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All maximal idempotent submonoids of $\mathrm{Hyp}_G(2)$

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Abstract. The purpose of this paper is to determine all maximal idempotent submonoids and some maximal compatible idempotent submonoids of the monoid of all generalized hypersubstitutions of type $\tau = (2)$.

1 Introduction

In Universal Algebra, identities are used to classify algebras into collections, called varieties and hyperidentities are use to classify varieties into collections, called hypervarities. The concept of a hypersubstitution is a tool to study hyperidentities and hypervarities. The notion of a hypersubstitution originated by K. Denecke, D. Lau, R. Pöschel and D. Schweigert [3]. In 2000, S. Leeratanavalee and K. Denecke generalized the concepts of a hypersubstitution and a hyperidentity to the concepts of a generalized hypersubstitution and a strong hyperidentity, respectively [4]. The set of all generalized hypersubstitutions together with a binary operation and the identity hypersubstitution forms a monoid. There are several published papers on algebraic properties of this monoid and its submonoids.

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The concept of regular subsemigroup plays an important role in the theory of semigroup. The concept of an idempotent submonoid is an example of a regular subsemigroup. In 2013, W. Puninagool and S. Leeratanavalee studied the natural partial order on the set $E(\mathrm{Hyp}_G(2))$ of all idempotent elements of $\mathrm{Hyp}_G(2)$, see [6]. In 2012, the authors studied the natural partial order on $\mathrm{Hyp}_G(2)$, see [7]. In this paper we determine all maximal idempotent submonoids and give some maximal compatible idempotent submonoids of $\mathrm{Hyp}_G(2)$ under this partial order.

2 Generalized hypersubstitutions

Let $n \in \mathbb{N}$ be a natural number and $X_n := \{x_1, x_2, \ldots, x_n\}$ be an n-element set. Let $\{f_i \mid i \in I\}$ be a set of n_i-ary operation symbols indexed by the set I. We call the sequence $\tau = (n_i)_{i \in I}$ of arities of f_i, the *type*. An n-ary term of type τ is defined inductively by the following.

(i) Every $x_i \in X_n$ is an n-ary term of type τ.

(ii) If $t_1, t_2, \ldots, t_{n_i}$ are n-ary terms of type τ, then $f_i(t_1, t_2, \ldots, t_{n_i})$ is an n-ary term of type τ.

We denote the smallest set which contains $x_1, \ldots, x_n$ and is closed under finite number of applications of (ii) by $W_\tau(X_n)$ and let $W_\tau(X) := \bigcup_{n=1}^{\infty} W_\tau(X_n)$ be the set of all terms of type τ.

A mapping σ from $\{f_i \mid i \in I\}$ into $W_\tau(X)$ which does not necessarily preserve the arity is called a *generalized hypersubstitution of type* τ. The set of all generalized hypersubstitutions of type τ is denoted by $\mathrm{Hyp}_G(\tau)$. In general, to combine two mappings together we use a composition of mappings. But in this case to combine two generalized hypersubstitutions we need the concept of a generalized superposition of terms and the extension of a generalized hypersubstitution which are defined by the following.

Definition 1 A generalized superposition of terms is a mapping $S^m : W_\tau(X)^{m+1} \longrightarrow W_\tau(X)$ where

(i) $S^m(x_j, t_1, \ldots, t_m) := t_j, 1 \leq j \leq m,$

(ii) $S^m(x_j, t_1, \ldots, t_m) := x_j, m < j \in \mathbb{N},$

(iii) $S^m(f_i(s_1, \ldots, s_{n_i}), t_1, \ldots, t_m) := f_i(S^m(s_1, t_1, \ldots, t_m), \ldots, S^m(s_{n_i}, t_1, \ldots, t_m)).$

Definition 2 Let $\sigma \in Hyp_G(\tau)$. The extension of σ is a mapping $\hat{\sigma} : W_\tau(X) \longrightarrow W_\tau(X)$ where

(i) $\hat{\sigma}[x] := x \in X$,

(ii) $\hat{\sigma}[f_i(t_1, \ldots, t_{n_i})] := S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$, for any n_i-ary operation symbol f_i where $\hat{\sigma}[t_j]$, $1 \leq j \leq n_i$ are already defined.

