$ ,  $\{b_n\}$  and  $\{c_n\}$  are sequences in  $(0, 1)$ . They showed that this process converges faster than the Agarwal et al. [2].

Very recently, Thakur et al. [33] introduced the following iterative process:

**Algorithm 6.** The sequence  $\{x_n\}$  defined by  $x_1 \in C$  and

$$\begin{aligned} x_{n+1} &= (1 - a_n)Tx_n + a_nTy_n, \\ y_n &= (1 - b_n)z_n + b_nTz_n, \\ z_n &= (1 - c_n)x_n + c_nTx_n, \quad n \geq 1 \end{aligned} \quad (9)$$

where  $\{a_n\}$ ,  $\{b_n\}$  and  $\{c_n\}$  are sequences in  $(0, 1)$ . They showed that this process converges faster than all of the Picard, the Mann, the Ishikawa, the Noor, the Agarwal et al. and the Abbas et al. processes for contractions in the sense of Berinde [5] and in support gave analytic proof by a numerical example.

We now modify (9) in a  $CAT(0)$  space as follows.

Let  $C$  be a nonempty closed convex subset of a complete  $CAT(0)$  space  $X$  and  $T: C \rightarrow C$  be a mapping. Suppose that  $\{x_n\}$  is a sequence generated iteratively by

$$\begin{aligned} x_{n+1} &= (1 - a_n)Tx_n \oplus a_nTy_n, \\ y_n &= (1 - b_n)z_n \oplus b_nTz_n, \\ z_n &= (1 - c_n)x_n \oplus c_nTx_n, \quad n \geq 1 \end{aligned} \quad (10)$$

where  $\{a_n\}$ ,  $\{b_n\}$  and  $\{c_n\}$  are sequences in  $(0, 1)$ .

If we put  $c_n = 0$  for all  $n \geq 1$ , then (10) reduces to the following iteration process

$$\begin{aligned} x_{n+1} &= (1 - a_n)Tx_n \oplus a_nTy_n, \\ y_n &= (1 - b_n)x_n \oplus b_nTx_n, \end{aligned} \quad (11)$$

where  $\{a_n\}$  and  $\{b_n\}$  are sequences in  $(0, 1)$  is called modified  $S$ -iteration process.

We recall the following.

Let  $(X, d)$  be a metric space and  $T: X \rightarrow X$  be a mapping. A mapping  $T: X \rightarrow X$  is called an  $\alpha$ -contraction if

$$d(Tx, Ty) \leq \alpha d(x, y), \quad (12)$$

where  $\alpha \in (0, 1)$  and for all  $x, y \in X$ .

The mapping  $T$  is called Kannan mapping [18] if there exists  $b \in (0, \frac{1}{2})$  such that

$$d(Tx, Ty) \leq b [d(x, Tx) + d(y, Ty)], \quad (13)$$

for all  $x, y \in X$ .

The mapping  $T$  is called Chatterjea mapping [12] if there exists  $c \in (0, \frac{1}{2})$  such that

$$d(Tx, Ty) \leq c [d(x, Ty) + d(y, Tx)], \quad (14)$$

for all  $x, y \in X$ .

In 1972, combining these three definitions, Zamfirescu [34] proved the following important result.

**Theorem Z.** Let  $(X, d)$  be a complete metric space and  $T: X \rightarrow X$  a mapping for which there exists the real number  $\alpha, b$  and  $c$  satisfying  $\alpha \in (0, 1)$ ,  $b, c \in (0, \frac{1}{2})$  such that for any pair  $x, y \in X$ , at least one of the following conditions holds:

$$(z_1) \quad d(Tx, Ty) \leq \alpha d(x, y),$$

$$(z_2) \quad d(Tx, Ty) \leq b [d(x, Tx) + d(y, Ty)],$$

$$(z_3) \quad d(Tx, Ty) \leq c [d(x, Ty) + d(y, Tx)].$$

Then  $T$  has a unique fixed point  $p$  and the Picard iteration  $\{x_n\}_{n=0}^{\infty}$  defined by  $x_{n+1} = Tx_n$ ,  $n = 0, 1, 2, \dots$  converges to  $p$  for any arbitrary but fixed  $x_0 \in X$ .

