

# Focal representation of $k$ -slant Helices in $\mathbb{E}^{m+1}$

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**Abstract.** The focal representation of a generic regular curve  $\gamma$  in  $\mathbb{E}^{m+1}$  consists of the centers of the osculating hyperplanes. A  $k$ -slant helix  $\gamma$  in  $\mathbb{E}^{m+1}$  is a (generic) regular curve whose unit normal vector  $V_k$  makes a constant angle with a fixed direction  $\vec{U}$  in  $\mathbb{E}^{m+1}$ . In the present paper we proved that if  $\gamma$  is a  $k$ -slant helix in  $\mathbb{E}^{m+1}$ , then the focal representation  $C_\gamma$  of  $\gamma$  in  $\mathbb{E}^{m+1}$  is an  $(m - k + 2)$ -slant helix in  $\mathbb{E}^{m+1}$ .

## 1 Introduction

Curves with constant slope, or so-called general helices (inclined curves), are well-known curves in the classical differential geometry of space curves. They are defined by the property that the tangent makes a constant angle with a fixed line (the axis of the general helix) (see, [1], [4], [7] and [8]). In [10], the definition is more restrictive: the fixed direction makes constant angle with these all the vectors of the Frenet frame. It is easy to check that the definition

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**2010 Mathematics Subject Classification:** 53A04, 53C42

**Key words and phrases:** Frenet curve, focal curve, slant helix

only works in the odd dimensional case. Moreover, in the same reference, it is proven that the definition is equivalent to the fact the ratios  $\frac{\kappa_2}{\kappa_1}, \frac{\kappa_4}{\kappa_3}, \dots$ ,  $\kappa_i$  being the curvatures, are constant. Further, J. Monterde has considered the Frenet curves in  $\mathbb{E}^m$  which have constant curvature ratios (i.e.,  $\frac{\kappa_2}{\kappa_1}, \frac{\kappa_3}{\kappa_2}, \frac{\kappa_4}{\kappa_3}, \dots$  are constant) [14]. The Frenet curves with constant curvature ratios are called ccr-curves. Obviously, ccr-curves are a subset of generalized helices in the sense of [10]. It is well known that *curves with constant curvatures* (W-curves) are well-known ccr-curves [12], [15].

Recently, Izumiya and Takeuchi have introduced the concept of slant helix in Euclidean 3-space  $\mathbb{E}^3$  by requiring that the normal lines make a constant angle with a fixed direction [11]. Further in [3] Ali and Turgut considered the generalization of the concept of slant helix to Euclidean  $n$ -space  $\mathbb{E}^n$ , and gave some characterizations for a non-degenerate slant helix. As a future work they remarked that it is possible to define a slant helix of type- $k$  as a curve whose unit normal vector  $V_k$  makes a constant angle with a fixed direction  $\vec{U}$  [9].

For a smooth curve (a source of light)  $\gamma$  in  $\mathbb{E}^{m+1}$ , the caustic of  $\gamma$  (defined as the envelope of the normal lines of  $\gamma$ ) is a singular and stratified hypersurface. The focal curve of  $\gamma$ ,  $C_\gamma$ , is defined as the singular stratum of dimension 1 of the caustic and it consists of the centers of the osculating hyperspheres of  $\gamma$ . Since the center of any hypersphere tangent to  $\gamma$  at a point lies on the normal plane to  $\gamma$  at that point, the focal curve of  $\gamma$  may be parametrized using the Frenet frame  $(t, n_1, n_2, \dots, n_m)$  of  $\gamma$  as follows:

$$C_\gamma(\theta) = (\gamma + c_1 n_1 + c_2 n_2 + \dots + c_m n_m)(\theta),$$

where the coefficients  $c_1, \dots, c_m$  are smooth functions that are called focal curvatures of  $\gamma$  [18].

This paper is organized as follows: Section 2 gives some basic concepts of the Frenet curves in  $\mathbb{E}^{m+1}$ . Section 3 tells about the focal representation of a generic curve given with a regular parametrization in  $\mathbb{E}^{m+1}$ . Further this section provides some basic properties of focal curves in  $\mathbb{E}^{m+1}$  and the structure of their curvatures. In the final section we consider  $k$ -slant helices in  $\mathbb{E}^{m+1}$ . We prove that if  $\gamma$  is a  $k$ -slant helix in  $\mathbb{E}^{m+1}$  then the focal representation  $C_\gamma$  of  $\gamma$  is an  $(m - k + 2)$ -slant helix in  $\mathbb{E}^{m+1}$ .

