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Properties of certain class of analytic functions with varying arguments defined by Ruscheweyh derivative

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Abstract. In the paper are studied the properties of the image of a class of analytic functions defined by the Ruscheweyh derivative trough the Bernardi operator.

1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic and univalent in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$.

Let $g \in \mathcal{A}$ where

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k. \tag{2}$$

The Hadamard product is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z). \tag{3}$$

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Ruscheweyh [4] defined the derivative $D^\gamma : A \to A$ by

$$D^\gamma f(z) = \frac{z}{(1-z)^{\gamma+1}} * f(z), (\gamma \geq -1). \tag{4}$$

In the particular case $n \in \mathbb{N}_0 = \{0, 1, 2 \dots\}$

$$D^n f(z) = \frac{z(z^{n-1}f(z))^{(n)}}{n!}. \tag{5}$$

The symbol $D^n f(z)(n \in \mathbb{N}_0)$ was called the n-th order Ruscheweyh derivative of $f(z)$ by Al-Amiri [1]. It is easy to see that

$$D^0 f(z) = f(z), D^1 f(z) = zf'(z)$$

$$D^n f(z) = z + \sum_{k=2}^{\infty} \delta(n,k) a_k z^k \tag{6}$$

where

$$\delta(n,k) = \binom{n+k-1}{n}. \tag{7}$$

Definition 1 *Let* f *and* g *be analytic functions in* U. *We say that the function* f *is subordinate to the function* g, *if there exist a function* w, *which is analytic in* U *and* $w(0) = 0; |w(z)| < 1; z \in U$, *such that* $f(z) = g(w(z)); \forall z \in U$. *We denote by* $\prec$ *the subordination relation.*

Attiya and Aouf defined in [2] the class $Q(n, \lambda, A, B)$ this way:

Definition 2 [2], [3] *For* $\lambda \geq 0; -1 \leq A < B \leq 1; 0 < B \leq 1; n \in \mathbb{N}_0$ *let* $Q(n, \lambda, A, B)$ *denote the subclass of* $\mathcal{A}$ *which contain functions* $f(z)$ *of the form* (1) *such that*

$$(1-\lambda)(D^n f(z))' + \lambda(D^{n+1} f(z))' \prec \frac{1+Az}{1+Bz}. \tag{8}$$

Definition 3 [5] *A function* $f(z)$ *of the form* (1) *is said to be in the class* $V(\theta_k)$ *if* $f \in A$ *and* $\arg(a_k) = \theta_k$ $, \forall k \geq 2$. *If* $\exists \delta \in \mathbb{R}$ *such that* $\theta_k + (k-1)\delta \equiv \pi (\text{mod } 2\pi), \forall k \geq 2$ *then* $f(z)$ *is said to be in the class* $V(\theta_k, \delta)$. *The union of* $V(\theta_k, \delta)$ *taken over all possible sequences* $\{\theta_k\}$ *and all possible real numbers* δ *is denoted by* V. *Let* $VQ(n, \lambda, A, B)$ *denote the subclass of* V *consisting of functions* $f(z) \in Q(n, \lambda, A, B)$.

Theorem 1 [3] *Let the function* f *defined by* (1) *be in* V. *Then* $f \in VQ(n, \lambda, A, B)$, *if and only if*

$$E(f) = \sum_{k=2}^{\infty} k\delta(n,k)C_k |a_k| \leq (B - A)(n+1) \tag{9}$$

where

$$C_k = (1 + B)[n + 1 + \lambda(k - 1)].$$

The extremal functions are

$$f_k(z) = z + \frac{(B - A)(n+1)}{kC_k\delta(n,k)} e^{i\theta_k} z^k, (k \geq 2).$$

Main results

Theorem 2 *Let*

$$F(z) = I_c f(z) = \frac{c+1}{z^c} \int_0^z f(t)t^{c-1}dt, c \in \mathbb{N}^*$$

If $f \in VQ(n, \lambda, 2\alpha - 1, B)$ *then* $F \in VQ(n, \lambda, 2\beta - 1, B)$, *where*

$$\beta = \beta(\alpha) = \frac{B + 1 + 2\alpha(c+1)}{2(c+2)} \geq \alpha.$$

The result is sharp.
Remark: The operator I_c *is the well-known Bernardi operator.*

Proof.
Let $f \in VQ(n, \lambda, 2\alpha - 1, B)$ and suppose it has the form (1). Then

$$F(z) = \frac{c+1}{z^c} \int_0^z \left(t + \sum_{k=2}^{\infty} a_k t^k \right) t^{c-1} dt =$$

$$= z + \sum_{k=2}^{\infty} \frac{c+1}{c+k} a_k z^k = z + \sum_{k=2}^{\infty} b_k z^k.$$

