Acta Univ. Sapientiae, Mathematica, 8, 1 (2016) 5–21

DOI: 10.1515/ausm-2016-0001

On submersion and immersion submanifolds of a quaternionic projective space

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Abstract. We study submanifolds of a quaternionic projective space, it is of great interest how to pull down some formulae deduced for submanifolds of a sphere to those for submanifolds of a quaternionic projective space.

1 Introduction

It is well known that an odd-dimensional sphere is a circle bundle over the quaternionic projective space. Consequently, many geometric properties of the quaternionic projective space are inherited from those of the sphere.

Let M be a connected real n-dimensional submanifold of real codimension p of a quaternionic Kähler manifold $\overline{M}^{n+p}$ with quaternionic Kähler structure $\{F, G, H\}$. If there exists an r-dimensional normal distribution ν of the normal bundle $TM^{\perp}$ such that

$$\begin{array}{lll} F\nu_x \subset \nu_x, & G\nu_x \subset \nu_x, & H\nu_x \subset \nu_x, \\ F\nu_x^{\perp} \subset T_xM, & G\nu_x^{\perp} \subset T_xM, & H\nu_x^{\perp} \subset T_xM, \end{array}$$

2010 Mathematics Subject Classification: 53C25, 53C40
Key words and phrases: quaternionic projective space, submersion and immersion submanifold

at each point x in M, then M is called a QR-submanifold of r QR-dimension, where $\nu^{\perp}$ denotes the complementary orthogonal distribution to ν in TM [2, 14, 16].

Equivalently, there exists distributions $(D_x, D_x^{\perp})$ of the tangent bundle TM, such that

$$\begin{array}{lll} FD_x \subset D_x, & GD_x \subset D_x, & HD_x \subset D_x, \\ FD_x^{\perp} \subset T_xM^{\perp}, & GD_x^{\perp} \subset T_xM^{\perp}, & HD_x^{\perp} \subset T_xM^{\perp}, \end{array}$$

where $D_x^{\perp}$ denotes the complementary orthogonal distribution to D_x in TM. Real hypersurfaces, which are typical examples of QR-submanifold with $r = 0$, have been investigated by many authors [3, 9, 14, 16, 18, 20] in connection with the shape operator and the induced almost contact 3-structure. Recently, Kwon and Pak have studied QR-submanifolds of $(p-1)$ QR-dimension isometrically immersed in a quaternionic projective space $QP^{\frac{n+p}{4}}$ [14, 16].
Pak and Sohn studied n-dimensional QR-submanifold of $(p-1)$ QR-dimension in a quaternionic projective space $QP^{\frac{(n+p)}{4}}$ [19].
Kim and Pak studied n-dimensional QR-submanifold of maximal QR-dimension isometrically immersed in a quaternionic projective space [13].

2 Preliminaries

Let $\overline{M}$ be a real $(n+p)$-dimensional quaternionic Kähler manifold. Then, by definition, there is a 3-dimensional vector bundle V consisting with tensor fields of type $(1,1)$ over $\overline{M}$ satisfying the following conditions (a), (b) and (c):
(a) In any coordinate neighborhood $\overline{\mathcal{U}}$, there is a local basis $\{F, G, H\}$ of V such that

$$\begin{gathered} F^2 = -I, \quad G^2 = -I, \quad H^2 = -I, \\ FG = -GF = H, \quad GH = -HG = F, \quad HF = -FH = G. \end{gathered} \tag{1}$$

(b) There is a Riemannian metric g which is hermite with respect to all of F, G and H.
(c) For the Riemannian connection $\overline{\nabla}$ with respect to g

$$\begin{pmatrix} \overline{\nabla}F \\ \overline{\nabla}G \\ \overline{\nabla}H \end{pmatrix} = \begin{pmatrix} 0 & r & -q \\ -r & 0 & p \\ q & -p & 0 \end{pmatrix} \begin{pmatrix} F \\ G \\ H \end{pmatrix} \tag{2}$$

where p, q and r are local 1-forms defined in $\overline{\mathcal{U}}$. Such a local basis $\{F, G, H\}$ is called a canonical local basis of the bundle V in $\overline{\mathcal{U}}$ (cf. [11, 12]).

For canonical local basis $\{F, G, H\}$ and $\{F', G', H'\}$ of V in coordinate neighborhoods of $\overline{\mathcal{U}}$ and $\overline{\mathcal{U}}'$, it follows that in $\overline{\mathcal{U}} \cap \overline{\mathcal{U}}'$

$$\begin{pmatrix} F' \\ G' \\ H' \end{pmatrix} = \left(s_{xy}\right) \begin{pmatrix} F \\ G \\ H \end{pmatrix} \qquad (x, y = 1, 2, 3)$$

where s_{xy} are local differentiable functions with $(s_{xy}) \in SO(3)$ as a consequence of (1). As is well known [11, 12], every quaternionic Kähler manifold is orientable.