## 2 Basic concepts

Let  $\gamma = \gamma(s) : I \rightarrow \mathbb{E}^{m+1}$  be a regular curve in  $\mathbb{E}^{m+1}$ , (i.e.,  $\|\gamma'(s)\|$  is nowhere zero) where  $I$  is an interval in  $\mathbb{R}$ . Then  $\gamma$  is called a *Frenet curve of osculating*

order  $d$ , ( $2 \leq d \leq m+1$ ) if  $\gamma'(s), \gamma''(s), \dots, \gamma^{(d)}(s)$  are linearly independent and  $\gamma'(s), \gamma''(s), \dots, \gamma^{(d+1)}(s)$  are no longer linearly independent for all  $s$  in  $I$  [18]. In this case,  $\text{Im}(\gamma)$  lies in a  $d$ -dimensional Euclidean subspace of  $\mathbb{E}^{m+1}$ . To each Frenet curve of rank  $d$  there can be associated the orthonormal  $d$ -frame  $\{t, n_1, \dots, n_{d-1}\}$  along  $\gamma$ , the Frenet  $r$ -frame, and  $d-1$  functions  $\kappa_1, \kappa_2, \dots, \kappa_{d-1}: I \rightarrow \mathbb{R}$ , the Frenet curvatures, such that

$$\begin{bmatrix} t' \\ n_1' \\ n_2' \\ \dots \\ n_{d-1}' \end{bmatrix} = v \begin{bmatrix} 0 & \kappa_1 & 0 & \dots & 0 \\ -\kappa_1 & 0 & \kappa_2 & \dots & 0 \\ 0 & -\kappa_2 & 0 & \dots & 0 \\ \dots & & & & \kappa_{d-1} \\ 0 & 0 & \dots & -\kappa_{d-1} & 0 \end{bmatrix} \begin{bmatrix} t \\ n_1 \\ n_2 \\ \dots \\ n_{d-1} \end{bmatrix} \quad (1)$$

where,  $v$  is the speed of  $\gamma$ . In fact, to obtain  $t, n_1, \dots, n_{d-1}$  it is sufficient to apply the Gram-Schmidt orthonormalization process to  $\gamma'(s), \gamma''(s), \dots, \gamma^{(d)}(s)$ . Moreover, the functions  $\kappa_1, \kappa_2, \dots, \kappa_{d-1}$  are easily obtained as by-product during this calculation. More precisely,  $t, n_1, \dots, n_{d-1}$  and  $\kappa_1, \kappa_2, \dots, \kappa_{d-1}$  are determined by the following formulas:

$$\begin{aligned} v_1(s) &: = \gamma'(s) \quad ; t := \frac{v_1(s)}{\|v_1(s)\|}, \\ v_\alpha(s) &: = \gamma^{(\alpha)}(s) - \sum_{i=1}^{\alpha-1} \langle \gamma^{(\alpha)}(s), v_i(s) \rangle \frac{v_i(s)}{\|v_i(s)\|^2}, \\ \kappa_{\alpha-1}(s) &: = \frac{\|v_\alpha(s)\|}{\|v_{\alpha-1}(s)\| \|v_1(s)\|}, \\ n_{\alpha-1} &: = \frac{v_\alpha(s)}{\|v_\alpha(s)\|}, \end{aligned} \quad (2)$$

where  $\alpha \in \{2, 3, \dots, d\}$  (see, [8]).

A Frenet curve of rank  $d$  for which  $\kappa_1, \kappa_2, \dots, \kappa_{d-1}$  are constant is called (generalized) screw line or helix [6]. Since these curves are trajectories of the 1-parameter group of the Euclidean transformations, so, F. Klein and S. Lie called them *W-curves* [12]. For more details see also [5].  $\gamma$  is said to have constant curvature ratios (that is to say, it is a ccr-curve) if all the quotients  $\frac{\kappa_2}{\kappa_1}, \frac{\kappa_3}{\kappa_2}, \frac{\kappa_4}{\kappa_3}, \dots, \frac{\kappa_i}{\kappa_{i-1}}$  ( $1 \leq i \leq m-1$ ) are constant [14], [15].

### 3 The focal representation of a curve in $\mathbb{E}^{m+1}$

The hyperplane normal to  $\gamma$  at a point is the union of all lines normal to  $\gamma$  at that point. The envelope of all hyperplanes normal to  $\gamma$  is thus a component

of the focal set that we call the main component (the other component is the curve  $\gamma$  itself, but we will not consider it) [16].