Since $f \in VQ(n, \lambda, 2\alpha - 1, B)$ we have

$$\sum_{k=2}^{\infty} k\delta(n,k)C_k |a_k| \leq [B - (2\alpha - 1)](n+1)$$

or equivalently

$$\frac{\sum_{k=2}^{\infty} k\delta(n,k)C_k\,|a_k|}{B-2\alpha+1} \leq n+1. \tag{10}$$

We know from Theorem 1 that $F \in VQ(n,\lambda,2\beta-1,B)$ if and only if

$$\sum_{k=2}^{\infty} k\delta(n,k)C_k\,|b_k| \leq [B-(2\beta-1)](n+1)$$

or

$$\frac{\sum_{k=2}^{\infty} k\delta(n,k)C_k\frac{c+1}{c+k}\,|a_k|}{B-2\beta+1} \leq n+1. \tag{11}$$

We note that the inequalities

$$\frac{k\delta(n,k)C_k\frac{c+1}{c+k}\,|a_k|}{B-2\beta+1} \leq \frac{k\delta(n,k)C_k\,|a_k|}{B-2\alpha+1}, \forall\ k \geq 2 \tag{12}$$

imply (11). From (12) we have

$$\frac{c+1}{(c+k)(B-2\beta+1)} \leq \frac{1}{B-2\alpha+1}$$

$$(c+1)(B-2\alpha+1) \leq (c+k)(B-2\beta+1), \forall\ k \geq 2$$

$$\beta \leq \frac{(k-1)(B+1)+2\alpha(c+1)}{2(c+k)}.$$

Let us consider the function

$$E(x) = \frac{(x-1)(B+1)+2\alpha(c+1)}{2(c+x)},$$

then its derivative is:

$$E'(x) = \frac{1}{2}\frac{(c+1)(B+1-2\alpha)}{(c+x)^2} > 0.$$

$E(x)$ is an increasing function. In our case we need $\beta \leq E(k)$ and for this reason we choose $\beta = \beta(\alpha) = E(2) = \dfrac{B+1+2\alpha(c+1)}{2(c+2)}$.

$$\beta(\alpha) > \alpha \Leftrightarrow B+1+2\alpha c+2\alpha > 2\alpha c+4\alpha \Leftrightarrow B+1-2\alpha > 0.$$

The result is sharp, because if

$$f_2(z) = z + \frac{(B - 2\alpha + 1)(n + 1)}{2C_2\delta(n, 2)} e^{i\theta_2} z^2,$$

then

$$F_2 = I_c f_2$$

belongs to $VQ(n, \lambda, 2\beta - 1, B)$ and its coefficients satisfy the corresponding inequality (9) with equality. Indeed,

$$F_2(z) = z + \frac{(B - 2\alpha + 1)(n + 1)}{2C_2\delta(n, 2)} \frac{c + 1}{c + 2} e^{i\theta_2} z^2 = z + \frac{(B - 2\beta(\alpha) + 1)(n + 1)}{2C_2\delta(n, 2)} e^{i\theta_2} z^2$$

and

$$E(F_2) = 2\delta(n, 2)C_2 \frac{(B - 2\beta(\alpha) + 1)(n + 1)}{2C_2\delta(n, 2)} = (B - 2\beta(\alpha) + 1)(n + 1).$$

□

Theorem 3 *If* $f \in VQ(n, \lambda, A, B)$ *then* $F \in VQ(n, \lambda, A^*, B)$, *where* $A^* = \dfrac{B + A(c + 1)}{c + 2} > A$. *The result is sharp.*

Proof. Let $f \in VQ(n, \lambda, A, B)$ and suppose it has the form (1). Then

$$F(z) = z + \sum_{k=2}^{\infty} \frac{c + 1}{c + k} a_k z^k = z + \sum_{k=2}^{\infty} b_k z^k.$$

Since $f \in VQ(n, \lambda, A, B)$ we have $\sum_{k=2}^{\infty} k\delta(n, k)C_k |a_k| \leq (B - A)(n + 1)$ or equivalently

$$\frac{\sum_{k=2}^{\infty} k\delta(n, k)C_k |a_k|}{B - A} \leq n + 1.$$

We know from Theorem 1 that $F \in VQ(n, \lambda, A^*, B)$ if and only if

$$\frac{\sum_{k=2}^{\infty} k\delta(n, k)C_k \frac{c+1}{c+k} |a_k|}{B - A^*} \leq n + 1, \forall k. \tag{13}$$

We note that

$$\frac{k\delta(n,k)C_k\frac{c+1}{c+k}\left|a_k\right|}{B-A^*} \leq \frac{k\delta(n,k)C_k\left|a_k\right|}{B-A} \tag{14}$$

implies (13). From (14) we have

$$\frac{c+1}{(c+k)(B-A^*)} \leq \frac{1}{B-A}$$
$$(c+1)(B-A) \leq (c+k)(B-A^*), \forall\ k \geq 2$$
$$A^* \leq \frac{B(k-1)+A(c+1)}{(c+k)}.$$

Let us consider the function

$$E(x) = \frac{B(x-1)+A(c+1)}{x+c};$$

its derivative is:

$$E'(x) = \frac{(B-A)(c+1)}{(x+c)^2} > 0.$$

$E(x)$ is an increasing function.

