

Analytical and numerical results of fractional differential-difference equations

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Abstract. In this paper, we examine the fractional differential-difference equation (FDDE) by employing the proposed sensitivity approach (SA) and Adomian transformation method (ADTM). In SA the nonlinear differential-difference equation is converted to infinite linear equations which have a wide criterion to solve for the analytical solution. By ADTM, the FDDE is converted into ordinary differential-difference equation that can be solved. We test both the techniques through some test problems which are arising in nonlinear dynamical systems and found that ADTM is equivalently appropriate and simpler method to handle than SA.

1 Introduction

A differential-difference equation (DDEs), first studied by Fermi et al. [1] in the 1950s is of enormous significance in describing physical phenomena of various fields such as mechanical engineering, biophysics, condensed matter physics, and in different physical problems like currents flow in electrical networks [2], particle vibrations in lattices, and pulses in biological chains [3]. Many forms of DDEs are discovered to analyze the discrete nonlinear system.

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In recent decades fractional derivatives are introduced to deal with non differentiable functions. The theory of using integrals and derivatives of an arbitrary order, fractional calculus, discussed about 300 years ago, have applications in fractional control of engineering systems, acoustics, damping laws, bioengineering and biomedical applications, electromagnetism, hydrology, signal processing, and many others [4, 5, 6]. It is used for examining stochastic processes forced by fractional Brownian processes [7, 8], non-random fractional phenomena in physics [9, 10, 11], the study of porous systems, and quantum mechanics [12, 13].

Recently, there have been a number of schemes committed to the solution of fractional differential equations. The Adomian decomposition method [14], homotopy perturbation method [15, 16, 17], homotopy analysis method [18, 19], Taylor matrix method [20] and many others have been used to solve the fractional differential equations.

In the present paper, the sensitivity approach [21, 22, 23] which has been presented to solve various kinds of optimal control problems and analysis of time delay systems has used. In this approach, a sensitivity parameter has introduced which transform the original nonlinear fractional differential-difference equation (FDDE) to a linear sequence of FDDEs. The system of equations then now consists of a linear term and a nonlinear series terms. Iterations have been done only for nonlinear series terms, i.e., the result of a sequence of linear FDDEs leads to nonlinear terms for compensation is extended to solve FDDEs. Also, the Adomian decomposition method (ADM) has been modified by a special transformation. The transformation has converted the fractional order differential-difference equation to ordinary differential-difference equation which then solved by the Adomian decomposition method.

2 Preliminaries

The modified Riemann-Liouville derivative of order α is defined, for a function $f(x)$, by

$$D_x^\alpha f(x) = \frac{1}{\Gamma(-\alpha)} \int_0^x (x-\eta)^{-1-\alpha} (f(\eta) - f(0)) d\eta; \quad \alpha < 0, \quad (1)$$

$$D_x^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-\eta)^{-\alpha} (f(\eta) - f(0)) d\eta; \quad 0 < \alpha < 1, \quad (2)$$

$$D_x^\alpha f(x) = (f^{(n)}(x))^{(\alpha-n)} d\eta; \quad n \leq \alpha \leq n+1, \quad n \geq 1, \quad (3)$$

where $f : \mathbf{R} \rightarrow \mathbf{R}$ is a continuous function. Properties of modified Riemann Liouville derivative are given as

$$D_x^\alpha x^{(\beta)} = \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \alpha + 1)} x^{\beta - \alpha}; \quad \beta > 0, \quad (4)$$

$$d^\alpha x(t) = \Gamma(\alpha + 1) dx(t); \quad \beta > 0, \quad (5)$$

3 Implementation of the methods

Consider the nonlinear FDDEs in the form of:

$$D_t^\alpha U_n(t) = \mathcal{N}(\dots, U_{n-1}(t), U_n(t), U_{n+1}(t), \dots); \quad (6)$$

with initial conditions

$$U_n(t_0) = f(n); \quad (7)$$

where $\mathcal{N}$ is the non linear function, $U_n(t)$ is an unknown function, $f(n)$ is the initial condition and t, n are independent variables. Assuming that a unique solution exist for Eq. (4). It is difficult to obtain exact solution of nonlinear FDDE (6). Only in some cases, we can obtain exact solution.