Now let M be an n-dimensional QR-submanifold of maximal QR-dimension, that is, of $(p-1)$ QR-dimension isometrically immersed in $\overline{M}$. Then by definition there is a unit normal vector field ξ such that $\nu_x^{\perp} = span\{\xi\}$ at each point x in M. We set

$$F\xi = -U, \qquad G\xi = -V, \qquad H\xi = -W. \tag{3}$$

Denoting by D_x the maximal quaternionic invariant subspace

$$T_xM \cap FT_xM \cap GT_xM \cap HT_xM,$$

of T_xM, we have $D_x^{\perp} \supset Span\{U, V, W\}$, where $D_x^{\perp}$ means the complementary orthogonal subspace to D_x in T_xM. But, using (1), we can prove that $D_x^{\perp} = Span\{U, V, W\}$ [2, 16]. Thus we have

$$T_xM = D_x \oplus Span\{U, V, W\}, \qquad \forall x \in M,$$

which together with (1) and (3) imply

$$FT_xM, GT_xM, HT_xM \subset T_xM \oplus Span\{\xi\}.$$

Therefore, for any tangent vector field X and for a local orthonormal basis $\{\xi_\alpha\}_{\alpha=1,\dots,p}$ $(\xi_1 := \xi)$ of normal vectors to M, we have

$$FX = \varphi X + u(X)\xi, \quad GX = \psi X + v(X)\xi, \quad HX = \theta X + \omega(X)\xi, \tag{4}$$

$$\begin{aligned} F\xi_\alpha &= -U_\alpha + P_1\xi_\alpha, \quad G\xi_\alpha = -V_\alpha + P_2\xi_\alpha, \\ H\xi_\alpha &= -W_\alpha + P_3\xi_\alpha, \quad (\alpha = 1, \dots, p). \end{aligned} \tag{5}$$

Then it is easily seen that $\{\varphi, \psi, \theta\}$ and $\{P_1, P_2, P_3\}$ are skew-symmetric endomorphisms acting on T_xM and $T_xM^{\perp}$, respectively.

Also, from the hermitian properties

$$\begin{aligned} g(FX, \xi_\alpha) &= -g(X, F\xi_\alpha), \quad g(GX, \xi_\alpha) = -g(X, G\xi_\alpha), \\ g(HX, \xi_\alpha) &= -g(X, H\xi_\alpha), \quad (\alpha = 1, \dots, p). \end{aligned}$$

It follows that

$$g(X, U_\alpha) = u(X)\delta_{1\alpha}, \ \ g(X, V_\alpha) = v(X)\delta_{1\alpha}, \ \ g(X, W_\alpha) = w(X)\delta_{1\alpha},$$

and hence

$$\begin{aligned} &g(X, U_1) = u(X), \ \ g(X, V_1) = v(X), \ \ g(X, W_1) = w(X), \\ &U_\alpha = 0, \ \ V_\alpha = 0, \ \ W_\alpha = 0, \ \ (\alpha = 2, \ldots p). \end{aligned} \tag{6}$$

On the other hand, comparing (3) and (5) with $\alpha = 1$, we have $U_1 = U, V_1 = V, W_1 = W$, which together with (3) and (6) imply

$$\begin{array}{lll} g(X, U) = u(X), & g(X, V) = v(X), & g(X, W) = w(X), \\ u(U) = 1, & v(V) = 1, & w(W) = 1, \\ F\xi = -U, & G\xi = -V, & H\xi = -W \\ F\xi_{\alpha = P_1 \xi_\alpha}, & G\xi_{\alpha = P_2 \xi_\alpha} & H\xi_{\alpha = P_3 \xi_\alpha}, \quad (\alpha = 2, \ldots, p). \end{array}$$

Now, let ∇ be the Levi-Civita connection on M and $\nabla^\perp$ the normal connection induced from $\overline{\nabla}$ in the normal bundle $TM^\perp$ of M. The Gauss and Weingarten formula are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \ \ \overline{\nabla}_X \xi_\alpha = -A_\alpha X + \nabla^\perp_X \xi_\alpha, \ \ (\alpha = 1, \ldots, p), \tag{7}$$

for any $X, Y \in \chi(M)$ and $\xi_\alpha \in \Gamma^\infty(T(M)^\perp)$, $(\alpha = 1, \ldots, p)$. h is the second fundamental form and A_α are shape operator corresponding to ξ_α.
We have the following Gauss equation

$$\begin{aligned} \overline{g}(\overline{R}(X, Y)Z, W) &= g(R(X, Y)Z, W) \\ &\quad - \sum_{i=1}^{p} \{g(A_a Y, Z)g(A_a X, W) - g(A_a X, Z)g(A_a Y, W)\}, \end{aligned} \tag{8}$$

and Codazzi and Ricci equations

$$\begin{aligned} \overline{g}(\overline{R}(X, Y)Z, \xi_a) &= (\nabla_X A_a)Y - (\nabla_Y A_a)X \\ &\quad - \sum_{b=1}^{p} \{s_{ba}(X)g(A_a Y, Z) - s_{ba}(Y)g(A_a X, Z)\}, \\ \overline{g}(\overline{R}(X, Y)\xi_a, \xi_b) &= g([A_b, A_a]X, Y) + \overline{g}(R^\perp(X, Y)\xi_a, \xi_b), \end{aligned} \tag{9}$$

where $\overline{R}$ and R are the curvature tensors of $\overline{M}$ and M, respectively. s_{ab} are called the coefficients of the third fundamental form of M in $\overline{M}$, such that satisfy

$$s_{ab} = -s_{ba}.$$

3 The principal circle bundle $S^{4n+3}(QP^n, S^3)$

Let Q^{n+1} be the $(n+1)$-dimensional quaternionic space with natural quaternionic Kähler structure $(\{F', G', H'\}, \langle , \rangle)$ and let S^{4n+3} be the unit sphere defined by

$$\begin{aligned} S^{4n+3} &= \{(w^1, \ldots, w^{n+1}) \in Q^{n+1} | \sum_{i=1}^{n+1} w^i (w^i)^* = 1\} \\ &= \{(x_1^1, x_2^1, x_3^1, x_4^1, \ldots, x_1^{n+1}, x_2^{n+1}, x_3^{n+1}, x_4^{n+1}) \in \mathbb{R}^{4n+4} | \\ &\sum_{i=1}^{n+1} (x_1^i)^2 + (x_2^i)^2 + (x_3^i)^2 + (x_4^i)^2 = 1\}. \end{aligned}$$

such that $w^* = (x_1, -x_2, -x_3, -x_4)$.
The unit normal vector field ξ to S^{4n+3} is given by

$$\xi = -\sum_{i=1}^{n+1} (x_1^i \frac{\partial}{\partial x_1^i} + x_2^i \frac{\partial}{\partial x_2^i} + x_3^i \frac{\partial}{\partial x_3^i} + x_4^i \frac{\partial}{\partial x_4^i}).$$