Proposition 1 ([4]) *For arbitrary* $t, t_1, t_2, \ldots, t_n \in W_\tau(X)$ *and for arbitrary generalized hypersubstitution* $\sigma, \sigma_1, \sigma_2$ *we have*

(i) $S^n(\hat{\sigma}[t], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]) = \hat{\sigma}[S^n(t, t_1, \ldots, t_n)]$,

(ii) $(\hat{\sigma}_1 \circ \sigma_2)\hat{} = \hat{\sigma}_1 \circ \hat{\sigma}_2$.

The binary operation of two generalized hypersubstitutions σ_1, σ_2 is defined by $\sigma_1 \circ_G \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ where $\circ$ denotes the usual composition of mappings. It turns out that $Hyp_G(\tau)$ together with the identity element σ_{id} where $\sigma_{id}(f_i) = f_i(x_1, \ldots, x_{n_i})$ is a monoid under $\circ_G$, see [4].

3 All Maximal idempotent submonoids of $Hyp_G(2)$

We recall first the definition of an idempotent element of a semigroup. Let S be a semigroup. An element $a \in S$ is called *idempotent* if $aa = a$. We denote the set of all idempotent elements of a semigroup S by $E(S)$. Let $E(S) \neq \emptyset$. Define $a \leq b (a, b \in E(S))$ iff $a = ab = ba$. Then $\leq$ is a partial order on $E(S)$. We call $\leq$ a natural partial order on $E(S)$. A natural partial order $\leq$ on a semigroup S is said to be a *compatible* if $a \leq b$ implies $ac \leq bc$ and $ca \leq cb$ for all $a, b, c \in S$. Throughout this paper, let f be a binary operation symbol of type $\tau = (2)$. By σ_t we denote a generalized hypersubstitution which maps f to the term $t \in W_{(2)}(X)$. For $t \in W_{(2)}(X)$ we introduce the following notation:

(i) $leftmost(t)$:= the first variable (from the left) occurring in t,

(ii) $rightmost(t)$:= the last variable occurring in t,

(iii) $var(t)$:= the set of all variables occurring in t.

Let $\sigma_t \in Hyp_G(2)$, we denote $R_1 := \{\sigma_t \mid t = f(x_1, t') \text{ where } t' \in W_{(2)}(X) \text{ and } x_2 \notin var(t')\}$, $R_2 := \{\sigma_t \mid t = f(t', x_2) \text{ where } t' \in W_{(2)}(X) \text{ and } x_1 \notin var(t')\}$, $R_3 := \{\sigma_t \mid t \in \{x_1, x_2, f(x_1, x_2)\}\}$ and $R_4 := \{\sigma_t \mid var(t) \cap \{x_1, x_2\} = \emptyset\}$.

In 2008, W. Puninagool and S. Leeratanavalee [5] proved that: $\bigcup_{i=1}^{4} R_i = E(\mathsf{Hyp}_G(2))$.

Example 1 Let $\sigma_s \in R_1$ and $\sigma_t \in R_2$ such that $s = f(x_1, s')$ and $t = f(t', x_2)$ where $s' = f(x_4, x_1)$ and $t' = f(x_2, x_6)$. Consider

$$\begin{aligned}(\sigma_s \circ_G \sigma_t)(f) &= \widehat{\sigma}_s[f(f(x_2, x_6), x_2)] \\ &= S^2(f(x_1, f(x_4, x_1)), \widehat{\sigma}_s[f(x_2, x_6)], \widehat{\sigma}_s[x_2]) \\ &= S^2(f(x_1, f(x_4, x_1), f(x_2, f(x_4, x_2)), x_2) \\ &= f(f(x_2, f(x_4, x_2)), f(x_4, f(x_2, f(x_4, x_2)))).\end{aligned}$$

So $\sigma_s \circ_G \sigma_t \notin \bigcup_{i=1}^{4} R_i$.

By the previous example, we have $\bigcup_{i=1}^{4} R_i$ is not a subsemigroup of $\mathsf{Hyp}_G(2)$.

Let $\sigma_t \in \mathsf{Hyp}_G(2)$, we denote $R'_1 := \{\sigma_t \mid t = f(x_1, t')$ where $t' \in W_{(2)}(X), x_2 \notin var(t')$ and $rightmost(t') \neq x_1\}$ and $R'_2 := \{\sigma_t \mid t = f(t', x_2)$ where $t' \in W_{(2)}(X), x_1 \notin var(t')$ and $leftmost(t') \neq x_2\}$.