In our case we need $A^* \leq E(k), \forall\ k \geq 2$ and for this reason we choose $A^* = E(2) = \dfrac{B+A(c+1)}{c+2}$.

We note that $A^* > A$, because

$$B + A(c+1) > A(c+2) \Leftrightarrow B > A.$$

The result is sharp, because if

$$f_2(z) = z + \frac{(B-A)(n+1)}{2C_2\delta(n,2)}e^{i\theta_2}z^2,$$

then

$$F_2 = I_c f_2$$

belongs to $VQ(n,\lambda,A^*,B)$ and its coefficients satisfy the corresponding inequality (9) with equality. Indeed,

$$F_2(z) = z + \frac{(B-A)(n+1)}{2C_2\delta(n,2)}\frac{c+1}{c+2}e^{i\theta_2}z^2 = z + \frac{(B-A^*)(n+1)}{2C_2\delta(n,2)}e^{i\theta_2}z^2$$

and

$$E(F_2) = 2\delta(n,2)C_2\frac{(B-A^*)(n+1)}{2C_2\delta(n,2)} = (B-A^*)(n+1).$$

□

Theorem 4 *If* $f \in VQ(n, \lambda, A, B)$ *then* $F \in VQ(n, \lambda, A, B^*)$, *where*

$$B^* = \frac{B(c+1)+A}{c+2} < B.$$

The result is sharp.

Proof. Let $f \in VQ(n, \lambda, A, B)$ and suppose it has the form (1).

Since $f \in VQ(n, \lambda, A, B)$ we have $\sum_{k=2}^{\infty} k\delta(n,k)C_k |a_k| \leq (B-A)(n+1)$ or equivalently

$$\frac{\sum_{k=2}^{\infty} k\delta(n,k)C_k |a_k|}{B-A} \leq n+1.$$

We know from *Theorem 1* that $F \in VQ(n, \lambda, A, B^*)$ if and only if

$$\sum_{k=2}^{\infty} k\delta(n,k)C_k |b_k| \leq (B^*-A)(n+1)$$

or

$$\frac{\sum_{k=2}^{\infty} k\delta(n,k)C_k \frac{c+1}{c+k} |a_k|}{B^*-A} \leq n+1. \tag{15}$$

We note that

$$\frac{k\delta(n,k)C_k \frac{c+1}{c+k} |a_k|}{B^*-A} \leq \frac{k\delta(n,k)C_k |a_k|}{B-A}, \forall k \tag{16}$$

implies (15). From (16) we have

$$\frac{c+1}{(c+k)(B^*-A)} \leq \frac{1}{B-A}$$
$$(c+1)(B-A) \leq (c+k)(B^*-A), \forall k \geq 2$$
$$\frac{B(c+1)+A(k-1)}{c+k} \leq B^*.$$

Let

$$E(x) = \frac{B(c+1)+A(x-1)}{x+c}$$

its derivative is:

$$E'(x) = \frac{(c+1)(A-B)}{(x+c)^2} < 0.$$

$E(x)$ is a decreasing function. In our case we need $E(k) \leq B^*$ and for this reason we choose $B^* = E(2) = \dfrac{A + B(c+1)}{c+2}$

$$B^* < B \Leftrightarrow A + Bc + B < Bc + 2B \Leftrightarrow A < B.$$

The result is sharp, because if

$$f_2(z) = z + \frac{(B-A)(n+1)}{2C_2\delta(n,2)} e^{i\theta_2} z^2,$$

then

$$F_2 = I_c f_2$$

belongs to $VQ(n, \lambda, A, B^*)$ and its coefficients satisfy the corresponding inequality (9) with equality. Indeed,

$$F_2(z) = z + \frac{(B-A)(n+1)}{2C_2\delta(n,2)} \frac{c+1}{c+2} e^{i\theta_2} z^2 = z + \frac{(B^*-A)(n+1)}{2C_2\delta(n,2)} e^{i\theta_2} z^2$$

and

$$E(F_2) = 2\delta(n,2) C_2 \frac{(B^*-A)(n+1)}{2C_2\delta(n,2)} = (B^*-A)(n+1).$$

□

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