Sensitivity approach (SA)

In this approach, a sensitivity parameter Λ , which varies between zero and unity, is introduced into nonlinear terms of FDDE. When $\Lambda = 0$, the nonlinear problem is transformed to a simple problem, which can be solved through analytic method. When $\Lambda = 1$, the original nonlinear problem is obtained. This transformation leads to solving a sequence of linear FDDEs instead of solving nonlinear FDDEs. Now, we embedded a sensitivity parameter Λ in Eqs. (6)–(7) and form the following sensitized FDDEs:

$$D_t^\alpha U_n(t, \Lambda) = \mathcal{N}(\dots, U_{n-1}(t, \Lambda), U_n(t, \Lambda), U_{n+1}(t, \Lambda), \dots) \quad (8)$$

with initial conditions

$$U_n(t, \Lambda) \Big|_{t=t_0} = f(n) \quad (9)$$

Where $0 \leq \Lambda \leq 1$. In the following explanation, we suppose that the solution of Eq. (6) is distinctively existed and $U_n(t, \Lambda)$ with Λ is infinitely differentiable

with respect to the Λ in the region of $\Lambda = 0$, and its Maclaurin series is convergent at $\Lambda = 1$. Apparently when $\Lambda = 1$ Eq. (8) is equivalent to the original problem Eq. (6). According to the assumption we can write:

$$u_n(t, \Lambda) = \sum_{j=1}^{\infty} \Lambda^j u_n^{(j)}(t) \quad (10)$$

where $(*)^{(j)} = \frac{1}{j!} \frac{\partial^j}{\partial \Lambda^j} (*) \Big|_{\Lambda=0}$. Now, we substitute Eq. (10) into Eq. (8) and equating terms with the same order of Λ on each side we have:

$$\Lambda^0 : D_t^\alpha u_n^{(0)}(t) = \mathcal{N}(u_n^{(0)}(t)), \quad u_n^{(0)}(t_0) = f(n), \quad (11)$$

$$\Lambda^1 : D_t^\alpha u_n^{(1)}(t) = \mathcal{N}(u_n^{(1)}(t)) + g_n^{(0)}(t), \quad u_n^{(1)}(t_0) = 0, \quad (12)$$

$$\Lambda^j : D_t^\alpha u_n^{(j)}(t) = \mathcal{N}(u_n^{(j)}(t)) + g_n^{(j-1)}(t), \quad u_n^{(j)}(t_0) = 0, \quad (13)$$

Where, $g^{(j-1)}(t)$ is the coefficient of $\Lambda^{(j-1)}$ in the expanding of $f(n)$ and can be resolve in the following manner:

$$g^{(j-1)}(t) = \frac{1}{(j-1)!} \frac{\mathcal{N}(..., u_{n-1}(t, \Lambda), u_n(t, \Lambda), u_{n+1}(t, \Lambda), \dots)}{\partial \Lambda^{(j-1)}} \Big|_{\Lambda=0}, \quad (14)$$

It should be noticed that Eq. (11) gives linear approximate and Eq. (6) gives correction term to linear approximate solution by keeping in consideration second order nonlinearity and so on. If the above process carries on, at each step, a system of non-homogeneous linear FDDEs is obtained in which non-homogeneous terms are known from the previous step. Hence, solving the presented sequence is a recursive process. After indentifying $U^j(t)$ for $j \geq 1$, it is obvious that $\Lambda = 1$ should be set in Eq. (8) and Eq. (9) so that they deform to the exact solution of Eq. (6) and so we have:

$$u_n(t, 1) = \sum_{j=0}^{\infty} u_n^{(j)}(t) \quad (15)$$

In this way, the original nonlinear FDDE has been converted into a sequence of linear FDDEs, which should be solved in a recursive development and this will overcomes the complexity of working with nonlinear FDDEs. It is clear from the above procedure, a nonlinear FDDE is transformed into a sequence

of linear FDDEs. In order to solve the Eq. (8), the following sensitized linear FDDE can be constructed:

$$D_t^\alpha u_n^{(0)}(t, \Lambda) = \mathcal{N}(\dots, u_{n-1}^{(0)}(t, \Lambda), u_n^{(0)}(t, \Lambda), u_{n+1}^{(0)}(t, \Lambda), \dots) \quad (16)$$

$$u_n^{(0)}(t, \Lambda) = f(n) \quad (17)$$

Assuming the solution of Eq. (16) as

$$u_n(t, \Lambda) = \sum_{j=1}^{\infty} \Lambda^j u_n^{(0,j)}(t) \quad (18)$$

Now by substituting Eq. (18) into Eq. (16) and equating terms with the same order of Λ on each side we have:

$$\Lambda^0 : D_t^\alpha u_n^{(0,0)}(t) = \mathcal{N}(\dots, u_{n-1}^{(0,0)}(t, \Lambda), u_n^{(0,0)}(t, \Lambda), u_{n+1}^{(0,0)}(t, \Lambda), \dots), \quad (19)$$

$$u_n^{(0,0)}(t_0) = f(n),$$

$$\Lambda^1 : D_t^\alpha u_n^{(0,1)}(t) = \mathcal{N}(\dots, u_{n-1}^{(0,1)}(t, \Lambda), u_n^{(0,1)}(t, \Lambda), u_{n+1}^{(0,1)}(t, \Lambda), \dots), \quad (20)$$

$$u_n^{(0,1)}(t_0) = 0,$$

$\vdots$

$$\Lambda^j : D_t^\alpha u_n^{(0,j)}(t) = \mathcal{N}(\dots, u_{n-1}^{(0,j)}(t, \Lambda), u_n^{(0,j)}(t, \Lambda), u_{n+1}^{(0,j)}(t, \Lambda), \dots), \quad (21)$$

$$u_n^{(0,j)}(t_0) = 0,$$

$\vdots$

By taking the inverse operator of D_t^α on applying Eqs. (19)–(21), we get the solution of FDDE as

$$u_n^0(t) = u_n^{(0,1)}(t) = \sum_{j=0}^{\infty} u_n^{(0,j)}(t) \quad (22)$$

In the same way, Eq. (6) has to be solved for $u_n^0(t)$ for $j \geq 1$. After some similar calculation we have:

$$u_n^{j,k}(t) = u_n^{(j,0)}(t) + u_n^{(j,1)}(t) + u_n^{(j,2)}(t) + u_n^{(j,3)}(t) + \dots \quad (23)$$

$$u_n^{(j,0)}(t) = 0,$$

$$\vdots$$

$$u_n^{(j,1)}(t) = -J_1^\alpha g_n^{j-1}(t) \quad (24)$$

$$\vdots$$

$$u_n^{(j,k)}(t) = -J_1^\alpha \mathcal{N}(\dots u_{n-1}^{(j,k)}(t), u_n^{(j,k)}(t), u_{n+1}^{(j,k)}(t) \dots)^j.$$

Since, the steps above are enough to find the analytical solution, however, only some iteration of sub-problems and the original problem are sufficient to get a satisfactory accurate solution. We can replace ∞ by any positive integers S and T in the above mentioned series which may help in obtaining an approximate closed-form solution

$$u_n(t) = \sum_{j=0}^S \sum_{k=0}^T u_n^{(j,k)}(t) \quad (25)$$

Adomian decomposition transformation method (ADTM)

Fractional complex transforms [24, 25] has now been become a useful tool to convert fractional differential equations to ordinary differential equations, which provides a very simple and easy solution approach. In the present method, the FDDE has transformed into ordinary DDE and then utilizing the Adomian decomposition method to solve for exact or analytic solutions. Duan [26, 27] has provided the efficient recurrence one variable formula to decompose the multivariable Adomian polynomials to solve the non-linear differential equation. Recalling the Eq. (6)

$$D_t^\alpha u_n(t) = \mathcal{N}(\dots, u_{n-1}(t), u_n(t), u_{n+1}(t), \dots) \quad (26)$$