The conditions  $(z_1) - (z_3)$  can be written in the following equivalent form

$$d(Tx, Ty) \leq h \max \left\{ d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Tx)}{2} \right\} \quad (15)$$

for all  $x, y \in X$  and  $0 < h < 1$ , has been obtained by Ćirić [10] in 1974.

A mapping satisfying (15) is called Ćirić quasi-contraction. It is obvious that each of the conditions  $(z_1) - (z_3)$  implies (15).

An operator  $T$  satisfying the contractive conditions  $(z_1) - (z_3)$  in the theorem  $Z$  is called  $Z$ -operator.

In 2000, Berinde [4] introduced a new class of operators on a normed space  $E$  satisfying

$$\|Tx - Ty\| \leq \delta \|x - y\| + L \|Tx - x\|, \quad (*)$$

for any  $x, y \in E$ ,  $0 \leq \delta < 1$  and  $L \geq 0$ .

He proved that this class is wider than the class of Zamfirescu operators and used the Mann iteration process to approximate fixed points of this class of operators in a normed space given in the form of following theorem.

**Theorem B.** Let  $C$  be a nonempty closed convex subset of a normed space  $E$ . Let  $T: C \rightarrow C$  be an operator satisfying  $(*)$ . Let  $\{x_n\}_{n=0}^{\infty}$  be defined by: for  $x_1 = x \in C$ , the sequence  $\{x_n\}_{n=0}^{\infty}$  given by (5) where  $\{a_n\}$  is a sequence in  $[0, 1]$ . If  $F(T) \neq \emptyset$  and  $\sum_{n=1}^{\infty} a_n = \infty$ , then  $\{x_n\}_{n=0}^{\infty}$  converges strongly to the unique fixed point of  $T$ .

In this paper, inspired and motivated by [33, 34], we study an iteration process (10) and establish strong convergence theorems to approximate the fixed point for Ćirić quasi contractive operator in the framework of  $CAT(0)$  spaces.

We need the following useful lemmas to prove our main result in this paper.

**Lemma 1** (See [27]) *Let  $X$  be a  $CAT(0)$  space.*

(i) *For  $x, y \in X$  and  $t \in [0, 1]$ , there exists a unique point  $z \in [x, y]$  such that*

$$d(x, z) = t d(x, y) \quad \text{and} \quad d(y, z) = (1 - t) d(x, y). \quad (A)$$

*We use the notation  $(1 - t)x \oplus ty$  for the unique point  $z$  satisfying (A).*

(ii) *For  $x, y \in X$  and  $t \in [0, 1]$ , we have*

$$d((1 - t)x \oplus ty, z) \leq (1 - t)d(x, z) + td(y, z).$$

**Lemma 2** (See [6]) *Let  $\{p_n\}_{n=0}^{\infty}$ ,  $\{q_n\}_{n=0}^{\infty}$ ,  $\{r_n\}_{n=0}^{\infty}$  be sequences of nonnegative numbers satisfying the following condition:*

$$p_{n+1} \leq (1 - s_n)p_n + q_n + r_n, \quad \forall n \geq 0,$$

*where  $\{s_n\}_{n=0}^{\infty} \subset [0, 1]$ . If  $\sum_{n=0}^{\infty} s_n = \infty$ ,  $\lim_{n \rightarrow \infty} q_n = O(s_n)$  and  $\sum_{n=0}^{\infty} r_n < \infty$ , then  $\lim_{n \rightarrow \infty} p_n = 0$ .*

## 2 Strong convergence theorems in CAT(0) Space

In this section, we establish strong convergence result of iteration process (10) to approximate a fixed point for Ciric quasi contractive operator in the framework of CAT(0) spaces.

