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Some generalization of integral inequalities for twice differentiable mappings involving fractional integrals

Mehmet Zeki Sarikaya
Department of Mathematics,
Faculty of Science and Arts,
Duzce University, Turkey
email: sarikayamz@gmail.com

Huseyin Budak
Department of Mathematics,
Faculty of Science and Arts,
Duzce University, Turkey
email: hsyn.budak@gmail.com

Abstract. In this paper, a general integral identity involving Riemann-Liouville fractional integrals is derived. By use this identity, we establish new some generalized inequalities of the Hermite-Hadamard's type for functions whose absolute values of derivatives are convex.

1 Introduction

The following definition for convex functions is well known in the mathematical literature:

The function $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$, is said to be convex if the following inequality holds

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$. We say that f is concave if $(-f)$ is convex.

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Many inequalities have been established for convex functions but the most famous inequality is the Hermite-Hadamard's inequality, due to its rich geometrical significance and applications(see, e.g.,[12, p.137], [6]). These inequalities state that if $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

Both inequalities hold in the reversed direction if f is concave. We note that Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Hadamard's inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found (see, for example, [6, 8, 9, 12], [14]-[16], [22], [23]) and the references cited therein.

In [16], Sarikaya et. al. established inequalities for twice differentiable convex mappings which are connected with Hadamard's inequality, and they used the following lemma to prove their results:

Lemma 1 *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I°, $a, b \in I^\circ$ with $a < b$. If $f'' \in L_1[a, b]$, then*

$$\begin{aligned} &\tfrac{1}{b-a}\int_a^b f(x)dx - f\left(\tfrac{a+b}{2}\right) \\ &\qquad = \tfrac{(b-a)^2}{2}\int_0^1 m(t)\left[f''(ta+(1-t)b)+f''(tb+(1-t)a)\right]dt, \end{aligned} \tag{2}$$

where

$$m(t) := \begin{cases} t^2, & t \in [0, \frac{1}{2}) \\ (1-t)^2, & t \in [\frac{1}{2}, 1]. \end{cases}$$

Also, the main inequalities in [16], pointed out as follows:

Theorem 1 *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I° with $f'' \in L_1[a, b]$. If $|f''|$ is convex on $[a, b]$, then*

$$\left|\tfrac{1}{b-a}\int_a^b f(x)dx - f\left(\tfrac{a+b}{2}\right)\right| \leq \tfrac{(b-a)^2}{24}\left[\tfrac{|f''(a)|+|f''(b)|}{2}\right]. \tag{3}$$

Theorem 2 *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I° such that $f'' \in L_1[a, b]$ where $a, b \in I$, $a < b$. If $|f''|^q$ is convex on $[a, b]$, $q > 1$, then*

$$\left|\tfrac{1}{b-a}\int_a^b f(x)dx - f\left(\tfrac{a+b}{2}\right)\right| \leq \tfrac{(b-a)^2}{8(2p+1)^{1/p}}\left[\tfrac{|f''(a)|^q+|f''(b)|^q}{2}\right]^{1/q} \tag{4}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

In the following we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper. More details, one can consult [7, 10, 11, 13].

Definition 1 *Let* $f \in L_1[a,b]$. *The Riemann-Liouville integrals* $J_{a+}^{\alpha}f$ *and* $J_{b-}^{\alpha}f$ *of order* $\alpha > 0$ *with* $a \geq 0$ *are defined by*

$$J_{a+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \quad x>a$$

and

$$J_{b-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, \quad x<b$$

respectively. Here, $\Gamma(\alpha)$ *is the Gamma function and* $J_{a+}^{0}f(x)=J_{b-}^{0}f(x)=f(x)$.

Meanwhile, Sarikaya et al. [19] presented the following important integral identity including the first-order derivative of f to establish many interesting Hermite-Hadamard type inequalities for convexity functions via Riemann-Liouville fractional integrals of the order $\alpha > 0$.

Lemma 2 *Let* $f:[a,b]\to\mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $0 \leq a < b$. *If* $f' \in L[a,b]$, *then the following equality for fractional integrals holds:*

$$\begin{aligned}&\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{\left(\frac{a+b}{2}\right)+}^{\alpha}f(b)+J_{\left(\frac{a+b}{2}\right)-}^{\alpha}f(a)\right]-f\left(\frac{a+b}{2}\right)\\&=\frac{b-a}{4}\left\{\int_0^1 t^{\alpha}f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt-\int_0^1 t^{\alpha}f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt\right\}\end{aligned} \tag{5}$$

with $\alpha > 0$.