From

$$\begin{aligned} \langle F'\xi, \xi \rangle &= \langle F'^2 \xi, F'\xi \rangle = -\langle \xi, F'\xi \rangle, \\ \langle G'\xi, \xi \rangle &= \langle G'^2 \xi, G'\xi \rangle = -\langle \xi, G'\xi \rangle, \\ \langle H'\xi, \xi \rangle &= \langle H'^2 \xi, H'\xi \rangle = -\langle \xi, H'\xi \rangle, \end{aligned}$$

it follows $\langle F'\xi, \xi \rangle = 0, \langle G'\xi, \xi \rangle = 0, \langle H'\xi, \xi \rangle = 0$, that is, $F'\xi, G'\xi, H'\xi \in T(S^{4n+3})$. We put

$$F'\xi = -\iota U', \quad G'\xi = -\iota V', \quad H'\xi = -\iota W', \tag{10}$$

where ι denotes the immersion of S^{4n+3} into Q^{n+1}. From the Hermitian property of $\langle , \rangle$, it is easily seen that U', V', W' are unit tangent vector field of S^{4n+3}.

We put

$$H_p(S^{4n+3}) = \{X' \in T_p(S^{4n+3}) | u'(X') = 0, v'(X') = 0, w'(X') = 0\}.$$

Then u', v', w' define a connection form of the principal bundle $S^{4n+3}(QP^n, S^3)$ and we have

$$T_p(S^{4n+3}) = H_p(S^{4n+3}) \oplus span\{U'_p, V'_p, W'_p\}.$$

We call $H_p(S^{4n+3})$ and $span\{U'_p, V'_p, W'_p\}$ the horizontal subspace and vertical subspace of $T_p(S^{4n+3})$, respectively. By definition, the horizontal subspace $H_p(S^{4n+3})$ is isomorphic to $T_{\pi(p)}(QP^n)$, where π is the natural projection from S^{4n+3} onto QP^n. Therefore, for a vector field X on QP^n, there exists unique horizontal vector field X' of S^{4n+3} such that $\pi(X') = X$. The vector field X' is called the horizontal lift of X and we denote it by X^*.

Proposition 1 *As a subspace of* $T_p(Q^{n+1})$, $H_p(S^{4n+3})$ *is* $\{F', G', H'\}$*-invariant subspace.*

Proof. By definition (10) of the vertical vector field $\{U', V', W'\}$, for $X' \in H_p(S^{4n+3})$, it follows

$$\begin{aligned}
\langle F'\iota X', \xi\rangle &= -\langle \iota X', F'\xi\rangle = \langle \iota X', \iota U'\rangle = 0,\\
\langle G'\iota X', \xi\rangle &= -\langle \iota X', G'\xi\rangle = \langle \iota X', \iota V'\rangle = 0,\\
\langle H'\iota X', \xi\rangle &= -\langle \iota X', H'\xi\rangle = \langle \iota X', \iota W'\rangle = 0.
\end{aligned}$$

This shows that $F'\iota X', G'\iota X', H'\iota X' \in T_p(S^{4n+3})$. In entirely the same way we compute

$$\begin{aligned}
\langle F'\iota X', \iota U'\rangle &= -\langle \iota X', F'\iota U'\rangle = \langle \iota X', -\xi\rangle = 0,\\
\langle G'\iota X', \iota V'\rangle &= -\langle \iota X', G'\iota V'\rangle = \langle \iota X', -\xi\rangle = 0,\\
\langle H'\iota X', \iota W'\rangle &= -\langle \iota X', H'\iota W'\rangle = \langle \iota X', -\xi\rangle = 0.
\end{aligned}$$

By use from relations

$$\begin{aligned}
F'V' = -H'\xi, \quad F'W' = G'\xi, \quad G'U' = H'\xi,\\
G'W' = F'\xi, \quad H'U' = -G'\xi, \quad H'V' = F'\xi,
\end{aligned}$$

we have

$$\begin{aligned}
\langle F'\iota X', \iota V'\rangle &= -\langle \iota X', F'\iota V'\rangle = \langle \iota X', H'\xi\rangle = -\langle \iota X', \iota W'\rangle = 0,\\
\langle F'\iota X', \iota W'\rangle &= -\langle \iota X', F'\iota W'\rangle = \langle \iota X', G'\xi\rangle = -\langle \iota X', \iota V'\rangle = 0,\\
\langle G'\iota X', \iota U'\rangle &= -\langle \iota X', G'\iota U'\rangle = \langle \iota X', H'\xi\rangle = -\langle \iota X', \iota W'\rangle = 0,\\
\langle G'\iota X', \iota W'\rangle &= -\langle \iota X', G'\iota W'\rangle = \langle \iota X', F'\xi\rangle = -\langle \iota X', \iota U'\rangle = 0,\\
\langle H'\iota X', \iota V'\rangle &= -\langle \iota X', H'\iota V'\rangle = \langle \iota X', F'\xi\rangle = -\langle \iota X', \iota U'\rangle = 0,\\
\langle H'\iota X', \iota U'\rangle &= -\langle \iota X', H'\iota U'\rangle = \langle \iota X', G'\xi\rangle = -\langle \iota X', \iota V'\rangle = 0.
\end{aligned}$$

and hence $F'\iota X', G'\iota X', H'\iota X' \in H_p(S^{4n+3})$, which completes the proof. $\square$