**Theorem 1** *Let  $C$  be a nonempty closed convex subset of a complete CAT(0) space  $X$  and let  $T: C \rightarrow C$  be an operator satisfying the condition (15). Let  $\{x_n\}$  be defined by the iteration scheme (10). If  $\sum_{n=1}^{\infty} a_n b_n = \infty$ , then  $\{x_n\}$  converges strongly to the unique fixed point of  $T$ .*

**Proof.** By Theorem Z, we know that  $T$  has a unique fixed point in  $C$ , say  $u$ . Consider  $x, y \in C$ . Since  $T$  is a operator satisfying (15), then if

$$\begin{aligned} d(Tx, Ty) &\leq \frac{h}{2}[d(x, Tx) + d(y, Ty)] \\ &\leq \frac{h}{2}[d(x, Tx) + d(y, x) + d(x, Tx) + d(Tx, Ty)], \end{aligned}$$

implies

$$\left(1 - \frac{h}{2}\right)d(Tx, Ty) \leq \frac{h}{2}d(x, y) + hd(x, Tx),$$

which yields (using the fact that  $0 < h < 1$ )

$$d(Tx, Ty) \leq \left(\frac{h/2}{1 - h/2}\right)d(x, y) + \left(\frac{h}{1 - h/2}\right)d(x, Tx). \quad (16)$$

If

$$\begin{aligned} d(Tx, Ty) &\leq \frac{h}{2}[d(x, Ty) + d(y, Tx)] \\ &\leq \frac{h}{2}[d(x, Tx) + d(Tx, Ty) + d(y, x) + d(x, Tx)], \end{aligned}$$

implies

$$\left(1 - \frac{h}{2}\right)d(Tx, Ty) \leq \frac{h}{2}d(x, y) + hd(x, Tx),$$

which also yields (using the fact that  $0 < h < 1$ )

$$d(Tx, Ty) \leq \left(\frac{h/2}{1 - h/2}\right)d(x, y) + \left(\frac{h}{1 - h/2}\right)d(x, Tx). \quad (17)$$

Denote

$$L_1 = \max \left\{ h, \frac{h/2}{1 - h/2} \right\} = h,$$

$$L_2 = \max \left\{ \frac{h}{1-h/2}, \frac{h}{1-h/2} \right\} = \frac{h}{1-h/2}.$$

Thus, in all cases,

$$\begin{aligned} d(Tx, Ty) &\leq L_1 d(x, y) + L_2 d(x, Tx) \\ &= h d(x, y) + \left( \frac{h}{1-h/2} \right) d(x, Tx) \end{aligned} \quad (18)$$

holds for all  $x, y \in C$ .

Also from (15) with  $y = u = Tu$ , we have

$$\begin{aligned} d(Tx, u) &\leq h \max \left\{ d(x, u), \frac{d(x, Tx)}{2}, \frac{d(x, u) + d(u, Tx)}{2} \right\} \\ &\leq h \max \left\{ d(x, u), \frac{d(x, Tx)}{2}, \frac{d(x, u) + d(u, Tx)}{2} \right\} \\ &\leq h \max \left\{ d(x, u), \frac{d(x, u) + d(u, Tx)}{2}, \frac{d(x, u) + d(u, Tx)}{2} \right\}. \end{aligned} \quad (19)$$

Since for non-negative real numbers  $a$  and  $b$ , we have

$$\frac{a+b}{2} \leq \max\{a, b\}. \quad (20)$$

Using (20) in (19), we have

$$d(Tx, u) \leq h d(x, u). \quad (21)$$

Now (21) gives

$$d(Tx_n, u) \leq h d(x_n, u) \quad (22)$$

$$d(Ty_n, u) \leq h d(y_n, u) \quad (23)$$

and

$$d(Tz_n, u) \leq h d(z_n, u). \quad (24)$$

Using (10), (21) and Lemma 1(ii), we have

$$\begin{aligned} d(z_n, u) &= d((1-c_n)x_n \oplus c_nTx_n, u) \\ &\leq (1-c_n)d(x_n, u) + c_nd(Tx_n, u) \\ &\leq (1-c_n)d(x_n, u) + c_nh d(x_n, u) \\ &= [1 - (1-h)c_n]d(x_n, u). \end{aligned} \quad (25)$$