It is remarkable that Sarikaya et al. [19] first give the following interesting integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 3 *Let* $f:[a,b]\to\mathbb{R}$ *be a positive function with* $0 \leq a < b$ *and* $f \in L_1[a,b]$. *If* f *is a convex function on* $[a,b]$, *then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right)\leq\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{\left(\frac{a+b}{2}\right)+}^{\alpha}f(b)+J_{\left(\frac{a+b}{2}\right)-}^{\alpha}f(a)\right]\leq\frac{f(a)+f(b)}{2} \tag{6}$$

with $\alpha > 0$.

For some recent results connected with fractional integral inequalities see ([1, 2, 3, 4, 5], [17], [18], [20], [21], [24])

In this paper, we expand the Lemma 2 to the case of including a twice differentiable function involving Riemann-Liouville fractional integrals and some other integral inequalities using the generalized identity is obtained for fractional integrals.

2 Main results

For our results, we give the following important fractional integrtal identity:

Lemma 3 *Let* $f : [a, b] \to \mathbb{R}$ *be twice differentiable mapping on* (a, b) *with* $0 \le a < b$*. If* $f'' \in L[a, b]$, *then the following equality for fractional integrals holds:*

$$\begin{aligned}
&(\alpha+1)(1-\lambda)^{\alpha}\lambda^{\alpha} f(\lambda a+(1-\lambda)b) \\
&\quad-\frac{(\alpha+1)\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\lambda^{\alpha+1} J^{\alpha}_{(\lambda a+(1-\lambda)b)-} f(a)+(1-\lambda)^{\alpha+1} J^{\alpha}_{(\lambda a+(1-\lambda)b)+} f(b)\right] \\
&=-(b-a)^{2}(1-\lambda)^{\alpha+1}\lambda^{\alpha+1}\left\{(1-\lambda)\int_0^1 t^{\alpha+1} f''[t(\lambda a+(1-\lambda)b)+(1-t)a]\,dt\right. \qquad (7) \\
&\quad\left.+\lambda\int_0^1 (1-t)^{\alpha+1} f''[tb+(1-t)(\lambda a+(1-\lambda)b)]\,dt\right\}
\end{aligned}$$

where $\lambda \in (0, 1)$ *and* $\alpha > 0$.

Proof. Integrating by parts

$$\begin{aligned}
&\int_0^1 t^{\alpha+1} f''[t(\lambda a+(1-\lambda)b)+(1-t)a]\,dt \\
&=\left.\frac{t^{\alpha+1} f'[t(\lambda a+(1-\lambda)b)+(1-t)a]}{(1-\lambda)(b-a)}\right|_0^1 \\
&\quad-\frac{\alpha+1}{(1-\lambda)(b-a)}\int_0^1 t^{\alpha} f'[t(\lambda a+(1-\lambda)b)+(1-t)a]\,dt \\
&=\frac{f'(\lambda a+(1-\lambda)b)}{(1-\lambda)(b-a)}-\frac{\alpha+1}{(1-\lambda)(b-a)}
\end{aligned}$$

$$\times \left[\frac{f(\lambda a + (1-\lambda)b)}{(1-\lambda)(b-a)} - \frac{\alpha}{(1-\lambda)(b-a)} \int_0^1 t^{\alpha-1} f\left[t(\lambda a + (1-\lambda)b) + (1-t)a\right] dt \right]$$

$$= \frac{f'(\lambda a + (1-\lambda)b)}{(1-\lambda)(b-a)} - \frac{(\alpha+1) f(\lambda a + (1-\lambda)b)}{(1-\lambda)^2(b-a)^2}$$

$$+ \frac{(\alpha+1)\alpha}{(1-\lambda)^{\alpha+2}(b-a)^{\alpha+2}} \int_a^{\lambda a+(1-\lambda)b} (x-a)^{\alpha-1} f(x)dx$$

$$= \frac{f'(\lambda a + (1-\lambda)b)}{(1-\lambda)(b-a)} - \frac{(\alpha+1) f(\lambda a + (1-\lambda)b)}{(1-\lambda)^2(b-a)^2}$$

