

Unified theory for certain generalized types of closed sets and some separation axioms

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Abstract. In this paper the notion of $\mu\nu$ g-closed sets and certain characterizations of such sets have been given. As an application of $\mu\nu$ g-closed sets, the notion of (μ, ν) -regular spaces and (μ, ν) -normal spaces have been introduced and some characterizations of such spaces are also given.

1 Introduction

For the last one decade or so, a new area of study has emerged and has been rapidly growing. The area is concerned with the investigations of generalized topological spaces and several classes of generalized types of open sets. Recently, a significant contribution to the theory of generalized open sets, was extended by A. Császár [1, 2, 3]. It is observed that a large number of papers are devoted to the study of generalized open sets, containing the class of open sets and possessing properties more or less similar to those of open sets.

We recall some notions defined in [2]. Let X be a non-empty set and $\exp X$ denote the power set of X . We call a class $\mu \subseteq \exp X$ a generalized topology [2], (briefly, GT) if $\emptyset \in \mu$ and union of elements of μ belong to μ . A set X with a GT μ on it is called a generalized topological space (briefly, GTS) and is denoted by (X, μ) . For a GTS (X, μ) , the elements of μ are called μ -open

2010 Mathematics Subject Classification: 54A05, 54D10, 54E55

Key words and phrases: $\mu\nu$ g-closed set, (μ, ν) -regular space, (μ, ν) -normal space

sets and the complements of μ -open sets are called μ -closed sets. For $A \subseteq X$, we denote by $c_\mu(A)$ the intersection of all μ -closed sets containing A , i.e., the smallest μ -closed set containing A ; and by $i_\mu(A)$ the union of all μ -open sets contained in A , i.e., the largest μ -open set contained in A (see [2, 3]).

It is easy to observe that i_μ and c_μ are idempotent and monotonic, where the operator $\gamma : \exp X \rightarrow \exp X$ is said to be idempotent if $A \subseteq X$ implies $\gamma(\gamma(A)) = \gamma(A)$ and monotonic if $A \subseteq B \subseteq X$ implies $\gamma(A) \subseteq \gamma(B)$. It is known from [3, 1] that if μ is a GT on X , $x \in X$ and $A \subseteq X$, then $x \in c_\mu(A)$ iff $x \in M \in \mu \Rightarrow M \cap A \neq \emptyset$. It is also well known from [3, 1] that $x \in i_\mu(A)$ if and only if there exists $U \in \mu$ with $x \in U$ such that $x \in U \subseteq A$ and $c_\mu(X \setminus A) = X \setminus i_\mu(A)$ and $i_\mu(X \setminus A) = X \setminus c_\mu(A)$.

2 $\mu\nu$ g-closed sets and $\mu\nu$ g-open sets

Definition 1 Let μ and ν be two GT's on a set X . Then $A \subseteq X$ is called $\mu\nu$ g-closed [4] if $c_\nu(A) \subseteq U$ whenever $A \subseteq U$ and $U \in \mu$. The complement of a $\mu\nu$ g-closed set is called a $\mu\nu$ g-open set.

Proposition 1 Let μ and ν be two GT's on a set X . Then for $A, B \subseteq X$ the following holds:

- (i) If A is ν -closed then A is $\mu\nu$ g-closed.
- (ii) If A is $\mu\nu$ g-closed and μ -open then A is ν -closed.
- (iii) If A is $\mu\nu$ g-closed and $A \subseteq B \subseteq c_\nu(A)$, then B is $\mu\nu$ g-closed.

Proof. (i) Let A be a ν -closed subset of X and $A \subseteq U \in \mu$. Then $c_\nu(A) = A \subseteq U$ and thus A is $\mu\nu$ g-closed.

(ii) Let A be a $\mu\nu$ g-closed, μ -open subset of X . Then $c_\nu(A) \subseteq A$ and hence A is ν -closed.