Again using (10), (24), (25) and Lemma 1(ii), we have

$$\begin{aligned}
 d(y_n, u) &= d((1 - b_n)z_n \oplus b_n Tz_n, u) \\
 &\leq (1 - b_n)d(z_n, u) + b_nd(Tz_n, u) \\
 &\leq (1 - b_n)d(z_n, u) + b_nh d(z_n, u) \\
 &= [(1 - (1 - h)b_n)]d(z_n, u) \\
 &\leq [(1 - (1 - h)b_n)][(1 - (1 - h)c_n)]d(x_n, u) \\
 &\leq [1 - (1 - h)b_n]d(x_n, u).
 \end{aligned} \tag{26}$$

Now using (10), (22), (23), (26) and Lemma 1(ii), we have

$$\begin{aligned}
 d(x_{n+1}, u) &= d((1 - a_n)Tx_n \oplus a_n Ty_n, u) \\
 &\leq (1 - a_n)d(Tx_n, u) + a_nd(Ty_n, u) \\
 &\leq (1 - a_n)h d(x_n, u) + a_nh d(y_n, u) \\
 &\leq (1 - a_n)h d(x_n, u) + a_nh[1 - (1 - h)b_n]d(x_n, u) \\
 &= [(1 - a_n)h + a_nh(1 - (1 - h)b_n)]d(x_n, u) \\
 &= h[1 - (1 - h)a_nb_n]d(x_n, u) \\
 &\leq [1 - (1 - h)a_nb_n]d(x_n, u) \\
 &= (1 - g_n)d(x_n, u)
 \end{aligned} \tag{27}$$

where  $g_n = (1 - h)a_nb_n$ , since  $0 < h < 1$  and by assumption of the theorem  $\sum_{n=1}^{\infty} a_nb_n = \infty$ , it follows that  $\sum_{n=1}^{\infty} g_n = \infty$ , therefore by Lemma 2, we get that  $\lim_{n \rightarrow \infty} d(x_n, u) = 0$ . Thus  $\{x_n\}$  converges strongly to a fixed point of  $T$ .

To show uniqueness of the fixed point  $u$ , assume that  $w_1, w_2 \in F(T)$  and  $w_1 \neq w_2$ .

Applying (15) and using the fact that  $0 < h < 1$ , we obtain

$$\begin{aligned}
 d(w_1, w_2) &= d(Tw_1, Tw_2) \\
 &\leq h \max \left\{ d(w_1, w_2), \frac{d(w_1, Tw_1) + d(w_2, Tw_2)}{2}, \right. \\
 &\quad \left. \frac{d(w_1, Tw_2) + d(w_2, Tw_1)}{2} \right\} \\
 &= h \max \left\{ d(w_1, w_2), \frac{d(w_1, w_1) + d(w_2, w_2)}{2}, \right. \\
 &\quad \left. \frac{d(w_1, w_2) + d(w_2, w_1)}{2} \right\}
 \end{aligned}$$

$$\begin{aligned}
&= h \max \left\{ d(w_1, w_2), 0, d(w_1, w_2) \right\} \\
&\leq h d(w_1, w_2) \\
&< d(w_1, w_2), \text{ since } 0 < h < 1,
\end{aligned}$$

which is a contradiction. Therefore  $w_1 = w_2$ . Thus  $\{x_n\}$  converges strongly to the unique fixed point of  $T$ . This completes the proof.  $\square$

**Theorem 2** *Let  $C$  be a nonempty closed convex subset of a complete  $CAT(0)$  space  $X$  and let  $T: C \rightarrow C$  be an operator satisfying the condition (15). Let  $\{x_n\}$  be defined by the iteration scheme (11). If  $\sum_{n=1}^{\infty} a_n b_n = \infty$ , then  $\{x_n\}$  converges strongly to the unique fixed point of  $T$ .*

**Proof.** The proof of Theorem 2 immediately follows from Theorem 1 by taking  $c_n = 0$  for all  $n \geq 1$ . This completes the proof.  $\square$

The contraction condition (12) makes  $T$  continuous function on  $X$  while this is not the case with contractive conditions (13), (14) and (18).

The contractive conditions (13) and (14) both included in the class of Zamfirescu operators and so their convergence theorems for iteration process (10) are obtained in Theorem 1 in the setting of  $CAT(0)$  space.