Therefore, the almost quaternionic structure $\{F, G, H\}$ can be induced on $T_{\pi(p)}$ (QP^n) and we set

$$(FX)^* = F'\iota X^*, \quad (GX)^* = G'\iota X^*, \quad (HX)^* = H'\iota X^*. \tag{11}$$

Next, using the Gauss formula (7) for the vertical vector field $\{U', V', W'\}$ and a horizontal vector fields X' of $T_p(S^{4n+3})$, we compute

$$\nabla^E_{X'}U' = \nabla'_{X'}U' + g'(A'X', U')\xi = \nabla'_{X'}U' + \langle X', U'\rangle\xi = \nabla'_{X'}U', \tag{12}$$

by similar computation for vector fields $\{V', W'\}$ we have

$$\begin{aligned} \nabla^E_{X'}V' &= \nabla'_{X'}V', \\ \nabla^E_{X'}W' &= \nabla'_{X'}W', \end{aligned} \tag{13}$$

where ∇^E denotes the Euclidean connection of E^{4n+4}, ∇' denotes the connection of S^{4n+3} and A' denotes the shape operator with respect to ξ. Now, using relations (10), (12) and (13) and the Weingarten formula (7), we conclude

$$\begin{aligned} \nabla'_{X'}U' &= -\nabla^E_{X'}F'\xi' = -(\nabla^E_{X'}F')\xi - F'\nabla^E_{X'}\xi \\ &= -(r(X')G'\xi - q(X')H'\xi) - F'\nabla^E_{X'}\xi \\ &= r(X')V' - q(X')W' + F'(\iota A'X') \\ &= r(X')V' - q(X')W' + F'\iota X' \end{aligned} \tag{14}$$

by similar computation for vector fields $\{V', W'\}$ we have

$$\begin{aligned} \nabla'_{X'}V' &= -\nabla^E_{X'}G'\xi' = -r(X')U' + p(X')W' + G'\iota X', \\ \nabla'_{X'}W' &= -\nabla^E_{X'}H'\xi' = q(X')U' - p(X')V' + H'\iota X'. \end{aligned} \tag{15}$$

Consequently, according to notation (11), relations (14) and (15) can be written as

$$\begin{aligned} \nabla'_{X^*}U' &= r(X^*)V' - q(X^*)W' + (FX)^*, \\ \nabla'_{X^*}V' &= -r(X^*)U' + p(X^*)W' + (GX)^*, \\ \nabla'_{X^*}W' &= q(X^*)U' - p(X^*)V' + (HX)^*. \end{aligned} \tag{16}$$

We note that, since by definition, the Lie derivative of a horizontal lift of a vector field with respect to a vertical vector field is zero, it follows

$$\begin{aligned} 0 &= L_{U'}X^* = [U', X^*] = \nabla'_{U'}X^* - \nabla'_{X^*}U', \\ 0 &= L_{V'}X^* = [V', X^*] = \nabla'_{V'}X^* - \nabla'_{X^*}V' \\ 0 &= L_{W'}X^* = [W', X^*] = \nabla'_{W'}X^* - \nabla'_{X^*}W' \end{aligned}$$

and using (16), we conclude

$$\begin{aligned}\nabla'_{U'}X^* &= r(X^*)V' - q(X^*)W' + (FX)^*,\\ \nabla'_{V'}X^* &= -r(X^*)U' + p(X^*)W' + (GX)^*,\\ \nabla'_{W'}X^* &= q(X^*)U' - p(X^*)V' + (HX)^*.\end{aligned} \tag{17}$$

We define a Riemannian metric g and a connection ∇ in QP^n respectively by

$$g(X,Y) = g'(X^*,Y^*), \tag{18}$$

$$\nabla_X Y = \pi(\nabla'_{X^*}Y^*). \tag{19}$$

Then $(\nabla'_X Y)^*$ is the horizontal part of $\nabla'_{X^*}Y^*$ and therefore

$$\begin{aligned}\nabla'_{X^*}Y^* = (\nabla'_X Y)^* &+ g'(\nabla'_{X^*}Y^*,U')U'\\ &+ g'(\nabla'_{X^*}Y^*,V')V' + g'(\nabla'_{X^*}Y^*,W')W'.\end{aligned} \tag{20}$$

Using relations (16) and (18), we compute

$$\begin{aligned}g'(\nabla'_{X^*}Y^*,U') &= -g'(Y^*,\nabla'_{X^*}U')\\ &= -g'(Y^*, r(X^*)V' - q(X^*)W' + (FX)^*) = -g(Y,FX),\\ g'(\nabla'_{X^*}Y^*,V') &= -g(Y,GX),\\ g'(\nabla'_{X^*}Y^*,V') &= -g(Y,HX),\end{aligned}$$

and, using (20), we conclude

$$\nabla'_{X^*}Y^* = (\nabla_X Y)^* - g(Y,FX)U' - g(Y,GX)V' - g(Y,HX)W'. \tag{21}$$

Proposition 2 ∇ *is the Levi-Civita connection for* g.