**Definition 1** *Given a generic curve (i.e., a Frenet curve of osculating order  $m+1$ )  $\gamma: \mathbb{R} \rightarrow \mathbb{E}^{m+1}$ , let  $F: \mathbb{E}^{m+1} \times \mathbb{R} \rightarrow \mathbb{R}$  be the  $(m+1)$ -parameter family of real functions given by*

$$F(q, \theta) = \frac{1}{2} \|q - \gamma(\theta)\|^2. \quad (3)$$

*The caustic of the family  $F$  is given by the set*

$$\left\{ q \in \mathbb{E}^{m+1} : \exists \theta \in \mathbb{R} : F'_q(\theta) = 0 \text{ and } F''_q(\theta) = 0 \right\} \quad (4)$$

[16].

**Proposition 1** [17] *The caustic of the family  $F(q, \theta) = \frac{1}{2} \|q - \gamma(\theta)\|^2$  coincides with the focal set of the curve  $\gamma: \mathbb{R} \rightarrow \mathbb{E}^{m+1}$ .*

**Definition 2** *The center of the osculating hypersphere of  $\gamma$  at a point lies in the hyperplane normal to the  $\gamma$  at that point. So we can write*

$$C_\gamma = \gamma + c_1 n_1 + c_2 n_2 + \cdots + c_m n_m, \quad (5)$$

*which is called the focal curve of  $\gamma$ , where  $c_1, c_2, \dots, c_m$  are smooth functions of the parameter of the curve  $\gamma$ . We call the function  $c_i$  the  $i^{\text{th}}$  focal curvature of  $\gamma$ . Moreover, the function  $c_1$  never vanishes and  $c_1 = \frac{1}{\kappa_1}$  [18].*

The focal curvatures of  $\gamma$ , parametrized by arc length  $s$ , satisfy the following “scalar Frenet equations” for  $c_m \neq 0$ :

$$\begin{aligned} 1 &= \kappa_1 c_1 \\ c_1 &= \kappa_2 c_2 \\ c_2 &= -\kappa_2 c_1 + \kappa_3 c_3 \\ &\vdots \\ c_{m-1} &= -\kappa_{m-1} c_{m-2} + \kappa_m c_m \\ c_m - \frac{(R_m^2)'}{2c_m} &= -\kappa_m c_{m-1} \end{aligned} \quad (6)$$

where  $R_m$  is the radius of the osculating  $m$ -sphere. In particular  $R_m^2 = \|C_\gamma - \gamma\|^2$  [18].

**Theorem 1** [16] Let  $\gamma : s \rightarrow \gamma(s) \in \mathbb{E}^{m+1}$  be a regular generic curve. Write  $\kappa_1, \kappa_2, \dots, \kappa_m$  for its Euclidean curvatures and  $\{t, n_1, n_2, \dots, n_m\}$  for its Frenet Frame. For each non-vertex  $\gamma(s)$  of  $\gamma$ , write  $\varepsilon(s)$  for the sign of  $(c'_m + c_{m-1}\kappa_m)(s)$  and  $\delta_\alpha(s)$  for the sign of  $(-1)^\alpha \varepsilon(s) \kappa_m(s)$ ,  $\alpha = 1, \dots, m$ . Then the following holds:

a) The Frenet frame  $\{T, N_1, N_2, \dots, N_m\}$  of  $C_\gamma$  at  $C_\gamma(s)$  is well-defined and its vectors are given by  $T = \varepsilon n_m$ ,  $N_\alpha = \delta_\alpha n_{m-1}$ , for  $l = 1, \dots, m-1$ , and  $N_m = \pm t$ . The sign in  $\pm t$  is chosen in order to obtain a positive basis.

b) The Euclidean curvatures  $K_1, K_2, \dots, K_m$  of the parametrized focal curve of  $\gamma$ ,  $C_\gamma : s \rightarrow C_\gamma(s)$ , are related to those of  $\gamma$  by:

$$\frac{K_1}{|\kappa_m|} = \frac{K_2}{\kappa_{m-1}} = \dots = \frac{|K_m|}{\kappa_1} = \frac{1}{|c'_m + c_{m-1}\kappa_m|}, \quad (7)$$

the sign of  $K_m$  is equal to  $\delta_m$  times the sign chosen in  $\pm t$ .