We denote $(MI)_{\mathsf{Hyp}_G(2)} = R'_1 \cup R'_2 \cup R_3 \cup R_4$, $(MI_1)_{\mathsf{Hyp}_G(2)} = R_1 \cup R_3 \cup R_4$ and $(MI_2)_{\mathsf{Hyp}_G(2)} = R_2 \cup R_3 \cup R_4$.

Proposition 2 $(MI)_{\mathsf{Hyp}_G(2)}$ *is an idempotent submonoid of* $\mathsf{Hyp}_G(2)$*.*

Proof. It is clear that $(MI)_{\mathsf{Hyp}_G(2)} \subseteq \mathsf{Hyp}_G(2)$ and every element in $(MI)_{\mathsf{Hyp}_G(2)}$ is idempotent. Next, we show that $(MI)_{\mathsf{Hyp}_G(2)}$ is a submonoid of $\mathsf{Hyp}_G(2)$.
Case 1: $\sigma_t \in R'_1$. Then $t = f(x_1, t')$ where $t' \in W_{(2)}(X)$ such that $x_2 \notin var(t')$ and $rightmost(t') \neq x_1$. Let $\sigma_s \in (MI)_{\mathsf{Hyp}_G(2)}$.

Case 1.1: $\sigma_s \in R'_1$. Then $s = f(x_1, s')$ where $x_2 \notin var(s')$ and $rightmost(s') \neq x_1$. Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_s)(f) &= \widehat{\sigma}_t[f(x_1, s')] \\ &= S^2(f(x_1, t'), x_1, \widehat{\sigma}_t[s']) \\ &= f(x_1, t') \quad \text{since } x_2 \notin var(t').\end{aligned}$$

Then $\sigma_t \circ_G \sigma_s \in R'_1 \subseteq (MI)_{\mathsf{Hyp}_G(2)}$.

Case 1.2: $\sigma_s \in R'_2$. Then $s = f(s', x_2)$ where $x_1 \notin var(s')$ and $leftmost(s') \neq x_2$. Consider $(\sigma_s \circ_G \sigma_t)(f) = \widehat{\sigma}_s[f(x_1, t')] = S^2(f(s', x_2), x_1, \widehat{\sigma}_s[t'])$

$= f(S^2(s', x_1, w), S^2(x_2, x_1, w))$, where $w = \widehat{\sigma}_s[t']$. Since $x_2 \notin var(t')$ and $rightmost(t') \neq x_1$, then $x_1, x_2 \notin var(w)$. Since $x_1 \notin var(s')$ and $x_1, x_2 \notin var(w)$, then $x_1, x_2 \notin var(S^2(s', x_1, w))$. Consider $(\sigma_t \circ_G \sigma_s)(f) = \widehat{\sigma}_t[f(s', x_2)] = S^2(f(x_1, t'), \widehat{\sigma}_t[s'], x_2) = f(S^2(x_1, u, x_2), S^2(t', u, x_2,))$, where $u = \widehat{\sigma}_t[s']$. Since $x_1 \notin var(s')$ and $leftmost(s') \neq x_2$, we have $x_1, x_2 \notin var(u)$. Since $x_2 \notin var(t')$ and $x_1, x_2 \notin var(u)$, we have $x_1, x_2 \notin var(S^2(t', u, x_2))$. Then $\sigma_s \circ_G \sigma_t, \sigma_t \circ_G \sigma_s \in R'_4 \subseteq (MI)_{Hyp_G(2)}$.

Case 1.3: $\sigma_s \in R_3$. Then $s = x_1$ or $s = x_2$ or $s = f(x_1, x_2)$.

If $s = x_1$, then $(\sigma_t \circ_G \sigma_s)(f) = \widehat{\sigma}_t[x_1] = x_1$ and $(\sigma_s \circ_G \sigma_t)(f) = \widehat{\sigma}_{x_1}[f(x_2, t')] = S^2(x_1, x_2, \widehat{\sigma}_{x_1}[t']) = x_2$.

If $s = x_2$, then $(\sigma_t \circ_G \sigma_s)(f) = \widehat{\sigma}_t[x_2] = x_2$ and $(\sigma_s \circ_G \sigma_t)(f) = \widehat{\sigma}_{x_2}[f(x_2, t')] = S^2(x_2, x_2, \widehat{\sigma}_{x_2}[t'])$.