with initial conditions

$$u_n(t_0) = f(n); \quad (27)$$

Let us suppose

$$T = \frac{t^\alpha}{\Gamma(\alpha + 1)} \quad (28)$$

Differentiating Eq. (28) and making use of modified Riemann Liouville derivative

$$\frac{d^\alpha u_n(t)}{dt^\alpha} = u'_n(T) \quad (29)$$

Putting in Eq. (26), which transform the FDDE into ordinary DDE

$$u'_{n,m}(T) = \mathcal{L}u_{n,m}(T) = \mathcal{N}(\dots, u_{n-1,m}(T), u_{n,m}(T), u_{n+1,m}(T), \dots) \quad (30)$$

where $\mathcal{L} = \frac{d}{dT}$ and $\mathcal{L}^{-1} = \int_{T_0}^T (*)dT$ are the linear operator the inverse operators respectively. Applying the inverse operator on both sides of the Eq. (26) with Eq. (27) gives

$$u_{n,m+1}(T) = f(n) + \mathcal{L}^{-1}[\mathcal{N}(\dots, u_{n-1,m}(T), u_{n,m}(T), u_{n+1,m}(T), \dots)] \quad (31)$$

In this section, three examples have presented to demonstrate the applicability of the suggested methods to solve nonlinear fractional differential-difference equations.

4 Test problems

Problem 1

Consider the following Volterra equation

$$D_t^\alpha u_n(t) = u_n(t)(u_{n+1}(t) - u_{n-1}(t)) \quad (32)$$

with initial conditions

$$u_n(0) = n \quad (33)$$

The exact solution for $\alpha = 1$ can be written as

$$u_n(t) = \frac{n}{(1-2t)}$$

For solving this equation, the following new equation is constructed with sensitivity parameter:

$$D_t^\alpha u_n(t, \Lambda) = \Lambda u_n(t, \Lambda)(u_{n+1}(t, \Lambda) - u_{n-1}(t, \Lambda)) \quad (34)$$

with initial conditions

$$u_n(0, \Lambda) = n \quad (35)$$

Now assume:

$$u_n(t, \lambda) = \sum_{j=0}^{\infty} \lambda^j u_n^{(0,j)}(t) \quad (36)$$

$$u_{n+1}(t, \lambda) = \sum_{j=0}^{\infty} \lambda^j u_{n+1}^{(0,j)}(t) \quad (37)$$

$$u_{n-1}(t, \lambda) = \sum_{j=0}^{\infty} \lambda^j u_{n-1}^{(0,j)}(t) \quad (38)$$

Substitute Eq. (36)–(38) in Eq. (34), it has been seen that the nonlinear original FDDEs are changed into a set of linear recursive FDDEs by using Eq. (19)–(21), in which at each step, the non-homogeneous terms are calculated from the preceding steps and process can be handled very simply which can solved the equation, we get the following series solution.

$$u_n(t) = n + \frac{2nt^\alpha}{\Gamma(\alpha+1)} + \frac{8nt^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{8nt^{3\alpha}(4\Gamma(\alpha+1)^2 + \Gamma(2\alpha+1))}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)} + \dots \quad (39)$$

By putting $\alpha = 1$, we recover ref. [24].

Now for the Adomian decomposition transformation method, Let us suppose

$$\eta = \frac{1}{\Gamma(\alpha+1)} t^\alpha \quad (40)$$

From modified Riemann Liouville derivative, we have

$$\frac{d^\alpha u_n(t)}{dt^\alpha} = u'_n(\eta) \quad (41)$$

Using Eq. (41) in Eq. (32)

$$u'_{n,m} = u_{n,m}(\eta)[u_{n+1,m}(\eta) - u_{n-1,m}(\eta)]$$

$$L u_{n,m} = A(u_{n,m}, u_{n+1,m}) - B(u_{n,m}, u_{n-1,m}) \quad (42)$$

Operating L^{-1} on both sides gives:

$$u_{n,m+1} = u_n(0) + L^{-1}(A(u_{n,m}, u_{n+1,m}) - B(u_{n,m}, u_{n-1,m})) \quad (43)$$