(iii) Let $B \subseteq U$ where U is a μ -open set. Then $A \subseteq U$ and hence by $\mu\nu$ g-closedness of A , $c_\nu(A) \subseteq U$. Now $c_\nu(A) \subseteq c_\nu(B) \subseteq c_\nu(c_\nu(A)) = c_\nu(A)$. Hence $c_\nu(A) = c_\nu(B)$. Therefore $c_\nu(B) \subseteq U$ and hence B is $\mu\nu$ g-closed. $\square$

Theorem 1 Let μ and ν be two GT's on X . Then $A \subseteq X$ is $\mu\nu$ g-closed if and only if $c_\nu(A) \cap F = \emptyset$ whenever $A \cap F = \emptyset$ and F is μ -closed.

Proof. Let A be a $\mu\nu$ g-closed subset of X and F be μ -closed with $A \cap F = \emptyset$. Then $A \subseteq X \setminus F$ where $X \setminus F$ is μ -open. Thus $c_\nu(A) \subseteq X \setminus F$. Therefore we have $c_\nu(A) \cap F = \emptyset$.

Conversely, let $A \subseteq U$ and U be μ -open. Then $A \cap (X \setminus U) = \emptyset$ where $X \setminus U$ is μ -closed. Thus by hypothesis, $c_v(A) \cap (X \setminus U) = \emptyset$ and hence $c_v(A) \subseteq U$ showing A to be $\mu\nu g$ -closed. $\square$

Definition 2 [5] Let (X, μ) be a GTS and $A \subseteq X$. Then the subset $\bigwedge_\mu(A)$ is defined by

$$\bigwedge_\mu(A) = \begin{cases} \bigcap \{G : A \subseteq G, G \in \mu\}, & \text{if there exists } G \in \mu \text{ such that } A \subseteq G; \\ X, & \text{otherwise.} \end{cases}$$

Theorem 2 Let μ and ν be two GT's on X . Then $A(\subseteq X)$ is $\mu\nu g$ -closed if and only if $c_v(A) \subseteq \bigwedge_\mu(A)$.

Proof. Suppose that A is $\mu\nu g$ -closed. Let $x \notin \bigwedge_\mu(A)$. Then there exists a μ -open set G with $x \notin G$ and $A \subseteq G$. Then $x \notin c_v(A)$ (as A is $\mu\nu g$ -closed). Thus $c_v(A) \subseteq \bigwedge_\mu(A)$.

Conversely, suppose that $c_v(A) \subseteq \bigwedge_\mu(A)$. Let $A \subseteq U$ where U is μ -open. Then $c_v(A) \subseteq \bigwedge_\mu(A) \subseteq \bigwedge_\mu(U) = U$. Thus A is $\mu\nu g$ -closed. $\square$

Theorem 3 Let μ and ν be two GT's on X . Then $A(\subseteq X)$ is called $\mu\nu g$ -closed if and only if $c_\mu(\{x\}) \cap A \neq \emptyset$ for each $x \in c_v(A)$.

Proof. Suppose that A is $\mu\nu g$ -closed and $c_\mu(\{x\}) \cap A = \emptyset$ for some $x \in c_v(A)$. Then $A \subseteq X \setminus c_\mu(\{x\})$ where $X \setminus c_\mu(\{x\})$ is μ -open. Thus $c_v(A) \subseteq X \setminus c_\mu(\{x\}) \subseteq X \setminus \{x\}$. This contradicts the fact that $x \in c_v(A)$.

Conversely, suppose that A be not $\mu\nu g$ -closed. Then $c_v(A) \setminus U \neq \emptyset$ for some μ -open set U with $A \subseteq U$. Let $x \in c_v(A) \setminus U$. Then $x \in c_v(A)$ and $x \notin U$. Then $c_\mu(\{x\}) \cap U = \emptyset$ and hence $c_\mu(\{x\}) \cap A \subseteq c_\mu(\{x\}) \cap U = \emptyset$. This shows that $c_\mu(\{x\}) \cap A = \emptyset$ for some $x \in c_v(A)$. $\square$

Theorem 4 Let μ and ν be two GT's on X . Then a subset $A(\subseteq X)$ is $\mu\nu g$ -open if and only if $F \subseteq i_v(A)$ whenever $F \subseteq A$ and F is μ -closed.