Proof. Let T be the torsion tensor field of ∇. Then we have

$$\begin{aligned}T(X,Y) &= \nabla_X Y - \nabla_Y X - [X,Y] = \pi(\nabla'_{X^*}Y^*) - \pi(\nabla'_{Y^*}X^*) - [\pi X^*, \pi Y^*]\\ &= \pi(\nabla'_{X^*}Y^* - \nabla'_{Y^*}X^* - [X^*,Y^*]) = \pi(T'(X^*,Y^*)) = 0,\end{aligned}$$

hence ∇ is torsion free. We now show that ∇ is a metric connection.

$$\begin{aligned}(\nabla_X g)(Y,Z) &= X(g(Y,Z)) - g(\nabla_X Y,Z) - g(Y,\nabla_X Z)\\ &= X^*(g'(Y^*,Z^*)) - g'((\nabla_X Y)^*,Z^*) - g'(Y^*,(\nabla_X Z)^*).\end{aligned}$$

Since Z^* is horizontal vector field, from relation (20), it follows that

$$\begin{aligned} g'((\nabla_X Y)^*, Z^*) &= g'(\nabla'_{X^*}Y^*, Z^*) - g'(\nabla'_{X^*}Y^*, U')g'(U', Z^*) \\ &\quad - g'(\nabla'_{X^*}Y^*, V')g'(V', Z^*) - g'(\nabla'_{X^*}Y^*, U')g'(U', Z^*) \\ &= g'(\nabla'_{X^*}Y^*, Z^*) \end{aligned}$$

and we compute

$$\begin{aligned} (\nabla_X g)(Y, Z) &= X^*(g'(Y^*, Z^*)) - g'(\nabla'_{X^*}Y^*, Z^*) - g'(\nabla'_{X^*}Z^*, Y^*) \\ &= (\nabla'_{X^*}g')(Y^*, Z^*), \end{aligned}$$

where we have used the fact that ∇' is the Levi-Civita connection for g'. Thus ∇ is the Levi-Civita connection for g and the proof is complete. □

Now, by (21), it follows

$$\begin{aligned} [X^*, Y^*] &= [X, Y]^* + g'([X^*, Y^*], U')U' \\ &\quad + g'([X^*, Y^*], V')V' + g'([X^*, Y^*], W')W' \\ &= [X, Y]^* + g'(\nabla'_{X^*}Y^* - \nabla'_{Y^*}X^*, U')U' \\ &\quad + g'(\nabla'_{X^*}Y^* - \nabla'_{Y^*}X^*, V')V' + g'(\nabla'_{X^*}Y^* - \nabla'_{Y^*}X^*, W')W' \\ &= [X, Y]^* + g'((\nabla'_X Y)^* - g(Y, FX)U' - g(Y, GX)V' \\ &\quad - g(Y, HX)W', U')U' - g'((\nabla_Y X)^* - g(FY, X)U' \\ &\quad - g(GY, X)V' - g(HY, X)W', U')U' + g'((\nabla_X Y)^* \\ &\quad - g(Y, FX)U' - g(Y, GX)V' - g(Y, HX)W', V')V' \\ &\quad - g'((\nabla_Y X)^* - g(FY, X)U' - g(GY, X)V' - g(HY, X)W', V')V' \\ &\quad + g'((\nabla_X Y)^* - g(Y, FX)U' - g(Y, GX)V' - g(Y, HX)W', W')W' \\ &\quad - g'((\nabla_Y X)^* - g(FY, X)U' - g(GY, X)V' - g(HY, X)W', W')W' \\ &= [X, Y]^* - 2g(Y, FX)U' - 2g(Y, GX)V' - 2g(Y, HX)W'. \end{aligned} \tag{22}$$

Consequently, using (16), (17), (21) and (22), the curvature tensor R of QP^n is calculated as follows:

$$\begin{aligned} R(X, Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \\ &= \pi\{\nabla'_{X^*}((\nabla_Y Z)^*) - \nabla'_{Y^*}((\nabla_X Z)^*) - \nabla'_{[X,Y]^*} Z^*)\} \\ &= \pi\{\nabla'_{X^*}(\nabla'_{Y^*}Z^* + g(Z, FY)U' + g(Z, GY)V' + g(Z, HY)W') \\ &\quad - \nabla'_{Y^*}(\nabla'_{X^*}Z^* + g(Z, FX)U' + g(Z, GX)V' + g(Z, HX)W') \\ &\quad - \nabla'_{[X^*,Y^*]+2g(Y,FX)U'+2g(Y,GX)V'+2g(Y,HX)W'} Z^*)\} \end{aligned}$$

$$
\begin{aligned}
&= \pi\{\nabla'_{X^*}\nabla'_{Y^*}Z^* + g(Z,FY)\nabla'_{X^*}U' + g(Z,GY)\nabla'_{X^*}V' \\
&\quad + g(Z,HY)\nabla'_{X^*}W' - \nabla'_{Y^*}\nabla'_{X^*}Z^* - g(Z,FX)\nabla'_{Y^*}U' \\
&\quad - g(Z,GX)\nabla'_{Y^*}V' - g(Z,HX)\nabla'_{X^*}W' - \nabla'_{[X^*,Y^*]}Z^* \\
&\quad - 2g(Y,FX)\nabla'_{U'}Z^* - 2g(Y,GX)\nabla'_{V'}Z^* - 2g(Y,HX)\nabla'_{W'}Z^*\} \\
&= \pi\{R'(X^*,Y^*)Z^* + g(Z,FY)(r(X^*)V' - q(X^*)W' + (FX)^*) \\
&\quad + g(Z,GY)(-r(X^*)U' + p(X^*)W' + (GX)^*) \\
&\quad + g(Z,HY)(q(X^*)U' - p(X^*)V' + (HX)^*) \\
&\quad + g(Z,FX)(r(Y^*)V' - q(Y^*)W' + (FY)^*) \\
&\quad + g(Z,GX)(-r(Y^*)U' + p(Y^*)W' + (GY)^*) \\
&\quad + g(Z,HX)(q(Y^*)U' - p(Y^*)V' + (HY)^*) \\
&\quad + 2g(Y,FX)(r(Z^*)V' - q(Z^*)W' + (FZ)^*) \\
&\quad + 2g(Y,GX)(-r(Z^*)U' + p(Z^*)W' + (GZ)^*) \\
&\quad + 2g(Y,HX)q(Z^*)U' - p(Z^*)V' + (HZ)^*\}.
\end{aligned}
$$