That is the Frenet formulas of  $C_\gamma$  at  $C_\gamma(s)$  are

$$\begin{aligned} T' &= \frac{1}{\Lambda} |\kappa_m| N_1 \\ N'_1 &= \frac{1}{\Lambda} (-|\kappa_m| T + \kappa_{m-1} N_2) \\ N'_2 &= \frac{1}{\Lambda} (-|\kappa_{m-1}| N_1 + \kappa_{m-2} N_3) \\ &\vdots \\ N'_{m-1} &= \frac{1}{\Lambda} (-\kappa_2 N_{m-2} \mp \delta_m \kappa_1 N_m) \\ N'_m &= \frac{1}{\Lambda} \mp \delta_m \kappa_1 N_{m-1} \end{aligned} \quad (8)$$

where  $\Lambda = |c'_m + c_{m-1}\kappa_m|$ .

**Corollary 1** Let  $\gamma = \gamma(s)$  be a regular generic curve in  $\mathbb{E}^{m+1}$  and  $C_\gamma : s \rightarrow C_\gamma(s)$  be the focal representation of  $\gamma$ . Then the Frenet frame of  $C_\gamma$  becomes as follows;

i) If  $m$  is even, then

$$\begin{aligned} T &= n_m \\ N_1 &= -n_{m-1} \\ N_2 &= n_{m-2} \\ &\dots \\ N_{m-1} &= -n_1 \\ N_m &= t \end{aligned} \quad (9)$$

ii) If  $m$  is odd, then

$$\begin{aligned}
 T &= n_m \\
 N_1 &= -n_{m-1} \\
 N_2 &= n_{m-2} \\
 &\dots \\
 N_{m-1} &= n_1 \\
 N_m &= -t.
 \end{aligned} \tag{10}$$

**Proof.** By the use of (7) with (8) we get the result.  $\square$

## 4 $k$ -Slant helices

Let  $\gamma = \gamma(s) : I \rightarrow \mathbb{E}^{m+1}$  be a regular generic curve given with arclength parameter. Further, let  $\vec{U}$  be a unit vector field in  $\mathbb{E}^{m+1}$  such that for each  $s \in I$  the vector  $\vec{U}$  is expressed as the linear combinations of the orthogonal basis  $\{V_1(s), V_2(s), \dots, V_{m+1}(s)\}$  with

$$\vec{U} = \sum_{j=1}^{m+1} a_j(s) V_j(s). \tag{11}$$

where  $a_j(s)$  are differentiable functions,  $1 \leq j \leq m+1$ .

Differentiating  $\vec{U}$  and using the Frenet equations (1), one can get

$$\frac{d\vec{U}}{ds} = \sum_{i=1}^{m+1} P_i(s) V_i(s), \tag{12}$$

where

$$\begin{aligned}
 P_1(s) &= a_1' - \kappa_1 a_2, \\
 P_i(s) &= a_i' + \kappa_{i-1} a_{i-1} - \kappa_i a_{i+1}, \quad 2 \leq i \leq m, \\
 P_{m+1}(s) &= a_{m+1}' + \kappa_m a_m.
 \end{aligned} \tag{13}$$

If the vector field  $\vec{U}$  is constant then the following system of ordinary differential equations are obtained

$$\begin{aligned}
 0 &= a_1' - \kappa_1 a_2, \\
 0 &= a_2' + \kappa_1 a_1 - \kappa_2 a_3, \\
 0 &= a_i' + \kappa_{i-1} a_{i-1} - \kappa_i a_{i+1}, \quad 3 \leq i \leq m, \\
 0 &= a_{m+1}' + \kappa_m a_m.
 \end{aligned} \tag{14}$$

**Definition 3** Recall that a unit speed generic curve  $\gamma = \gamma(s) : I \rightarrow \mathbb{E}^{m+1}$  is called a  $k$ -type slant helix if the vector field  $V_k$  ( $1 \leq k \leq m+1$ ) makes a constant angle  $\theta_k$  with the fixed direction  $\vec{U}$  in  $\mathbb{E}^{m+1}$ , that is

$$\langle \vec{U}, V_k \rangle = \cos \theta_k, \quad \theta_k \neq \frac{\pi}{2}. \quad (15)$$

A 1-type slant helix is known as cylindrical helix [2] or generalized helix [13], [4]. For the characterization of generalized helices in  $(n+2)$ -dimensional Lorentzian space  $\mathbb{L}^{n+2}$  see [19].