$$+ \frac{(\alpha+1)\Gamma(\alpha+1)}{(1-\lambda)^{\alpha+2}(b-a)^{\alpha+2}} J^{\alpha}_{(\lambda a+(1-\lambda)b)^-} f(a)$$

that is,

$$\begin{aligned}
&-\int_0^1 t^{\alpha+1} f''\left[t(\lambda a + (1-\lambda)b) + (1-t)a\right] dt \\
&= -\frac{f'(\lambda a + (1-\lambda)b)}{(1-\lambda)(b-a)} + \frac{(\alpha+1) f(\lambda a + (1-\lambda)b)}{(1-\lambda)^2(b-a)^2} \\
&\quad - \frac{(\alpha+1)\Gamma(\alpha+1)}{(1-\lambda)^{\alpha+2}(b-a)^{\alpha+2}} J^{\alpha}_{(\lambda a+(1-\lambda)b)^-} f(a)
\end{aligned} \tag{8}$$

and similarly we have

$$\begin{aligned}
&-\int_0^1 (1-t)^{\alpha+1} f''\left[tb + (1-t)(\lambda a + (1-\lambda)b)\right] dt \\
&= \frac{f'(\lambda a + (1-\lambda)b)}{\lambda(b-a)} + \frac{(\alpha+1) f(\lambda a + (1-\lambda)b)}{\lambda^2(b-a)^2} \\
&\quad - \frac{(\alpha+1)\alpha}{\lambda^{\alpha+2}(b-a)^{\alpha+2}} \int_{\lambda a+(1-\lambda)b}^{b} (b-x)^{\alpha-1} f(x)dx \qquad (9) \\
&= \frac{f'(\lambda a + (1-\lambda)b)}{\lambda(b-a)} + \frac{(\alpha+1) f(\lambda a + (1-\lambda)b)}{\lambda^2(b-a)^2} \\
&\quad - \frac{(\alpha+1)\Gamma(\alpha+1)}{\lambda^{\alpha+2}(b-a)^{\alpha+2}} J^{\alpha}_{(\lambda a+(1-\lambda)b)^+} f(b).
\end{aligned}$$

Adding (8) and (9) we have (7). This completes the proof. □

Corollary 1 *Under the assumptions Lemma 3 with $\lambda = \frac{1}{2}$, then it follows that*

$$\frac{-(b-a)^2}{8}\left\{\int_0^1 t^{\alpha+1} f''\left[t\left(\frac{a+b}{2}\right)+(1-t)a\right]dt\right.$$
$$\left.+\int_0^1 (1-t)^{\alpha+1} f''\left[tb+(1-t)\frac{a+b}{2}\right]dt\right\}$$
$$=(\alpha+1)f\left(\frac{a+b}{2}\right)-\frac{(\alpha+1)\Gamma(\alpha+1)}{(b-a)^\alpha 2^{1-\alpha}}\left[J^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a)+J^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)\right].$$

Remark 1 *If we choose $\alpha = 1$ in Corollary 1, we have*

$$f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx$$
$$=\frac{-(b-a)^2}{16}\left\{\int_0^1 t^2 f''\left[t\left(\frac{a+b}{2}\right)+(1-t)a\right]dt\right.$$
$$\left.+\int_0^1 (1-t)^2 f''\left[tb+(1-t)\frac{a+b}{2}\right]dt\right\}.$$

Theorem 4 *Let f:$[a,b]\to\mathbb{R}$ be twice differentiable mapping on (a,b) with $0\le a<b$. If $|f''|^q$, $q\ge 1$ is convex on $[a,b]$, then the following inequality for fractional integrals holds:*

$$\left|(\alpha+1)(1-\lambda)^\alpha\lambda^\alpha f(\lambda a+(1-\lambda)b)-\frac{(\alpha+1)\Gamma(\alpha+1)}{(b-a)^\alpha}\right.$$
$$\left.\times\left[\lambda^{\alpha+1}J^\alpha_{(\lambda a+(1-\lambda)b)^-}f(a)+(1-\lambda)^{\alpha+1}J^\alpha_{(\lambda a+(1-\lambda)b)^+}f(b)\right]\right|$$
$$\le\frac{(b-a)^2(1-\lambda)^{\alpha+1}\lambda^{\alpha+1}}{(\alpha+2)^{1-\frac{1}{q}}}\left\{(1-\lambda)\left(\frac{(\alpha+2)|f''(\lambda a+(1-\lambda)b)|^q+|f''(a)|^q}{\alpha+3}\right)^{\frac{1}{q}}\right.$$
$$\left.+\lambda\left(\frac{(\alpha+2)|f''(\lambda a+(1-\lambda)b)|^q+|f''(b)|^q}{\alpha+3}\right)^{\frac{1}{q}}\right\}. \tag{10}$$

where $\lambda\in(0,1)$ and $\alpha>0$.