Proof. Suppose that A is $\mu\nu g$ -open. Let $F \subseteq A$ and F be μ -closed. Then $X \setminus A \subseteq X \setminus F \in \mu$ and $X \setminus A$ is $\mu\nu g$ -closed. Thus $X \setminus i_v(A) = c_v(X \setminus A) \subseteq X \setminus F$ and hence $F \subseteq i_v(A)$.

Conversely, let $X \setminus A \subseteq U$ where U is μ -open. Then $X \setminus U \subseteq A$ and $X \setminus U$ is μ -closed. Thus by the hypothesis, $X \setminus U \subseteq i_v(A)$ and thus $c_v(X \setminus A) = X \setminus i_v(A) \subseteq U$. Hence A is $\mu\nu g$ -open. $\square$

Definition 3 Let μ and ν be two GT's on X . Then μ and ν is said to have the property $(*)$ if $A \in \mu$, $B \in \nu$ implies that $A \cup B \in \mu$.

Theorem 5 Let μ and ν be two GT's on X satisfying the property $(*)$. Then the following are equivalent:

- (1) A is $\mu\nu g$ -closed.
- (2) $c_\nu(A) \setminus A$ does not contain any non-empty μ -closed set.
- (3) $c_\nu(A) \setminus A$ is $\mu\nu g$ -open.

Proof. (1) $\Rightarrow$ (2): Suppose that A is a $\mu\nu g$ -closed set. Let $F \subseteq c_\nu(A) \setminus A$ and F be μ -closed. Then $A \subseteq X \setminus F$ where $X \setminus F$ is μ -open and hence, $c_\nu(A) \subseteq X \setminus F$. Therefore, we have $F \subseteq X \setminus c_\nu(A)$ and hence, $F \subseteq c_\nu(A) \cap (X \setminus c_\nu(A)) = \emptyset$. Thus $F = \emptyset$.

(2) $\Rightarrow$ (3): Let us assume that $F \subseteq c_\nu(A) \setminus A$ and F be μ -closed. By (2), we have $F = \emptyset$ and $F \subseteq i_\nu[c_\nu(A) \setminus A]$. Hence by Theorem 4, $c_\nu(A) \setminus A$ is $\mu\nu g$ -open.

(3) $\Rightarrow$ (1): Suppose that $A \subseteq U$ and U is μ -open. Then, $c_\nu(A) \setminus U \subseteq c_\nu(A) \setminus A$. By (3), $c_\nu(A) \setminus A$ is $\mu\nu g$ -open. Since μ and ν have the property $(*)$, $c_\nu(A) \setminus U$ is μ -closed (as $c_\nu(A)$ is ν -closed and $X \setminus U$ is μ -closed). By Theorem 4, we have $c_\nu(A) \setminus U \subseteq i_\nu(c_\nu(A) \setminus A) = \emptyset$. [In fact if $i_\nu(c_\nu(A) \setminus A) \neq \emptyset$, then there exists some $x \in i_\nu(c_\nu(A) \setminus A)$. Then, there exists $G \in \nu$ such that $x \in G \subseteq c_\nu(A) \setminus A$. Since $G \subseteq X \setminus A$, we have $G \cap A = \emptyset$ and $G \in \nu$. Thus $G \cap c_\nu(A) = \emptyset$ and $G \subseteq X \setminus c_\nu(A)$. Therefore, we obtain $G \subseteq c_\nu(A) \cap (X \setminus c_\nu(A)) = \emptyset$.] Therefore, we have $c_\nu(A) \subseteq U$ and hence A is $\mu\nu g$ -closed. $\square$

Theorem 6 Let μ and ν be two GT's on X satisfying the property $(*)$. A subset A of X is $\mu\nu g$ -open if and only if $G = X$ whenever G is μ -open and $i_\nu(A) \cup (X \setminus A) \subseteq G$.

Proof. Let A be a $\mu\nu g$ -open set and G be μ -open with $i_\nu(A) \cup (X \setminus A) \subseteq G$. Then $X \setminus G \subseteq c_\nu(X \setminus A) \setminus (X \setminus A)$. Since $X \setminus A$ is $\mu\nu g$ -closed and $X \setminus G$ is μ -closed, by Theorem 5, $X \setminus G = \emptyset$ and hence $G = X$.

