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A study of the absence of arbitrage opportunities without calculating the risk-neutral probability

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Abstract. In this paper, we establish the property of conditional full support for two processes: the Ornstein Uhlenbeck and the stochastic integral in which the Brownian Bridge is the integrator and we build the absence of arbitrage opportunities without calculating the risk-neutral probability.

1 Introduction

Stochastic portfolio theory is a section of mathematical finance. It is introduced by Fernholz [1, 2], and then further developed by Fernholz, Karatzas and Kardaras [3]. It analyses the results of portfolio by a new and different structure.

The conditional full support (CFS) is a simple condition on asset prices which specifies that from any time, the asset price path can continue arbitrarily close to any given path with positive conditional probability. The conditional full support's notion is introduced by Guasoni et al. (2008) [16] who proves

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that the fractional Brownian motion with arbitrary Hurst parameter has a desired property.

This latter is generalized by Cherny (2008) [17] who proves that any Brownian moving average satisfies the conditional full support condition. Then, the (CSF) property is established for Gaussian processes with stationary increments by Gasbarra (2011) [18].

Let's note that, by the main result of Guasoni et al. (2008) [16], the CFS generates the consistent price systems which admit a martingale measure. In 2014, Attila Herczegh et al. provide a new result on conditional full support in higher dimensions [19].

By the main result of Guasoni, Résonyi, and Schachermayer [16], the CFS generates the consistent price systems which admit a martingale measure.

M. S. Pakkanan in 2009 [7] presents conditions that imply the conditional full support for the process $Z := H + K * W$, where W is a Brownian motion and H is a continuous process.

This paper is organized as follows. Section 2 presents some basic concepts from stochastic portfolio theory and some results on consistent price system. In section 3, we present the conditions that imply the conditional full support (CFS) property for processes $Z := H+K*W$. In section 4, we establish our main result on the conditional full support for the processes: the Ornstein Uhlenbeck and the stochastic integral such that the Brownian Bridge is the integrator and we build the absence of arbitrage opportunities without calculating the risk-neutral probability in the case of existence of the consistent price systems. Finnaly we give a conclusion.

2 Reminder

2.1 Markets and portfolios

We shall place ourselves in a model M for a financial market of the form

$$\begin{aligned} dB(t) &= B(t)r(t)dt, \qquad B(0) = 1, \\ dS_i(t) &= S_i(t)\left(b_i(t)dt + \sum_{\nu=1}^{d} \sigma_{i\nu}(t)dW_\nu(t)\right), \\ S_i(0) &= s_i > 0;\ i = 1, \dots, n, \end{aligned} \tag{1}$$

consisting of a money-market $B(.)$ and of n stocks, whose prices $S_1(.); \dots; S_n(.)$ are driven by the d-dimensional Brownian motion $W(.) = W_1(.); \dots; W_d(.))'$ with $d \geq n$.

The following notations are adopted; The interest-rate process $r(.)$ for the money-market, the vector-valued process $b(.) = (b_1(.);\ldots;b_n(.))'$ of rates of return for the various stocks, and the (n*d)-matrix-valued process $\sigma(.) = (\sigma_{i\nu}(.))_{1<i<n,1\leq\nu\leq d}$ of stock-price volatilities.

Definition 1 *A portfolio $\pi(.) = (\pi_1(.),\ldots,\pi_n(.))'$ is an $\mathbb{F}$-progressively measurable process, bounded uniformly in (t,w), with values in the set*

$$\bigcup_{k\in\mathbb{N}}\{(\pi_1,\ldots,\pi_n)\in\mathbb{R}^n|\pi_1^2+\ldots+\pi_n^2\leq k^2, \pi_1+\ldots+\pi_n=1\}.$$

The Market Portfolio

The stock price $S_i(t)$ can be interpreted as the capitalization of the i^{th} company at time t, and the quantities

$$S(t) = S_1(t)+\ldots+S_n(t) \quad \text{and} \quad \mu_i(t) = \frac{S_i(t)}{S(t)}, \quad i=1,\ldots,n \tag{2}$$

as the total capitalization of the market and the relative capitalizations of the individual companies, respectively.

Clearly, $0<\mu_i(t)<1$, $\forall\, i=1,\ldots,n$ and $\sum_{i=1}^n \mu_i(t) = 1$.