Proof. Firstly, we suppose that $q = 1$. Using Lemma 3 and convexity of $|f''|^q$, we find that

$$\Big|(\alpha+1)(1-\lambda)^{\alpha}\lambda^{\alpha}f(\lambda a+(1-\lambda)b)-\frac{(\alpha+1)\Gamma(\alpha+1)}{(b-a)^{\alpha}}$$
$$\times\left[\lambda^{\alpha+1}J^{\alpha}_{(\lambda a+(1-\lambda)b)^-}f(a)+(1-\lambda)^{\alpha+1}J^{\alpha}_{(\lambda a+(1-\lambda)b)^+}f(b)\right]\Big|$$
$$\leq (b-a)^2(1-\lambda)^{\alpha+1}\lambda^{\alpha+1}\left\{(1-\lambda)\int_0^1 t^{\alpha+1}\left|f''\left[t(\lambda a+(1-\lambda)b)+(1-t)a\right]\right|dt\right.$$
$$\left.+\lambda\int_0^1(1-t)^{\alpha+1}\left|f''\left[tb+(1-t)(\lambda a+(1-\lambda)b)\right]\right|dt\right\}$$
$$\leq (b-a)^2(1-\lambda)^{\alpha+1}\lambda^{\alpha+1}\left\{(1-\lambda)\int_0^1 t^{\alpha+1}\left[t\left|f''(\lambda a+(1-\lambda)b)\right|+(1-t)\left|f''(a)\right|\right]dt\right.$$
$$\left.+\lambda\int_0^1(1-t)^{\alpha+1}\left[t\left|f''(b)\right|+(1-t)\left|f''(\lambda a+(1-\lambda)b)\right|\right]dt\right\}$$
$$=\frac{(b-a)^2(1-\lambda)^{\alpha+1}\lambda^{\alpha+1}}{\alpha+2}\left\{(1-\lambda)\left(\frac{(\alpha+2)\left|f''(\lambda a+(1-\lambda)b)\right|+\left|f''(a)\right|}{\alpha+3}\right)\right.$$
$$\left.+\lambda\left(\frac{(\alpha+2)\left|f''(\lambda a+(1-\lambda)b)\right|+\left|f''(b)\right|^q}{\alpha+3}\right)\right\}.$$

Secondly, we suppose that $q > 1$. Using Lemma 3 and power mean inequality, we have

$$\left\{(1-\lambda)\int_0^1 t^{\alpha+1}f''\left[t(\lambda a+(1-\lambda)b)+(1-t)a\right]dt\right.$$
$$\left.+\lambda\int_0^1(1-t)^{\alpha+1}f''\left[tb+(1-t)(\lambda a+(1-\lambda)b)\right]dt\right\}$$
$$\leq(1-\lambda)\left(\int_0^1 t^{\alpha+1}\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\alpha+1}\left|f''\left[t(\lambda a+(1-\lambda)b)+(1-t)a\right]\right|^q dt\right)^{\frac{1}{q}}$$
$$+\lambda\left(\int_0^1(1-t)^{\alpha+1}\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)^{\alpha+1}\left|f''\left[tb+(1-t)(\lambda a+(1-\lambda)b)\right]\right|^q dt\right)^{\frac{1}{q}}. \tag{11}$$