**Remark 1** *Our result extends the corresponding result of [30] to the case of three-step iteration process (10) and from uniformly convex Banach space to the setting of  $CAT(0)$  spaces.*

**Remark 2** *Theorem 1 also extends Theorem B to the case of three-step iteration process (10) and from normed space to the setting of  $CAT(0)$  spaces.*

### 3 Application to contraction of integral type

**Theorem 3** *Let  $C$  be a nonempty closed convex subset of a complete  $CAT(0)$  space  $X$  and let  $T: C \rightarrow C$  be an operator satisfying the following condition:*

$$\int_0^{d(Tx, Ty)} \mu(t) dt \leq h \int_0^{\max \left\{ d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Tx)}{2} \right\}} \mu(t) dt \quad (28)$$

for all  $x, y \in X$  and  $0 < h < 1$ , where  $\mu: [0, +\infty) \rightarrow [0, +\infty)$  is a Lebesgue-integrable mapping which is summable (i.e. with finite integral) on each compact subset of  $[0, +\infty)$ , nonnegative, and such that for each  $\varepsilon > 0$ ,  $\int_0^\varepsilon \mu(t) dt > 0$ . Let  $\{x_n\}$  be defined by the iteration process (10). If  $\sum_{n=1}^{\infty} a_n b_n = \infty$ , then  $\{x_n\}$  converges strongly to the unique fixed point of  $T$ .

**Proof.** The proof of Theorem 3 follows from Theorem 1 by taking  $\mu(t) = 1$  over  $[0, +\infty)$  since the contractive condition of integral type transforms into a general contractive condition (15) not involving integrals. This completes the proof.  $\square$

**Example 1** Let  $X = \{0, 1, 2, 3, 4, 5\}$  and  $d$  be the usual metric of reals. Let  $T: X \rightarrow X$  be given by

$$\begin{cases} T(x) = 5, & \text{if } x = 0 \\ = 3, & \text{otherwise.} \end{cases}$$

Again let  $\mu: [0, +\infty) \rightarrow [0, +\infty)$  be given by  $\mu(t) = 1$  for all  $t \in [0, +\infty)$ . Then  $\mu: [0, +\infty) \rightarrow [0, +\infty)$  is a Lebesgue-integrable mapping which is summable (i.e. with finite integral) on each compact subset of  $[0, +\infty)$ , nonnegative, and such that for each  $\varepsilon > 0$ ,  $\int_0^\varepsilon \mu(t) dt > 0$ .

Let us take  $x = 0$ ,  $y = 1$ . Then from condition (28), we have

$$\begin{aligned} 2 = \int_0^{d(Tx, Ty)} \mu(t) dt &\leq h \int_0^{\max \left\{ d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Tx)}{2} \right\}} \mu(t) dt \\ &= h \max \left\{ 1, \frac{7}{2}, \frac{7}{2} \right\} \end{aligned}$$

which implies  $h \geq \frac{4}{7}$ . Now if we take  $0 < h < 1$ , then condition (28) is satisfied and 3 is of course a unique fixed point of  $T$ .

The following corollaries are special cases of Theorem 3.

**Corollary 1** Let  $C$  be a nonempty closed convex subset of a complete  $CAT(0)$  space  $X$  and let  $T: C \rightarrow C$  be an operator satisfying the following condition:

$$\int_0^{d(Tx, Ty)} \mu(t) dt \leq h \int_0^{d(x, y)} \mu(t) dt \quad (29)$$

for all  $x, y \in X$  and  $h \in (0, 1)$ , where  $\mu: [0, +\infty) \rightarrow [0, +\infty)$  is a Lebesgue-integrable mapping which is summable (i.e. with finite integral) on each compact subset of  $[0, +\infty)$ , nonnegative, and such that for each  $\varepsilon > 0$ ,  $\int_0^\varepsilon \mu(t) dt > 0$ . Let  $\{x_n\}$  be defined by the iteration process (10). If  $\sum_{n=1}^\infty a_n b_n = \infty$ , then  $\{x_n\}$  converges strongly to the unique fixed point of  $T$ .

Condition (29) is called Branciari [7] contractive condition of integral type.

Putting  $\mu(t) = 1$  in the condition (29), we get Banach contraction condition.