We give the following result;

**Theorem 2** Let  $\gamma = \gamma(s)$  be a regular generic curve in  $\mathbb{E}^{m+1}$ . If  $C_\gamma : s \rightarrow C_\gamma(s)$  is the focal representation of  $\gamma$  then the following statements are valid;

i) If  $\gamma$  is a 1-slant helix then the focal representation  $C_\gamma$  of  $\gamma$  is an  $(m+1)$ -slant helix in  $\mathbb{E}^{m+1}$ .

ii) If  $\gamma$  is an  $(m+1)$ -slant helix then the focal representation  $C_\gamma$  of  $\gamma$  is a 1-slant helix in  $\mathbb{E}^{m+1}$ .

iii) If  $\gamma$  is a  $k$ -slant helix ( $2 < k < m$ ) then the focal representation  $C_\gamma$  of  $\gamma$  is an  $(m-k+2)$ -slant helix in  $\mathbb{E}^{m+1}$ .

**Proof.** i) Suppose  $\gamma$  is a 1-slant helix in  $\mathbb{E}^{m+1}$ . Then by Definition 3 the vector field  $V_1$  makes a constant angle  $\theta_1$  with the fixed direction  $\vec{U}$  defined in (11), that is

$$\langle \vec{U}, V_1 \rangle = \cos \theta_1, \quad \theta_1 \neq \frac{\pi}{2}. \quad (16)$$

For the focal representation  $C_\gamma(s)$  of  $\gamma$ , we can choose the orthogonal basis

$$\{V_1(s) = t, V_2(s) = n_1, \dots, V_{m+1}(s) = n_m\}$$

such that the equalities (9) or (10) is hold. Hence, we get,

$$\langle \vec{U}, V_1 \rangle = \langle \vec{U}, t \rangle = \langle \vec{U}, \pm N_m \rangle = \text{cons.} \quad (17)$$

where  $\{T, N_1, N_2, \dots, N_m\}$  is the Frenet frame of  $C_\gamma$  at point  $C_\gamma(s)$ . From the equality (17) it is easy to see that  $C_\gamma$  is an  $(m+1)$ -slant helix of  $\mathbb{E}^{m+1}$ .

ii) Suppose  $\gamma$  is an  $(m+1)$ -slant helix in  $\mathbb{E}^{m+1}$ . Then by Definition 3 the vector field  $V_{m+1}$  makes a constant angle  $\theta_{m+1}$  with the fixed direction  $\vec{U}$  defined in (11), that is

$$\langle \vec{U}, V_{m+1} \rangle = \cos \theta_{m+1}, \quad \theta_{m+1} \neq \frac{\pi}{2}. \quad (18)$$

For the focal representation  $C_\gamma(s)$  of  $\gamma$ , one can get

$$\langle \vec{U}, V_{m+1} \rangle = \langle \vec{U}, n_m \rangle = \langle \vec{U}, T \rangle = \text{cons.} \quad (19)$$

where  $\{V_1 = t, V_2 = n_1, \dots, V_{m+1} = n_m\}$  and  $\{T, N_1, N_2, \dots, N_m\}$  are the Frenet frame of  $\gamma$  and  $C_\gamma$ , respectively. From the equality (19) it is easy to see that  $C_\gamma$  is a 1-slant helix of  $\mathbb{E}^{m+1}$ .

iii) Suppose  $\gamma$  is a  $k$ -slant helix in  $\mathbb{E}^{m+1}$  ( $2 \leq k \leq m$ ). Then by Definition 3 the vector field  $V_k$  makes a constant angle  $\theta_k$  with the fixed direction  $\vec{U}$  defined in (11), that is

$$\langle \vec{U}, V_k \rangle = \cos \theta_k, \quad \theta_k \neq \frac{\pi}{2}, \quad 2 \leq k \leq m. \quad (20)$$

Let  $C_\gamma(s)$  be the focal representation of  $\gamma$ . Then using the equalities (9) or (10) we get

$$\langle \vec{U}, V_k \rangle = \langle \vec{U}, n_{k-1} \rangle = \langle \vec{U}, N_{m-k+1} \rangle = \text{cons.}, \quad 2 \leq k \leq m \quad (21)$$

where

$$\{V_1 = t, V_2 = n_1, \dots, V_{m+1} = n_m\}$$

and

$$\{\tilde{V}_1 = T, \tilde{V}_2 = N_1, \dots, \tilde{V}_{m-k+2} = N_{m-k+1}, \dots, \tilde{V}_{m+1} = N_m\}$$

are the Frenet frame of  $\gamma$  and  $C_\gamma$ , respectively. From the equality (21) it is easy to see that  $C_\gamma$  is an  $(m - k + 2)$ -slant helix of  $\mathbb{E}^{m+1}$ .  $\square$

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*Received: October 22, 2014*