Since $x_1 \notin var(t')$ and $rightmost(t') \neq x_2$, then $S^2(x_2, x_2, \widehat{\sigma}_{x_2}[t']) = x_i \notin \{x_1, x_2\}$.

If $s = f(x_1, x_2)$, then $\sigma_s = \sigma_{id}$ such that $\sigma_t \circ_G \sigma_{id} = \sigma_t = \sigma_{id} \circ_G \sigma_t$.

Therefore $\sigma_s \circ_G \sigma_t, \sigma_s \circ_G \sigma_t \in (MI)_{Hyp_G(2)}$.

Case 1.4: $\sigma_s \in R_4$. Then $s = f(s_1, s_2)$ where $x_1, x_2 \notin var(s)$. Consider $(\sigma_t \circ_G \sigma_s)(f) = \widehat{\sigma}_t[f(s_1, s_2)] = S^2(f(x_2, t'), \widehat{\sigma}_t[s_1], \widehat{\sigma}_t[s_2]) = f(S^2(x_2, w_1, w_2), S^2(t', w_1, w_2))$, where $w_1 = \widehat{\sigma}_t[s_1]$ and $w_2 = \widehat{\sigma}_t[s_2]$. Then $x_1, x_2 \notin var(w_1) \cup var(w_2)$. The consequence is $x_1, x_2 \notin var(S^2(t', w_1, w_2))$.
Since $x_1, x_2 \notin var(w_2) \cup var(S^2(t', w_1, w_2))$, so that $\sigma_t \circ_G \sigma_s \in R'_4 \subseteq (MI)_{Hyp_G(2)}$.
Consider $(\sigma_s \circ_G \sigma_t)(f) = \widehat{\sigma}_s[f(x_2, t')] = S^2(f(s_1, s_2), x_2, \widehat{\sigma}_s[t']) = f(s_1, s_2)$ since $x_1, x_2 \notin var(s)$. So that $\sigma_s \circ_G \sigma_t \in R_4 \subseteq (MI)_{Hyp_G(2)}$.

Case 2: $\sigma_t \in R'_2$ and $\sigma_s \in R'_2 \cup R_3 \cup R_4$. It can be proved similarly as in Case 1. Then we have $\sigma_t \circ_G \sigma_s, \sigma_s \circ_G \sigma_t \in (MI)_{Hyp_G(2)}$.

Case 3: $\sigma_t \in R_3$ and $\sigma_s \in R_3 \cup R_4$. It can be proved similarly as in Case 1.3. Then we have $\sigma_t \circ_G \sigma_s, \sigma_s \circ_G \sigma_t \in (MI)_{Hyp_G(2)}$.

Case 4: $\sigma_t \in R_4$ and $\sigma_s \in R_4$. Then $\sigma_t \circ_G \sigma_s = \sigma_t \in R_4 \subseteq (MI)_{Hyp_G(2)}$.

Therefore $(MI)_{Hyp_G(2)}$ is a submonoid of $Hyp_G(2)$. □

Corollary 1 $(MI_1)_{Hyp_G(2)}$ *and* $(MI_2)_{Hyp_G(2)}$ *are idempotent submonoids of* $Hyp_G(2)$.

Proposition 3 $(MI)_{Hyp_G(2)}$ *is a maximal idempotent submonoid of* $Hyp_G(2)$.

Proof. Let K be a proper idempotent submonoid of $Hyp_G(2)$ such that $(MI)_{Hyp_G(2)} \subseteq K \subset Hyp_G(2)$. Let $\sigma_t \in K$. Then σ_t is an idempotent element.

Case 1: $\sigma_t \in R_1 \backslash R'_1$. Then $t = f(x_1, t')$ where $x_2 \notin var(t')$ and $rightmost(t') =$