Where the nonlinear terms $A(U_{n,m}, U_{n+1,m})$ and $B(U_{n,m}, U_{n-1,m})$ can be decomposed as follows:

$$A_{n,m} = \sum_{i=0}^{m-1} U_{n,i} U_{n+1,m-1-i} \quad B_{n,m} = \sum_{i=0}^{m-1} U_{n,i} U_{n-1,m-1-i} \quad (44)$$

$$A_{n,1} = U_{n,0} U_{n+1,0}, \quad B_{n,1} = U_{n,0} U_{n-1,0},$$

$$A_{n,2} = U_{n,1} U_{n+1,0} + U_{n,0} U_{n+1,1}, \quad B_{n,2} = U_{n,1} U_{n-1,0} + U_{n,0} U_{n-1,1},$$

$$A_{n,3} = U_{n,2} U_{n+1,0} + U_{n,1} U_{n+1,1} + U_{n,0} U_{n+1,2},$$

$$B_{n,3} = U_{n,2} U_{n-1,0} + U_{n,1} U_{n-1,1} + U_{n,0} U_{n-1,2},$$

$$A_{n,4} = U_{n,3} U_{n+1,0} + U_{n,2} U_{n+1,1} + U_{n,1} U_{n+1,2} + U_{n,0} U_{n+1,3}$$

$$B_{n,4} = U_{n,3} U_{n-1,0} + U_{n,2} U_{n-1,1} + U_{n,1} U_{n-1,2} + U_{n,0} U_{n-1,3} \quad (45)$$

The solution of the transformed problem is

$$U_n = 2n\eta + 4n\eta^2 + 8n\eta^3 + 16n\eta^4 + 32n\eta^5 + 64n\eta^6 + \dots \quad (46)$$

Now, replacing

$$\eta = \frac{1}{\Gamma(\alpha + 1)} t^\alpha$$

in Eq.(46), we get

$$\begin{aligned} U_n = & 2n \frac{t^\alpha}{\Gamma(\alpha + 1)} + 4n \left(\frac{t^\alpha}{\Gamma(\alpha + 1)} \right)^2 + 8n \left(\frac{t^\alpha}{\Gamma(\alpha + 1)} \right)^3 \\ & + 16n \left(\frac{t^\alpha}{\Gamma(\alpha + 1)} \right)^4 + \dots \end{aligned} \quad (47)$$

Table 1: Numerical comparison of problem 1 at $t = 0.01$

n	$\alpha = 0.5$		$\alpha = 0.75$		$\alpha = 1$		
	SA	ADTM	SA	ADTM	SA	ADTM	Exact
-20	-27.1982	-25.8288	-21.5089	-21.4780	-20.4082	-20.4082	-20.4082
-10	-13.5991	-12.9144	-10.7544	-10.7390	-10.2041	-10.2041	-10.2041
10	13.5991	12.9144	10.7544	10.7390	10.2041	10.2041	10.2041
20	27.1982	25.8288	21.5089	21.4780	20.4082	20.4082	20.4082

Problem 2

Let us consider hybrid nonlinear difference equation of the Korteweg-de Vries (KdV) equations

$$D_t^\alpha U_n(t) = (1 - (U_n(t))^2)(U_{n+1}(t) - U_{n-1}(t)) \quad (48)$$

with initial conditions

$$U_n(0) = \tanh(k) \tanh(kn) \quad (49)$$

The exact solution of Eq. (48) for can be written as:

$$U_n(t) = \tanh(k) \tanh(kn + 2 \tanh(k)t) \quad (50)$$

Eq. (48) can be simplified as follows:

$$D_t^\alpha U_n(t) = (U_{n+1}(t) - U_{n-1}(t)) - (U_n(t))^2 U_{n+1}(t) + (U_n(t))^2 U_{n-1}(t) \quad (51)$$

The solution of Eq. (48) by SA is given by

$$U_n(t) = \tanh(k) \tanh(kn) + \frac{1}{\Gamma(\alpha + 1)} t^\alpha (\cosh(2k) + \cosh(2kn)) \left(\frac{1}{\cosh(kn - k)} \right)$$

$$\left(\frac{1}{(\cosh(kn))^2} \right) \left(\frac{1}{\cosh(kn + k)} \right) (\tanh(k))^2 + \dots \quad (52)$$