Hence, using convexity of $|f''|^q$ and (11) we obtain

$$\begin{aligned}
&\Bigg|(\alpha+1)(1-\lambda)^{\alpha}\lambda^{\alpha}f(\lambda a+(1-\lambda)b)-\frac{(\alpha+1)\Gamma(\alpha+1)}{(b-a)^{\alpha}} \\
&\quad\times\left[\lambda^{\alpha+1}J^{\alpha}_{(\lambda a+(1-\lambda)b)-}f(a)+(1-\lambda)^{\alpha+1}J^{\alpha}_{(\lambda a+(1-\lambda)b)+}f(b)\right]\Bigg| \\
&\leq\frac{(b-a)^2(1-\lambda)^{\alpha+1}\lambda^{\alpha+1}}{(\alpha+2)^{1-\frac{1}{q}}}\left\{(1-\lambda)\left(\int_0^1 t^{\alpha+1}\left[t\left|f''(\lambda a+(1-\lambda)b)\right|+(1-t)\left|f''(a)\right|\right]dt\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\lambda\left(\int_0^1(1-t)^{\alpha+1}\left[t\left|f''(b)\right|+(1-t)\left|f''(\lambda a+(1-\lambda)b)\right|\right]dt\right)^{\frac{1}{q}}\right\} \\
&\leq\frac{(b-a)^2(1-\lambda)^{\alpha+1}\lambda^{\alpha+1}}{(\alpha+2)^{1-\frac{1}{q}}}\left\{(1-\lambda)\left(\frac{(\alpha+2)\left|f''(\lambda a+(1-\lambda)b)\right|+\left|f''(a)\right|}{(\alpha+2)(\alpha+3)}\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\lambda\left(\frac{(\alpha+2)\left|f''(\lambda a+(1-\lambda)b)\right|+\left|f''(b)\right|^q}{(\alpha+2)(\alpha+3)}\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

This completes the proof. □

Corollary 2 *Under assumption Theorem 4 with* $\lambda=\frac{1}{2}$, *we obtain*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}2^{1-\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}f(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)+}f(b)\right]\right| \\
&\leq\frac{(b-a)^2}{8(\alpha+1)(\alpha+2)^{1-\frac{1}{q}}}\left\{\left(\frac{(\alpha+2)\left|f''\left(\frac{a+b}{2}\right)\right|^q+\left|f''(a)\right|^q}{\alpha+3}\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(\frac{(\alpha+2)\left|f''\left(\frac{a+b}{2}\right)\right|^q+\left|f''(b)\right|^q}{\alpha+3}\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Remark 2 *If we choose* $\alpha=1$ *in Corollary 2, we have*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq\frac{(b-a)^2}{16\times 3^{1-\frac{1}{q}}}\left\{\left(\frac{3\left|f''\left(\frac{a+b}{2}\right)\right|^q+\left|f''(a)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3\left|f''\left(\frac{a+b}{2}\right)\right|^q+\left|f''(b)\right|^q}{4}\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Theorem 5 *Let* f*:*$[a, b] \to \mathbb{R}$ *be twice differentiable mapping on* (a, b) *with* $0 \le a < b$*. If* $|f''|^q$ *is convex on* $[a, b]$ *for same fixed* $q > 1$*, then the following inequality for fractional integrals holds:*

$$\Big|(\alpha+1)(1-\lambda)^\alpha \lambda^\alpha f(\lambda a+(1-\lambda)b) - \frac{(\alpha+1)\Gamma(\alpha+1)}{(b-a)^\alpha} \times \left[\lambda^{\alpha+1} J^\alpha_{(\lambda a+(1-\lambda)b)^-} f(a) + (1-\lambda)^{\alpha+1} J^\alpha_{(\lambda a+(1-\lambda)b)^+} f(b)\right]\Big| \le \frac{(b-a)^2 (1-\lambda)^{\alpha+1}\lambda^{\alpha+1}}{(p(\alpha+1)+1)^{\frac{1}{p}}} \left\{(1-\lambda)\left(\frac{|f''(\lambda a+(1-\lambda)b)|^q + |f''(a)|^q}{2}\right)^{\frac{1}{q}} + \lambda\left(\frac{|f''(\lambda a+(1-\lambda)b)|^q + |f''(b)|^q}{2}\right)^{\frac{1}{q}}\right\}. \tag{12}$$

where $\frac{1}{p}+\frac{1}{q}=1$, $\lambda \in (0,1)$ *and* $\alpha > 0$.