x_1. Choose $\sigma_s \in R_2' \subseteq K$, then $s = f(s', x_2)$ such that $x_1 \notin var(s')$ and $leftmost(s') \neq x_2$. Consider $(\sigma_s \circ_G \sigma_t)(f) = \widehat{\sigma}_s[f(x_1, t')] = S^2(f(s', x_2), x_1, \widehat{\sigma}_s[t'])$ $= f(S^2(s', x_1, w), S^2(x_2, x_1, w))$ where $w = \widehat{\sigma}_s[t']$. Since $x_2 \in var(s)$ and $rightmost(t') = x_1$, we have $x_1 \in var(w)$ and $S^2(s', x_1, w) \in W_{(2)}(X) \setminus X$. Since $x_1 \in var(w)$, $\sigma_s \circ_G \sigma_t$ is not idempotent. So $\sigma_t \in R_1'$.
Case 2: $\sigma_t \in R_2 \setminus R_2'$. Then $t = f(t', x_2)$ where $x_1 \notin var(t')$ and $leftmost(t') = x_2$. Choose $\sigma_s \in R_1' \subseteq K$, then $s = f(x_1, s')$ such that $x_2 \notin var(s')$ and $rightmost(s') \neq x_1$. Consider $(\sigma_s \circ_G \sigma_t)(f) = \widehat{\sigma}_s[f(t', x_2)] = S^2(f(x_1, s'), \widehat{\sigma}_s[t'], x_2) = f(S^2(x_1, w, x_2), S^2(s', w, x_2))$, where $w = \widehat{\sigma}_s[t']$. Since $x_1 \in var(s)$ and $leftmost(t') = x_2$, we have $x_2 \in var(w)$ and $S^2(s', w, x_2) \in W_{(2)}(X) \setminus X$. Since $x_2 \in var(w)$, $\sigma_s \circ_G \sigma_t$ is not idempotent. So $\sigma_t \in R_2'$. Then $\sigma_t \in (MI)_{\mathsf{Hyp}_G(2)}$. Therefore $K \subseteq (MI)_{\mathsf{Hyp}_G(2)}$ and thus $K = (MI)_{\mathsf{Hyp}_G(2)}$. □

Proposition 4 $(MI_1)_{\mathsf{Hyp}_G(2)}$ *is a maximal idempotent submonoid of* $\mathsf{Hyp}_G(2)$.

Proof. Let K be a proper idempotent submonoid of $\mathsf{Hyp}_G(2)$ such that $(MI_1)_{\mathsf{Hyp}_G(2)} \subseteq K \subset \mathsf{Hyp}_G(2)$. Let $\sigma_t \in K$. Then σ_t is an idempotent element. If $\sigma_t \in R_2$. Then $t = f(t', x_2)$ where $x_1 \notin var(t')$. Choose $\sigma_s \in R_1$ such that $s = f(x_1, s')$ where $x_2 \notin var(s')$, $s' \in W_{(2)}(X) \setminus X$ and $rightmost(s') = x_1$. Consider $(\sigma_t \circ_G \sigma_s)(f) = \widehat{\sigma}_t[f(x_1, s')] = S^2(f(t', x_2), x_1, \widehat{\sigma}_t[s']) = f(S^2(t', x_1, w), S^2(x_2, x_1, w))$, where $w = \widehat{\sigma}_t[s']$. Since $x_2 \in var(t)$, we have $x_1 \in var(w)$ and $S^2(t', x_1, w) \in W_{(2)}(X) \setminus X$. Since $x_1 \in var(w)$, $\sigma_t \circ_G \sigma_s$ is not idempotent, so $\sigma_t \in (MI_1)_{\mathsf{Hyp}_G(2)}$. Therefore $K = (MI_1)_{\mathsf{Hyp}_G(2)}$. □

Proposition 5 $(MI_2)_{\mathsf{Hyp}_G(2)}$ *is a maximal idempotent submonoid of* $\mathsf{Hyp}_G(2)$.

Proof. The proof is similar to the proof of Proposition 4. □

Corollary 2 $\{(MI)_{\mathsf{Hyp}_G(2)}, (MI_1)_{\mathsf{Hyp}_G(2)}, (MI_2)_{\mathsf{Hyp}_G(2)}\}$ *is the set of all maximal idempotent submonoids of* $\mathsf{Hyp}_G(2)$.

Proposition 6 ([6]) *Let* σ_t *be an idempotent element. Then* $\sigma_{x_1} \leq \sigma_t$ *if and only if* $leftmost(t) = x_1$.

Proposition 7 ([6]) *Let* σ_t *be an idempotent element. Then* $\sigma_{x_2} \leq \sigma_t$ *if and only if* $rightmost(t) = x_2$.