Since the curvature tensor R' of S^{4n+3} satisfies

$$R'(X^*,Y^*)Z^* = g'(Y^*,Z^*)X^* - g'(X^*,Z^*)Y^* = g(Y,Z)X^* - g(X,Z)Y^*, \quad (23)$$

we conclude that the curvature tensor of QP^n is given by

$$
\begin{aligned}
R(X,Y)Z = {} & g(Y,Z)X - g(X,Z)Y \\
& + g(FY,Z)FX - g(FX,Z)FY - 2g(FX,Y)FZ \\
& + g(GY,Z)GX - g(GX,Z)GY - 2g(GX,Y)GZ \\
& + g(HY,Z)HX - g(HX,Z)HY - 2g(HX,Y)HZ.
\end{aligned} \quad (24)
$$

4 Submanifolds of quaternionic manifolds

Let M be an n-dimensional submanifold of $QP^{\frac{n+p}{4}}$ and $\pi^{-1}(M)$ be the circle bundle over M which is compatible with the Hopf map

$$\pi : S^{n+p+3} \longrightarrow QP^{\frac{n+p}{4}}.$$

Then $\pi^{-1}(M)$ is a submanifold of S^{n+p+3}. The compatibility with the Hopf map is expressed by $\pi o \iota' = \iota o \pi$ where ι' and ι are the immersions of M into $QP^{\frac{n+p}{4}}$ and $\pi^{-1}(M)$ into S^{n+p+3}, respectively.

Let ξ_a, $a = 1, \dots, p$ be orthonormal normal local fields to M in $QP^{\frac{n+p}{4}}$ and

ξ^*_a be the horizontal lifts of ξ_a. Then ξ^*_a are mutually orthonormal normal local fields to $\pi^{-1}(M)$ in S^{n+p+3}. At each point $y \in \pi^{-1}(M)$ we compute

$$\begin{aligned}
&g^S(\iota' X^*, \xi^*_a) = g^S((\iota X)^*, \xi^*_a) = \overline{g}(\iota X, \xi_a) = 0,\\
&g^S(\iota' U, \xi^*_a) = g^S(U', \xi^*_a) = 0,\\
&g^S(\iota' V, \xi^*_a) = g^S(V', \xi^*_a) = 0,\\
&g^S(\iota' W, \xi^*_a) = g^S(W', \xi^*_a) = 0,\\
&g^S(\xi^*_a \xi^*_b) = \overline{g}(\xi_a, \xi_b) = \delta_{ab},
\end{aligned}$$

where g^S and $\overline{g}$ denote the Riemannian metric on S^{n+p+3} and $QP^{\frac{n+p}{4}}$, respectively. Here $U' = \iota' U, V' = \iota' V, W' = \iota' W$ are unit tangent vector field of S^{n+p+3} defined by relation (10).
Now, let $\nabla^S, \nabla', \overline{\nabla}$ and ∇ be the Riemannian connections of S^{n+p+3}, $\pi^{-1}(M)$, $QP^{\frac{n+p}{4}}$ and M, respectively. By means of the Gauss formula (7) and relations (4) and (21), we compute

$$\begin{aligned}
\nabla^S_{X^*} \iota' Y^* = \nabla^S_{X^*} (\iota Y)^* &= (\overline{\nabla}_X \iota Y)^* - \overline{g}(F\iota X, \iota Y)\iota' U\\
&\quad - \overline{g}(G\iota X, \iota Y)\iota' V - \overline{g}(H\iota X, \iota Y)\iota' W\\
&= (\iota \nabla_X Y + h(X, Y))^* - \overline{g}(\iota \varphi X, \iota Y) U'\\
&\quad - g(\iota \psi X, \iota Y) V' - g(\iota \theta X, \iota Y) W'\\
&= \iota' (\nabla_X Y)^* + (h(X, Y))^*\\
&\quad - g(\varphi X, Y) U' - g(\psi X, Y) V' - g(\theta X, Y) W'
\end{aligned} \tag{25}$$

where g is the metric on M. On the other hand, we also have

$$\begin{aligned}
\nabla^S_{X^*} \iota' Y^* = \iota' \nabla'_{X^*} Y^* + h'(X^*, Y^*) &= \iota'((\nabla_X Y)^* - g(\varphi X, Y) U\\
&\quad - g(\psi X, Y) V - g(\theta X, Y) W) + h'(X^*, Y^*)
\end{aligned} \tag{26}$$

where h and h' denote the second fundamental form of M and $\pi^{-1}(M)$, respectively. Comparing the vertical part and horizontal part of relations (25) and (26), we conclude

$$(h(X, Y))^* = h'(X^*, Y^*), \tag{27}$$

that is,

$$\sum_{a=1}^{p} g'(A'_a X^*, Y^*) \xi^*_a = (\sum_{a=1}^{p} g(A_a X, Y) \xi_a)^* = \sum_{a=1}^{p} g(A_a X, Y) \xi^*_a,$$

where A_a and A'_a are the shape operators with respect to normal vector fields ξ_a and ξ^*_a of M and $\pi^{-1}(M)$, respectively. Consequently, we have

$$g'(A'_a X^*, Y^*) = g(A_a X, Y), \quad (a = 1, ..., p).$$

Next, using the weingarten formula, we calculate $\nabla^S_{X^*}\xi^*_a$ as follows

$$\nabla^S_{X^*}\xi^*_a = -\iota' A'_a X^* + \nabla'^{\perp}_{X^*}\xi^*_a = -\iota' A'_a X^* + \sum_{b=1}^{p} s'_{ab}(X^*)\xi^*_b. \tag{28}$$