Now, applying the Adomian transformation method the solution of Eq. (48) with Eq. (49) is

$$U_n = \tanh(k) \tanh(kn) + \eta (\cosh(2k) + \cosh(2kn)) \left(\frac{1}{\cosh(kn - k)} \right) \left(\frac{1}{(\cosh(kn))^2} \right)$$

$$\left(\frac{1}{\cosh(kn+k)}\right)(\tanh(k))^2 \dots \quad (53)$$

Hence, the solution of the original problem is given by

$$\begin{aligned} U_n = & \tanh(k) \tanh(kn) + \frac{1}{\Gamma(\alpha+1)} t^\alpha (\cosh(2k) + \cosh(2kn)) \\ & \left(\frac{1}{\cosh(kn-k)}\right) \left(\frac{1}{(\cosh(kn))^2}\right) \\ & \left(\frac{1}{\cosh(kn+k)}\right) (\tanh(k))^2 \dots \end{aligned} \quad (54)$$

Table 2: Numerical comparison of problem 2 for $k = 0.1$ $t = 0.01$

n	$\alpha = 0.5$		$\alpha = 0.75$		$\alpha = 1$		
	SA	ADTM	SA	ADTM	SA	ADTM	Exact
-20	-0.291307	-0.291307	-0.291308	-0.291308	-0.291309	-0.291309	-0.291309
-10	-0.289080	-0.289134	-0.289502	-0.289509	-0.289695	-0.289694	-0.289694
0	0.060562	0.059690	0.032840	0.032701	0.016973	0.0169534	0.0169534
10	0.290275	0.290363	0.290150	0.290162	0.290030	0.290030	0.290030
20	0.291310	0.291310	0.291310	0.29131	0.291309	0.291309	0.291309

Problem 3

Consider the following fractional differential-difference problem

$$D_t^\alpha U_n(t) = (U_n(t))^2 (U_{n+1}(t) - U_{n-1}(t)) \quad (55)$$

with initial condition

$$U_n(0) = 1 - \frac{1}{n^2} \quad (56)$$

The exact solution of Eq. (55) for $\alpha = 1$ can be written as:

$$U_n(t) = 1 - \frac{1}{(n+2t)^2} \quad (57)$$

By the sensitivity approach and ADTM the solution of the problem are given by

$$U_n(t) = 1 - \frac{1}{n^2} + \frac{4t^\alpha}{n^3\Gamma(\alpha+1)} - \frac{24t^{2\alpha}}{n^4\Gamma(2\alpha+1)} + \dots \quad (58)$$

$$U_n(t) = 1 - \frac{1}{n^2} + \frac{4t^\alpha}{n^3\Gamma(\alpha+1)} - \frac{12t^{2\alpha}}{n^4(\Gamma(\alpha+1))^2} + \dots \quad (59)$$

Table 3: Numerical comparison of problem 3 at $t = 0.3$

n	$\alpha = 0.5$		$\alpha = 0.75$		$\alpha = 1$		
	SA	ADTM	SA	ADTM	SA	ADTM	Exact
-20	0.997137	0.99716	0.997259	0.997264	0.997343	0.997343	0.997343
-10	0.986425	0.98698	0.987878	0.987971	0.988683	0.988683	0.988683
10	0.991922	0.992079	0.991510	0.991555	0.988683	0.991100	0.991100
20	0.99777	0.997783	0.997707	0.997707	0.997644	0.997644	0.997644

5 Conclusions

In this study, we extend the sensitivity approach for solving the fractional differential difference equation and proposed a new Adomian decomposition transformation method, and obtain the analytical solution of Volterra and mKDV lattice equations. The solution shows that, both techniques are quite useful for solving a variety of linear and nonlinear fractional problems, but ADTM provides an easy and reliable scheme to be implemented on various problems. The comparison has also been done which is almost approximate to the exact solution. Numerical examples show that the suggested scheme is clearly quite efficient and potent technique in finding the solutions of the proposed equations (see table 1-3).

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