Proof. Using Lemma 3, convexity of $|f''|^q$ well-known Hölder's inequality, we have

$$\begin{aligned}
&\Big|(\alpha+1)(1-\lambda)^\alpha \lambda^\alpha f(\lambda a+(1-\lambda)b) - \frac{(\alpha+1)\Gamma(\alpha+1)}{(b-a)^\alpha} \\
&\quad \times \left[\lambda^{\alpha+1} J^\alpha_{(\lambda a+(1-\lambda)b)^-} f(a) + (1-\lambda)^{\alpha+1} J^\alpha_{(\lambda a+(1-\lambda)b)^+} f(b)\right]\Big| \\
&\le (b-a)^2 (1-\lambda)^{\alpha+1}\lambda^{\alpha+1} \left\{(1-\lambda)\left(\int_0^1 t^{p(\alpha+1)}\right)^{\frac{1}{p}}\right. \\
&\quad \left(\int_0^1 |f''[t(\lambda a+(1-\lambda)b)+(1-t)a]|^q\, dt\right)^{\frac{1}{q}} \\
&\quad \left. +\lambda\left(\int_0^1 (1-t)^{p(\alpha+1)}\right)^{\frac{1}{p}} \left(\int_0^1 |f''[tb+(1-t)(\lambda a+(1-\lambda)b)]|^q\, dt\right)^{\frac{1}{q}}\right\} \\
&\le (b-a)^2 (1-\lambda)^{\alpha+1}\lambda^{\alpha+1} \\
&\quad \times \left\{(1-\lambda)\frac{1}{(p(\alpha+1)+1)^{\frac{1}{p}}} \left(\int_0^1 \left[t\,|f''(\lambda a+(1-\lambda)b)|^q + (1-t)\,|f''(a)|^q\right] dt\right)^{\frac{1}{q}}\right.
\end{aligned}$$

$$+\lambda\frac{1}{(p(\alpha+1)+1)^{\frac{1}{p}}}\left(\int_0^1\left[t\left|f''(b)\right|^q+(1-t)\left|f''(\lambda a+(1-\lambda)b)\right|^q\right]dt\right)^{\frac{1}{q}}\Bigg\}$$

$$=\frac{(b-a)^2(1-\lambda)^{\alpha+1}\lambda^{\alpha+1}}{(p(\alpha+1)+1)^{\frac{1}{p}}}\left\{(1-\lambda)\left(\frac{|f''(\lambda a+(1-\lambda)b)|^q+|f''(a)|^q}{2}\right)^{\frac{1}{q}}\right.$$

$$\left.+\lambda\left(\frac{|f''(\lambda a+(1-\lambda)b)|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}}\right\}.$$

□

Corollary 3 *Under assumption Theorem 5 with* $\lambda=\frac{1}{2}$, *we obtain*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}2^{1-\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)\right]\right|$$

$$\leq\frac{(b-a)^2}{8(\alpha+1)(p(\alpha+1)+1)^{\frac{1}{p}}}\left\{\left(\frac{\left|f''\left(\frac{a+b}{2}\right)\right|^q+|f''(a)|^q}{2}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{\left|f''\left(\frac{a+b}{2}\right)\right|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}}\right\}.$$

Remark 3 *If we choose* $\alpha=1$ *in Corollary 3, we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq\frac{(b-a)^2}{16(2p+1)^{\frac{1}{p}}}\left\{\left(\frac{\left|f''\left(\frac{a+b}{2}\right)\right|^q+|f''(a)|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|f''\left(\frac{a+b}{2}\right)\right|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}}\right\}.$$

Theorem 6 *Let* $f:[a,b]\to\mathbb{R}$ *be twice differentiable mapping on* (a,b) *with* $0\leq a<b$. *If* $|f''|^q$ *is convex on* $[a,b]$ *for same fixed* $q>1$, *then the following inequality for fractional integrals holds:*

$$\left|(\alpha+1)(1-\lambda)^{\alpha}\lambda^{\alpha}f(\lambda a+(1-\lambda)b)-\frac{(\alpha+1)\Gamma(\alpha+1)}{(b-a)^{\alpha}}\right.$$

$$\left.\times\left[\lambda^{\alpha+1}J^{\alpha}_{(\lambda a+(1-\lambda)b)^-}f(a)+(1-\lambda)^{\alpha+1}J^{\alpha}_{(\lambda a+(1-\lambda)b)^+}f(b)\right]\right|$$

$$\leq(b-a)^2(1-\lambda)^{\alpha+1}\lambda^{\alpha+1}$$

$$\left\{(1-\lambda)\left(\frac{(q(\alpha+1)+1)|f''(\lambda a+(1-\lambda)b)|^q+|f''(a)|^q}{(q(\alpha+1)+1)(q(\alpha+1)+2)}\right)^{\frac{1}{q}}\right.$$
$$\left.+\lambda\left(\frac{(q(\alpha+1)+1)|f''(\lambda a+(1-\lambda)b)|^q+|f''(b)|^q}{(q(\alpha+1)+1)(q(\alpha+1)+2)}\right)^{\frac{1}{q}}\right\}. \tag{13}$$

where $\lambda\in(0,1)$ *and* $\alpha>0$.