where $\nabla'^{\perp}$ is normal connection $\pi^{-1}(M)$ in S^{n+p+3}.
On the other hand, from relation (21), it follows

$$\begin{aligned}
\nabla^S_{X^*}\xi^*_a &= (\overline{\nabla}_X\xi_a)^* - \overline{g}(F\iota X, \xi_a)\iota' U - \overline{g}(G\iota X, \xi_a)\iota' V - \overline{g}(H\iota X, \xi_a)\iota' W \\
&= (-\iota A_a X + \nabla^{\perp}_X \xi_a)^* - \sum_{b=1}^{p}\{u^b(X)\overline{g}(\xi_a, \xi_b)U' \\
&\quad + v^b(X)\overline{g}(\xi_a, \xi_b)V' + w^b(X)\overline{g}(\xi_a, \xi_b)W'\} \\
&= -\iota'(A_a X)^* + \sum_{b=1}^{p}(s_{ab}(X^*)\xi_b)^* - u^a(X)U' \\
&\quad - v^a(X)V' - w^a(X)W',
\end{aligned} \tag{29}$$

where $\nabla^{\perp}$ is normal connection M in $QP^{\frac{n+p}{4}}$.
We have put

$$\begin{aligned}
F\iota x &= \iota\varphi X + \sum_{a=1}^{p} u^a(X)\xi_a, \\
G\iota x &= \iota\psi X + \sum_{a=1}^{p} v^a(X)\xi_a, \\
H\iota x &= \iota\theta X + \sum_{a=1}^{p} w^a(X)\xi_a.
\end{aligned} \tag{30}$$

Comparing relations (28) and (29), we obtain

$$\begin{aligned}
A'_a X^* &= (A_a X)^* + u^a(X)U' + v^a(X)V' + w^a(X)W' \\
&= (A_a X)^* + g(U_a, X)U' + g(V_a, X)V + g(W_a, X)W', \\
\nabla'^{\perp}_{X^*}\xi^*_a &= (\nabla^{\perp}_X \xi_a)^*,
\end{aligned} \tag{31}$$

that is, $s'_{ab}(X^*) = s_{ab}(X)^*$, where U_a, V_a, W_a are defined by

$$\begin{aligned} F\xi_a &= -U_a + \sum_{b=1}^{p} P_{1_{ab}}\xi_b, \\ G\xi_a &= -V_a + \sum_{b=1}^{p} P_{2_{ab}}\xi_b, \\ H\xi_a &= -W_a + \sum_{b=1}^{p} P_{3_{ab}}\xi_b. \end{aligned} \tag{32}$$

where, $\sum_{b=1}^{p} P_{i_{ab}}\xi_b = P_i\xi_a, \quad (i = 1, 2, 3)$. Now, we consider $\nabla^S_U \xi^*_a$ and using relations (17) and (32) imply

$$\begin{aligned} \nabla^S_U \xi^*_a &= (F\xi_a)^* = -\iota U_{a^*} + P_1\xi^*_a, \\ \nabla^S_V \xi^*_a &= (G\xi_a)^* = -\iota V_{a^*} + P_2\xi^*_a, \\ \nabla^S_W \xi^*_a &= (H\xi_a)^* = -\iota W_{a^*} + P_3\xi^*_a. \end{aligned} \tag{33}$$

On the other hand, from the Weingarten formula, it follows

$$\begin{aligned} \nabla^S_U \xi^*_a &= -\iota' A'_a U + \nabla'^{\perp}_U \xi^*_a = -\iota' A'_a U + \sum_{b=1}^{p} s'_{ab}(U)\xi^*_b, \\ \nabla^S_V \xi^*_a &= -\iota' A'_a V + \nabla'^{\perp}_V \xi^*_a = -\iota' A'_a V + \sum_{b=1}^{p} s'_{ab}(V)\xi^*_b, \\ \nabla^S_W \xi^*_a &= -\iota' A'_a W + \nabla'^{\perp}_W \xi^*_a = -\iota' A'_a W + \sum_{b=1}^{p} s'_{ab}(W)\xi^*_b. \end{aligned} \tag{34}$$

Consequently, using (33) and (34), we obtain

$$\begin{aligned} &A'_a U = U^*_a, \;\; A'_a V = V^*_a, \;\; A'_a W = W^*_a, \\ &s'_{ab}(U) = P_1, \;\; s'_{ab}(V) = P_2, \;\; s'_{ab}(W) = P_3, \end{aligned} \tag{35}$$

$$\begin{aligned} &\nabla'^{\perp}_U \xi^*_a = (FX)^* + \iota U_{a^*}, \\ &\nabla'^{\perp}_V \xi^*_a = (GX)^* + \iota V_{a^*}, \\ &\nabla'^{\perp}_W \xi^*_a = (HX)^* + \iota W_{a^*}. \end{aligned} \tag{36}$$

The first relations of (31) and (35), we get

$$\begin{aligned} g'(A'_a A'_b X^*, Y^*) = g(A_a A_b X, Y) &+ u^b(X)u^a(Y) \\ &+ v^b(X)v^a(Y) + w^b(X)w^a(Y), \end{aligned} \tag{37}$$

and especially,

$$g'(A'^2_a X^*, Y^*) = g(A^2_a X, Y) + u^a(X)u^a(Y) + v^a(X)v^a(Y) + w^a(X)w^a(Y), \tag{38}$$