**Proof.** The proof of corollary 1 immediately follows from Theorem 1 by taking  $\mu(t) = 1$  over  $[0, +\infty)$  and

$$\max \left\{ d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Tx)}{2} \right\} = d(x, y)$$

since the contractive condition of integral type transforms into a general contractive condition (12) not involving integrals. This completes the proof.  $\square$

**Corollary 2** *Let  $C$  be a nonempty closed convex subset of a complete  $CAT(0)$  space  $X$  and let  $T: C \rightarrow C$  be an operator satisfying the following condition:*

$$\int_0^{d(Tx, Ty)} \mu(t) dt \leq b \int_0^{[d(x, Tx) + d(y, Ty)]} \mu(t) dt \quad (30)$$

for all  $x, y \in X$  and  $b \in (0, \frac{1}{2})$ , where  $\mu: [0, +\infty) \rightarrow [0, +\infty)$  is a Lebesgue-integrable mapping which is summable (i.e. with finite integral) on each compact subset of  $[0, +\infty)$ , nonnegative, and such that for each  $\varepsilon > 0$ ,  $\int_0^\varepsilon \mu(t) dt > 0$ . Let  $\{x_n\}$  be defined by the iteration process (10). If  $\sum_{n=1}^\infty a_n b_n = \infty$ , then  $\{x_n\}$  converges strongly to the unique fixed point of  $T$ .

Condition (30) is called Kannan contractive condition [18] of integral type.

Putting  $\mu(t) = 1$  in the condition (30), we get Kannan contraction condition.

**Proof.** The proof of corollary 2 immediately follows from Theorem 1 by taking  $\mu(t) = 1$  over  $[0, +\infty)$  and

$$\max \left\{ d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Tx)}{2} \right\} = \frac{d(x, Tx) + d(y, Ty)}{2}$$

since the contractive condition of integral type transforms into a general contractive condition (13) not involving integrals. This completes the proof.  $\square$

**Corollary 3** *Let  $C$  be a nonempty closed convex subset of a complete  $CAT(0)$  space  $X$  and let  $T: C \rightarrow C$  be an operator satisfying the following condition:*

$$\int_0^{d(Tx, Ty)} \mu(t) dt \leq c \int_0^{[d(x, Ty) + d(y, Tx)]} \mu(t) dt \quad (31)$$

for all  $x, y \in X$  and  $c \in (0, \frac{1}{2})$ , where  $\mu: [0, +\infty) \rightarrow [0, +\infty)$  is a Lebesgue-integrable mapping which is summable (i.e. with finite integral) on each compact subset of  $[0, +\infty)$ , nonnegative, and such that for each  $\varepsilon > 0$ ,  $\int_0^\varepsilon \mu(t) dt > 0$ . Let  $\{x_n\}$  be defined by the iteration process (10). If  $\sum_{n=1}^\infty a_n b_n = \infty$ , then  $\{x_n\}$  converges strongly to the unique fixed point of  $T$ .

Condition (31) is called Chatterjæe contractive condition [12] of integral type.

Putting  $\mu(t) = 1$  in the condition (31), we get Chatterjæe contraction condition.

**Proof.** The proof of corollary 3 immediately follows from Theorem 1 by taking  $\mu(t) = 1$  over  $[0, +\infty)$  and

$$\max \left\{ d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Tx)}{2} \right\} = \frac{d(x, Ty) + d(y, Tx)}{2}$$

since the contractive condition of integral type transforms into a general contractive condition (14) not involving integrals. This completes the proof.  $\square$

Now, we give the examples in support of above corollaries.

**Example 2** Let  $X$  be the real line with the usual metric  $d$  and suppose  $C = [0, 1]$ . Define  $T: C \rightarrow C$  by  $T(x) = \frac{x+1}{2}$  for all  $x \in C$ . Obviously  $T$  is self-mapping with a unique fixed point 1. Again let  $\mu: [0, +\infty) \rightarrow [0, +\infty)$  be given by  $\mu(t) = 1$  for all  $t \in [0, +\infty)$ . Then  $\mu: [0, +\infty) \rightarrow [0, +\infty)$  is a Lebesgue-integrable mapping which is summable (i.e. with finite integral) on each compact subset of  $[0, +\infty)$ , nonnegative, and such that for each  $\varepsilon > 0$ ,  $\int_0^\varepsilon \mu(t) dt > 0$ .