Conversely let us assume that $F \subseteq A$ and F be μ -closed. Since μ and ν have the property $(*)$, we have $i_\nu(A) \cup (X \setminus A) \subseteq i_\nu(A) \cup (X \setminus F)$ and $i_\nu(A) \cup (X \setminus F)$ is μ -open. Thus by the hypothesis, $X = i_\nu(A) \cup (X \setminus F)$. Hence, $F = F \cap (i_\nu(A) \cup (X \setminus F)) = F \cap i_\nu(A) \subseteq i_\nu(A)$. Thus from Theorem 4 it follows that A is $\mu\nu g$ -open. $\square$

Theorem 7 Let μ and ν be two GT's on X . For any $x \in X$, $\{x\}$ is μ -closed or $\mu\nu g$ -open.

Proof. Suppose that $\{x\}$ is not μ -closed. Then $X \setminus \{x\}$ is not μ -open. Then either there does not exist any μ -open set containing $X \setminus \{x\}$ or the only μ -open set containing $X \setminus \{x\}$ is X itself. Therefore, $c_\nu(X \setminus \{x\}) \subseteq X$ and hence, $X \setminus \{x\}$ is $\mu\nu$ g-closed. Thus $\{x\}$ is $\mu\nu$ g-open. $\square$

3 (μ, ν) -regular space and (μ, ν) -normal space

Definition 4 Let μ and ν be two GT's on X . Then (X, μ, ν) is said to be (μ, ν) -regular if for each μ -closed set F of X not containing x , there exist disjoint ν -open sets U and V such that $x \in U$ and $F \subseteq V$.

Theorem 8 Let μ and ν be two GT's on X . Then the followings are equivalent:

- (i) X is (μ, ν) -regular.
- (ii) For each $x \in X$ and each $U \in \mu$ containing x there exists $V \in \nu$ containing x such that $x \in V \subseteq c_\nu(V) \subseteq U$.
- (iii) For each μ -closed set F of X , $\cap\{c_\nu(V) : F \subseteq V \in \nu\} = F$.
- (iv) For each subset A of X and each $U \in \mu$ with $A \cap U \neq \emptyset$, there exists a $V \in \nu$ such that $A \cap V \neq \emptyset$ and $c_\nu(V) \subseteq U$.
- (v) For each non-empty subset A of X and each μ -closed subset F of X with $A \cap F = \emptyset$, there exist $V, W \in \nu$ such that $A \cap V \neq \emptyset$, $F \subseteq W$ and $W \cap V = \emptyset$.
- (vi) For each μ -closed set F with $x \notin F$ there exists $U \in \mu$ and a $\mu\nu$ g-open set V such that $x \in U$, $F \subseteq V$ and $U \cap V = \emptyset$.
- (vii) For each $A \subseteq X$ and each μ -closed set F with $A \cap F = \emptyset$ there exists $U \in \mu$ and a $\mu\nu$ g-open set V such that $A \cap U \neq \emptyset$, $F \subseteq V$ and $U \cap V = \emptyset$.
- (viii) For each μ -closed set F of X , $F = \cap\{c_\nu(V) : F \subseteq V, V \text{ is } \mu\nu\text{g-open}\}$.

Proof. (i) $\Rightarrow$ (ii): Let U be a μ -open set containing x . Then $x \notin X \setminus U$, where $X \setminus U$ is μ -closed. Then by (i) there exist $G, V \in \nu$ such that $X \setminus U \subseteq G$ and $x \in V$ and $G \cap V = \emptyset$. Thus $V \subseteq X \setminus G$ and so $x \in V \subseteq c_\nu(V) \subseteq X \setminus G \subseteq U$.

(ii) $\Rightarrow$ (iii): Let $X \setminus F \in \mu$ be such that $x \notin F$. Then by (ii) there exists $U \in \nu$ containing x such that $x \in U \subseteq c_\nu(U) \subseteq X \setminus F$. So, $F \subseteq X \setminus c_\nu(U) = V$ (say) $\in \nu$ and $U \cap V = \emptyset$. Thus $x \notin c_\nu(V)$. Thus $F \supseteq \cap\{c_\nu(V) : F \subseteq V \in \nu\}$.