Proof. Using Lemma 3, convexity of $|f''|^q$ well-known Hölder's inequality, we have

$$\left|(\alpha+1)(1-\lambda)^\alpha\lambda^\alpha f(\lambda a+(1-\lambda)b)-\frac{(\alpha+1)\Gamma(\alpha+1)}{(b-a)^\alpha}\right.$$
$$\left.\times\left[\lambda^{\alpha+1}J^\alpha_{(\lambda a+(1-\lambda)b)-}f(a)+(1-\lambda)^{\alpha+1}J^\alpha_{(\lambda a+(1-\lambda)b)+}f(b)\right]\right|$$
$$\leq(b-a)^2(1-\lambda)^{\alpha+1}\lambda^{\alpha+1}\left\{(1-\lambda)\left(\int_0^1 1^p\right)^{\frac{1}{p}}\left(\int_0^1 t^{q(\alpha+1)}\left|f''\left[t(\lambda a+(1-\lambda)b)+(1-t)a\right]\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\lambda\left(\int_0^1 1^p\right)^{\frac{1}{p}}\left(\int_0^1(1-t)^{q(\alpha+1)}\left|f''\left[tb+(1-t)(\lambda a+(1-\lambda)b)\right]\right|^q dt\right)^{\frac{1}{q}}\right\}$$
$$\leq(b-a)^2(1-\lambda)^{\alpha+1}\lambda^{\alpha+1}\left\{(1-\lambda)\left(\int_0^1 t^{q(\alpha+1)}\left[t\left|f''(\lambda a+(1-\lambda)b)\right|^q+(1-t)\left|f''(a)\right|^q\right]dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\lambda\left(\int_0^1(1-t)^{q(\alpha+1)}\left[t\left|f''(b)\right|^q+(1-t)\left|f''(\lambda a+(1-\lambda)b)\right|^q\right]dt\right)^{\frac{1}{q}}\right\}$$
$$=(b-a)^2(1-\lambda)^{\alpha+1}\lambda^{\alpha+1}\left\{(1-\lambda)\left(\frac{(q(\alpha+1)+1)|f''(\lambda a+(1-\lambda)b)|^q+|f''(a)|^q}{(q(\alpha+1)+1)(q(\alpha+1)+2)}\right)^{\frac{1}{q}}\right.$$
$$\left.+\lambda\left(\frac{(q(\alpha+1)+1)|f''(\lambda a+(1-\lambda)b)|^q+|f''(b)|^q}{(q(\alpha+1)+1)(q(\alpha+1)+2)}\right)^{\frac{1}{q}}\right\}.$$

□

Corollary 4 *Under assumption Theorem 6 with* $\lambda=\frac{1}{2}$, *we obtain*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha 2^{1-\alpha}}\left[J^\alpha_{\left(\frac{a+b}{2}\right)-}f(a)+J^\alpha_{\left(\frac{a+b}{2}\right)+}f(b)\right]\right|$$
$$\leq\frac{(b-a)^2}{8(\alpha+1)}\left\{\left(\frac{(q(\alpha+1)+1)\left|f''\left(\frac{a+b}{2}\right)\right|^q+|f''(a)|^q}{(q(\alpha+1)+1)(q(\alpha+1)+2)}\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{(q(\alpha+1)+1)\left|f''\left(\frac{a+b}{2}\right)\right|^q+|f''(b)|^q}{(q(\alpha+1)+1)(q(\alpha+1)+2)}\right)^{\frac{1}{q}}\Bigg\}.$$

Remark 4 *If we choose* $\alpha = 1$ *in Corollary 4, we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{(b-a)^2}{16}\left\{\left(\frac{(2q+1)\left|f''\left(\frac{a+b}{2}\right)\right|^q+|f''(a)|^q}{(2q+1)(2q+2)}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{(2q+1)\left|f''\left(\frac{a+b}{2}\right)\right|^q+|f''(b)|^q}{(2q+1)(2q+2)}\right)^{\frac{1}{q}}\right\}.$$

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