If  $x, y \in [0, 1]$ , then we have

$$d(Tx, Ty) = \left| \frac{x-y}{2} \right|.$$

Let us take  $x = 0, y = 1$ . Then from condition (29), we have

$$\frac{1}{2} = \int_0^{d(Tx, Ty)} \mu(t) dt \leq h \cdot 1 = h \int_0^{d(x, y)} \mu(t) dt$$

which implies  $h \geq \frac{1}{2}$ . Now if we take  $0 < h < 1$ , then condition (29) is satisfied and 1 is of course a unique fixed point of  $T$ .

**Example 3** Let  $X$  be the real line with the usual metric  $d$  and suppose  $C = [0, 1]$ . Define  $T: C \rightarrow C$  by  $T(x) = \frac{x}{4}$  for all  $x \in C$ . Obviously  $T$  is self-mapping with a unique fixed point  $0$ . Again let  $\mu: [0, +\infty) \rightarrow [0, +\infty)$  be given by  $\mu(t) = 1$  for all  $t \in [0, +\infty)$ . Then  $\mu: [0, +\infty) \rightarrow [0, +\infty)$  is a Lebesgue-integrable mapping which is summable (i.e. with finite integral) on each compact subset of  $[0, +\infty)$ , nonnegative, and such that for each  $\varepsilon > 0$ ,  $\int_0^\varepsilon \mu(t) dt > 0$ .

If  $x, y \in [0, 1]$ , then we have

$$d(Tx, Ty) = \left| \frac{x - y}{4} \right|.$$

Let us take  $x = 0$ ,  $y = 1$ . Then from condition (30), we have

$$\frac{1}{4} = \int_0^{d(Tx, Ty)} \mu(t) dt \leq b \cdot \frac{3}{4} = b \int_0^{[d(x, Tx) + d(y, Ty)]} \mu(t) dt$$

which implies  $b \geq \frac{1}{3}$ . Now if we take  $0 < b < \frac{1}{2}$ , then condition (30) is satisfied and  $0$  is of course a unique fixed point of  $T$ .

**Example 4** Let  $X$  be the real line with the usual metric  $d$  and suppose  $C = [0, 1]$ . Define  $T: C \rightarrow C$  by  $T(x) = \frac{x}{4}$  for all  $x \in C$ . Obviously  $T$  is self-mapping with a unique fixed point  $0$ . Again let  $\mu: [0, +\infty) \rightarrow [0, +\infty)$  be given by  $\mu(t) = 1$  for all  $t \in [0, +\infty)$ . Then  $\mu: [0, +\infty) \rightarrow [0, +\infty)$  is a Lebesgue-integrable mapping which is summable (i.e. with finite integral) on each compact subset of  $[0, +\infty)$ , nonnegative, and such that for each  $\varepsilon > 0$ ,  $\int_0^\varepsilon \mu(t) dt > 0$ .

If  $x, y \in [0, 1]$ , then we have

$$d(Tx, Ty) = \left| \frac{x - y}{4} \right|.$$

Let us take  $x = 0$ ,  $y = 1$ . Then from condition (31), we have

$$\frac{1}{4} = \int_0^{d(Tx, Ty)} \mu(t) dt \leq c \cdot \frac{5}{4} = c \int_0^{[d(x, Ty) + d(y, Tx)]} \mu(t) dt$$

which implies  $c \geq \frac{1}{5}$ . Now if we take  $0 < c < \frac{1}{2}$ , then condition (31) is satisfied and  $0$  is of course a unique fixed point of  $T$ .

## 4 Conclusion

The Ćirić quasi contractive operator [10] is more general than Banach contraction, Kannan contraction, Chatterjea contraction and Zamfirescu operators. Thus the results obtained in this paper are improvement and generalization of several known results from the existing literature (see, e.g., [4, 30] and some others).

## Acknowledgements

The author would like to thanks the anonymous referee for his careful reading and valuable suggestions on the manuscript.

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Received: 14 October 2014