(iii) $\Rightarrow$ (iv): Let $U \in \mu$ with $x \in U \cap A$. Then $x \notin X \setminus U$ and hence by (iii) there exists a ν -open set W such that $X \setminus U \subseteq W$ and $x \notin c_\nu(W)$. We put

$V = X \setminus c_v(W)$, which is a v -open set containing x and hence $A \cap V \neq \emptyset$ (as $x \in A \cap V$). Now $V \subseteq X \setminus W$ and so $c_v(V) \subseteq X \setminus W \subseteq U$.

(iv) $\Rightarrow$ (v): Let F be a μ -closed set as in the hypothesis of (v). Then $X \setminus F$ is a μ -open set and $(X \setminus F) \cap A \neq \emptyset$. Then there exists $V \in \mathcal{v}$ such that $A \cap V \neq \emptyset$ and $c_v(V) \subseteq X \setminus F$. If we put $W = X \setminus c_v(V)$, then $F \subseteq W$ and $W \cap V = \emptyset$.

(v) $\Rightarrow$ (i): Let F be a μ -closed set not containing x . Then by (v), there exist $W, V \in \mathcal{v}$ such that $F \subseteq W$ and $x \in V$ and $W \cap V = \emptyset$.

(i) $\Rightarrow$ (vi): Obvious as every v -open set is $\mu v g$ -open (by Proposition 1).

(vi) $\Rightarrow$ (vii): Let F be a μ -closed set such that $A \cap F = \emptyset$ for any subset A of X . Thus for $a \in A$, $a \notin F$ and hence by (vi), there exists $U \in \mu$ and a $\mu v g$ -open set V such that $a \in U$, $F \subseteq V$ and $U \cap V = \emptyset$ and $A \cap U \neq \emptyset$.

(vii) $\Rightarrow$ (i): Let $x \notin F$, where F be μ -closed. Since $\{x\} \cap F = \emptyset$, by (vii) there exist $U \in \mu$ and a $\mu v g$ -open set W such that $x \in U$, $F \subseteq W$ and $U \cap W = \emptyset$. Now put $V = i_v(W)$. Then $F \subseteq V$ (by Theorem 4) and $U \cap V = \emptyset$.

(iii) $\Rightarrow$ (viii): We have $F \subseteq \cap \{c_v(V) : F \subseteq V \text{ and } V \text{ is } \mu v g\text{-open}\} \subseteq \cap \{c_v(V) : F \subseteq V \text{ and } V \text{ is } v\text{-open}\} = F$.

(viii) $\Rightarrow$ (i): Let F be a μ -closed set in X not containing x . Then by (viii) there exists a $\mu v g$ -open set W such that $F \subseteq W$ and $x \in X \setminus c_v(W)$. Since F is μ -closed and W is $\mu v g$ -open, $F \subseteq i_v(W)$ (by Theorem 4). Take $V = i_v(W)$. Then $F \subseteq V$, $x \in X \setminus c_v(V) = U$ (say) (as $(X \setminus F) \cap V = \emptyset$ and $U \cap V = \emptyset$). $\square$

Definition 5 Let μ and v be two GT's on a set X . Then (X, μ, v) is said to be (μ, v) -normal if for disjoint μ -closed sets F_1 and F_2 , there exist $U_1, U_2 \in v$ such that $F_1 \subseteq U_1, F_2 \subseteq U_2$ with $U_1 \cap U_2 = \emptyset$.