For $x \in M$, let $\{e_1, \dots, e_n\}$be an orthonormal basis of $T_x M$ and y be a point of $\pi^{-1}(M)$ such that $\pi(y) = x$. We take an orthonormal basis $\{e^*_1, \dots, e^*_n, U, V, W\}$ of $T_y(\pi^{-1}(M))$. Then, using the first relations (35) and (39), we compute

$$\begin{aligned}\sum_{a=1}^{p} \operatorname{trace} A'^2_a &= \sum_{a=1}^{p} \Big\{ \sum_{i=1}^{n} g'(A'^2_a e^*_i, e^*_i) + g'(A'^2_a U, U) \\ &\quad + g'(A'^2_a V, V) + g'(A'^2_a W, W)\} \\ &= \sum_{a=1}^{p} \Big\{ \sum_{i=1}^{n} g(A'^2_a e_i, e_i) + u^a(e_i)u^a(e_i) \\ &\quad + v^a(e_i)v^a(e_i) + w^a(e_i)w^a(e_i) \\ &\quad + g'(A'_a U, A'_a U) + g'(A'_a V, A'_a V) + g'(A'_a W, A'_a W)\} \\ &= \sum_{a=1}^{p} \{\operatorname{trace} A^2_a + 2g(U_a, U_a) + 2g(V_a, V_a) + 2g(W_a, W_a)\},\end{aligned} \tag{39}$$

we conclude

Proposition 3 *Under the above assumptions, the following inequality*

$$\sum_{a=1}^{p} \operatorname{trace} A'^2_a \geq \sum_{a=1}^{p} \operatorname{trace} A^2_a$$

is always valid. The equality holds, if and only if M *is a* $\{F, G, H\}$*- invariant submanifold.*

Corollary 1 [8] M *is a totally geodesic submanifold if and only if relation* $A_\xi = 0$ *holds for any normal vector field* ξ *of* M*. Particularly,* M *is totally geodesic if and only if* $A_1 = \dots = A_p = 0$ *for an orthonormal frame field* $\xi_1, \dots, \xi_p$ *of* $T^\perp(M)$

Proposition 4 *Under the condition stated above, if* $\pi^{-1}(M)$ *is a totally geodesic submanifold of* S^{n+p+3}*, then* M *is a totally geodesic* $\{F, G, H\}$*- invariant submanifold.*

Proof. Since $\pi^{-1}(M)$ is a totally geodesic submanifold of S^{n+p+3}, using Corollary (26), it follows $A'_a = 0$. The first Relation (31) then implies $A_a = 0$ and $U_a = V_a = W_a = 0$, which, using relation (32), completes the proof. $\square$

Further, for the normal curvature of M in $QP^{\frac{n+p}{4}}$, using relation (24) and the second relation (9), we obtain

$$\begin{aligned}\overline{g}(R^{\perp}(X,Y)\xi_a,\xi_b) = g([A_a,A_b]X,Y) &+ u^b(X)u^a(Y)\\ &- u^a(X)u^b(Y) + v^b(X)v^a(Y) - v^a(X)v^b(Y)\\ &+ w^b(X)w^a(Y) - w^a(X)w^b(Y)\\ &- 2g(\varphi X,Y)P_1 - 2g(\psi X,Y)P_2 - 2g(\theta X,Y)P_3\end{aligned} \tag{40}$$

Therefore, if M be a totally geodesic submanifold of $\{F,G,H\}$- invariant submanifold, we conclude

$$\overline{g}(R^{\perp}(X,Y)\xi_a,\xi_b) = -2g(\varphi X,Y)P_1 - 2g(\psi X,Y)P_2 - 2g(\theta X,Y)P_3 \tag{41}$$

In this case the normal space $T_x^{\perp}(M)$ is also $\{F,G,H\}$- invariant and P_1, P_2, P_3 never vanish. We have thus proved

Proposition 5 *The normal curvature of a totally geodesic submanifold of* $\{F,G,H\}$- *invariant submanifold of a quaternionic projective space never vanishes.*

This proposition show that the normal connection of the quaternionic projective space which is immersed standardly in a higher dimensional quaternionic projective space not flat.

Finally, we give a relation between the normal curvature $R^{\perp}$ and $R'^{\perp}$ of M and $\pi^{-1}(M)$, respectively, where M is a n-dimensional submanifold of $QP^{\frac{n+p}{4}}$ and $\pi^{-1}(M)$ is the circle bundle over M which is compatible with the Hopf map π. Using relation (37), we obtain

$$\begin{aligned} g'([A'_a,A'_b]X^*,Y^*) &= g([A_a,A_b]X,Y) + u^b(X)u^a(Y) - u^a(X)u^b(Y)\\ &+ v^b(X)v^a(Y) - v^a(X)v^b(Y) + w^b(X)w^a(Y) - w^a(X)w^b(Y),\end{aligned}$$

and therefore, from the second relation (9), it follows

$$\begin{aligned} &-g^S(R'^S(\iota' X^*,\iota' Y^*)\xi^*_a,\xi^*_b) + g^S(R'^{\perp}(X^*,Y^*)\xi^*_a,\xi^*_b)\\ &\quad = -\overline{g}(\overline{R}(\iota X,\iota Y)\xi_a,\xi_b) + \overline{g}(R^{\perp}(X,Y)\xi_a,\xi_b) + u^b(X)u^a(Y) - u^a(X)u^b(Y)\\ &\qquad + v^b(X)v^a(Y) - v^a(X)v^b(Y) + w^b(X)w^a(Y) - w^a(X)w^b(Y).\end{aligned}$$

Using the expression (23) and (24) , for curvature of S^{n+p+3} and $QP^{\frac{n+p}{4}}(C)$, respectively and relations (30) and (32) imply

$$g^S(R^\perp(X^*,Y^*)\xi_a^*,\xi_b^*) = \overline{g}(R^\perp(X,Y)\xi_a,\xi_b) + 2g(\varphi X,Y)P_1 + 2g(\psi X,Y)P_2 + 2g(\theta X,Y)P_3$$

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Received: October 26, 2014