Proposition 8 *For each* $t \in W_{(2)}(X)$ *where* $x_2 \notin var(t)$, $\{\sigma_{x_1}, \sigma_{id}, \sigma_{f(x_1,t)}\}$ *is a maximal compatible idempotent submonoid of* $\mathsf{Hyp}_G(2)$.

Proof. By using Proposition 6, $\sigma_{x_1} \leq \sigma_{f(x_1,t)}$. Then $\sigma_{x_1} = \sigma_{x_1} \circ_G \sigma_{f(x_1,t)} = \sigma_{f(x_1,t)} \circ_G \sigma_{x_1}$ and σ_{id} is the identity element. We have $\{\sigma_{x_1}, \sigma_{id}, \sigma_{f(x_1,t)}\}$ is an idempotent submonoid of $\mathsf{Hyp}_G(2)$. Since

$$\sigma_{f(x_1,t)} \circ_G \sigma_{x_1} = \sigma_{x_1} \circ_G \sigma_{f(x_1,t)} = \sigma_{x_1} \circ_G \sigma_{x_1} = \sigma_{x_1} \leq \sigma_{f(x_1,t)} = \sigma_{f(x_1,t)} \circ_G \sigma_{f(x_1,t)}.$$

We have $\{\sigma_{x_1}, \sigma_{id}, \sigma_{f(x_1,t)}\}$ is a compatible idempotent submonoid of $\mathsf{Hyp}_G(2)$.

Let K be a proper compatible idempotent submonoid of $\mathsf{Hyp}_G(2)$ such that $\{\sigma_{x_1}, \sigma_{id}, \sigma_{f(x_1,t)}\} \subseteq K \subset \mathsf{Hyp}_G(2)$. Let $\sigma_s \in K$. Then σ_s is an idempotent element.

Case 1: $\sigma_s \in R_1 \setminus \{\sigma_{x_1}, \sigma_{id}, \sigma_{f(x_1,t)}\}$. Then $s = f(x_1, s')$ where $x_2 \notin var(s')$. Since K is a compatible idempotent submonoid and $\sigma_{f(x_1,t)} \leq \sigma_{id}$, we have $\sigma_{f(x_1,t)} \circ_G \sigma_{f(x_1,s')} = \sigma_{f(x_1,t)} \leq \sigma_{f(x_1,s')} = \sigma_{id} \circ_G \sigma_{f(x_1,s')}$ which is a contradiction.

Case 2: $\sigma_s \in R_2$. Then $s = f(s', x_2)$ where $x_1 \notin var(s')$. Since K is a compatible idempotent submonoid and $\sigma_{f(s',x_2)} \leq \sigma_{id}$, we have $\sigma_{x_1} \circ_G \sigma_{f(s',x_2)} = \sigma_{leftmost(s')} \leq \sigma_{x_1} = \sigma_{x_1} \circ_G \sigma_{f(x_1,s')}$. So $leftmost(s') = x_1$ which is a contradiction.

Case 3: $\sigma_s = \sigma_{x_2}$. Since K is a compatible idempotent submonoid and $\sigma_{x_1} \leq \sigma_{id}$, we have $\sigma_{x_2} \circ_G \sigma_{x_1} = \sigma_{x_1} \leq \sigma_{x_2} = \sigma_{x_2} \circ_G \sigma_{id}$ which is a contradiction.

Case 4: $\sigma_s \in R_4$. Then $s = f(s_1, s_2) \in W_{(2)}X \setminus X$ where $x_1, x_2 \notin var(s)$. Since K is a compatible idempotent submonoid and $\sigma_{x_1} \leq \sigma_{id}$, we have $\sigma_s \circ_G \sigma_{x_1} = \sigma_{x_1} \leq \sigma_s = \sigma_s \circ_G \sigma_{id}$ which is a contradiction.

Therefore $K = \{\sigma_{x_1}, \sigma_{id}, \sigma_{f(x_1,t)}\}$ is a maximal compatible idempotent submonoid of $\mathsf{Hyp}_G(2)$. □

Proposition 9 *For each* $t \in W_{(2)}X$ *where* $x_1 \notin var(t)$, $\{\sigma_{x_2}, \sigma_{id}, \sigma_{f(t,x_2)}\}$ *is a maximal compatible idempotent submonoid of* $\mathsf{Hyp}_G(2)$.

Proof. The proof is similar to the proof of Proposition 8. □

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