Theorem 9 Let μ and v be two GT's on X . Then the following properties are equivalent:

- (i) (X, μ, v) is (μ, v) -normal;
- (ii) for any two disjoint μ -closed sets F_1, F_2 , there exist $\mu v g$ -open sets V_1, V_2 such that $F_1 \subseteq V_1, F_2 \subseteq V_2$ and $V_1 \cap V_2 = \emptyset$;
- (iii) for any μ -closed set F and any μ -open set U containing F , there exists a $\mu v g$ -open set V such that $F \subseteq V \subseteq c_v(V) \subseteq U$;
- (iv) for any μ -closed set F and any μ -open set U containing F , there exists a v -open set G such that $F \subseteq G \subseteq c_v(G) \subseteq U$;
- (v) for any disjoint μ -closed sets F_1, F_2 , there exists a $\mu v g$ -open set V such that $F_1 \subseteq V$ and $c_v(V) \cap F_2 = \emptyset$;
- (vi) for any disjoint μ -closed sets F_1, F_2 , there exists a v -open set G such that $F_1 \subseteq G$ and $c_v(G) \cap F_2 = \emptyset$.

Proof. (i) $\Rightarrow$ (ii): Follows from (i) as, every ν -open set is $\mu\nu g$ -open.

(ii) $\Rightarrow$ (iii): Let F be a μ -closed set and U be any μ -open set containing F . Then F and $X \setminus U$ are disjoint μ -closed sets and by (ii) there exist $\mu\nu g$ -open sets V_1, V_2 such that $F \subseteq V_1, X \setminus U \subseteq V_2$ with $V_1 \cap V_2 = \emptyset$. Since V_2 is $\mu\nu g$ -open, by Theorem 4, $X \setminus U \subseteq i_\nu(V_2)$. Hence, $c_\nu(V_1) \cap i_\nu(V_2) = \emptyset$. Therefore, we obtain $F \subseteq V_1 \subseteq c_\nu(V_1) \subseteq X \setminus i_\nu(V_2) \subseteq U$. Put $V = V_1$, then we obtain $F \subseteq V \subseteq c_\nu(V) \subseteq U$.

(iii) $\Rightarrow$ (iv): Let F be a μ -closed set and U be any μ -open set containing F . Then by (iii) there exists a $\mu\nu g$ -open set V such that $F \subseteq V \subseteq c_\nu(V) \subseteq U$. By Theorem 4, $F \subseteq i_\nu(V)$. Put $G = i_\nu(V)$. Then G is a ν -open set. Furthermore, we obtain $F \subseteq G \subseteq c_\nu(G) \subseteq c_\nu(V) \subseteq U$.

(iv) $\Rightarrow$ (v): Let F_1, F_2 be any two disjoint μ -closed sets. Since $X \setminus F_2$ is a μ -open set containing F_1 , by (iv) there exists a ν -open set V such that $F_1 \subseteq V \subseteq c_\nu(V) \subseteq X \setminus F_2$. By Proposition 1, V is $\mu\nu g$ -open. Furthermore, we have $F_1 \subseteq V$ and $c_\nu(V) \cap F_2 = \emptyset$.

(v) $\Rightarrow$ (vi): Let F_1, F_2 be any disjoint μ -closed sets. Then there exists a $\mu\nu g$ -open set V such that $F_1 \subseteq V$ and $c_\nu(V) \cap F_2 = \emptyset$. By Theorem 4, $F_1 \subseteq i_\nu(V)$. Set $G = i_\nu(V)$. Then $G \in \nu, F_1 \subseteq G$ and $c_\nu(G) \cap F_2 = \emptyset$.

(vi) $\Rightarrow$ (i): Let F_1, F_2 be any two disjoint μ -closed sets. Then by (vi) there exists $G \in \nu$ such that $F_1 \subseteq G$ and $c_\nu(G) \cap F_2 = \emptyset$. Now, put $U_1 = G$ and $U_2 = X \setminus c_\nu(G)$. Then U_1 and U_2 are disjoint ν -open sets, $F_1 \subseteq U_1$ and $F_2 \subseteq U_2$. This shows that (X, μ, ν) is (μ, ν) -normal. $\square$

4 Conclusion

Interchanging μ and ν by different weak forms of open sets we can characterize different weak forms of generalized open sets and different weak forms of regular and normal spaces. If $\mu = \nu$, then we get the results obtained in [6].

Acknowledgment

The author acknowledges the financial support from UGC, New Delhi.

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Received: May 25, 2015