The resulting wealth process $V^{w,\mu}(.)$ satisfies

$$\frac{dV^{w,\mu}(t)}{V^{w,\mu}(t)} = \sum_{i=1}^n \mu_i(t)\frac{dS_i(t)}{S_i(t)} = \sum_{i=1}^n \frac{dS_i(t)}{S(t)} = \frac{dS(t)}{S(t)}.$$

2.2 Conditional full support

Definition 2 *Let $\mathcal{O}\subset\mathbb{R}^n$ be an open set and $(S(t))_{t\in[0,T]}$ be a continuous adapted process taking values in $\mathcal{O}$. We say that S has conditional full support in $\mathcal{O}$ if for all $t\in[0,T]$ and for all open set $G\subset C([0,T],\mathcal{O})$,*

$$\mathbb{P}(S\in G|\mathfrak{F}_t)>0, \quad \text{a.s. on the event} \quad (S|_{[0,t]}\in\{g|_{[0,t]}:g\in G\}). \tag{3}$$

We will also say that S has full support in $\mathcal{O}$, or simply full support when $\mathcal{O}=\mathbb{R}^n$, if (3) holds for $t=0$ and for all open subset of $C([0,T],\mathcal{O})$.

Recall also, the notion of consistent price system.

Definition 3 *Let $\varepsilon > 0$. An $\varepsilon-\mathrm{consistent}$ price system to S is a pair $(\widetilde{S}, \mathbf{Q})$, where $\mathbf{Q}$ is a probability measure equivalent to $\mathbf{P}$ and $\widetilde{S}$ is a $\mathbf{Q}-\mathrm{martingale}$ in the filtration $\mathfrak{F}$, such that*

$$\frac{1}{1+\varepsilon} \leq \frac{\widetilde{S}_i(t)}{S_i(t)} \leq 1+\varepsilon, \quad \text{almost surely for all } t \in [0,T] \text{ and } i = 1, \dots, n.$$

Note that $\widetilde{S}$ is a martingale under $\mathbf{Q}$, hence we may asuume that it is càdlàg, but it is not required in the definition that $\widetilde{S}$ is continuous.

Theorem 1 [14] *Let $\mathcal{O} \subset (0,\infty)^n$ be the open set defined by*

$$\mathcal{O} = \mathcal{O}(\delta) = \left\{ x \in (0,\infty)^n : \max_j \frac{x_j}{x_1 + \ldots + x_n} < 1 - \delta \right\} \tag{4}$$

and assume that the price process takes values and has conditional full support in $\mathcal{O}$. Then for any $\varepsilon > 0$ there is an $\varepsilon-\mathrm{consistent}$ price system $(\widetilde{S}, \mathbf{Q})$ such that $\widetilde{S}$ takes values in $\mathcal{O}$.

To check the condition of Theorem 2.1 we apply the next Theorem. In comparison to the existing results, we mention that our findings seem to be new in the sense that we do not assume that our process solves a stochastic differential equation as it is done in Stroock and Varadhan [11] and it is not only for one dimensional processes as it is in Pakkanen [7].

Theorem 2 [14] *Let X be a n-dimensional Itô proces on $[0,T]$, such that*

$$dX_i(t) = \mu_i(t)dt + \sum_{\nu=1}^{n} \sigma_{i\nu}(t) dW_\nu(t).$$

Assume that $|\mu|$ is bounded and σ satisfies

$$\varepsilon|\xi|^2 \leq |\sigma'(t)\xi|^2 \leq M|\xi|^2, \quad \text{a.s. for all } t \in [0,T] \text{ and } \xi \in \mathbb{R}^n \text{ and } \varepsilon, M > 0.$$

Then X has conditional full support.

2.3 Consistent Price System and Conditional Full support

Theorem 3 [14] *Let $\mathcal{O} \subset \mathbb{R}^n$ be an open set and $(S(t))_{t\in[0,T]}$ be an $\mathcal{O}$-valued, continuous adapted process having conditional full support in $\mathcal{O}$.*

Besides, let $(\varepsilon_t)_{t\in[0,T]}$ *be a continuous positive process, that satisfies*

$$|\varepsilon_t - \varepsilon_s| \le L_s \sup_{s\le u\le t}|S(u) - S(s)|, \quad \text{for all } 0 \le s \le t \le T,$$

with some progressively measurable finite valued $(L_s)_{s\in[0,T]}$.

Then S *admits an* ε*-consistent price system in the sense that, there is an equivalent probability* Q *on* $\mathfrak{F}_t$ *and a process* $(\widetilde{S}(t))_{t\in[0,T]}$ *taking values in* $\mathcal{O}$ *such that* $\widetilde{S}$ *is* Q *martingale, bounded in* $L^2(Q)$ *and finally* $|S(t) - \widetilde{S}(t)| \le \varepsilon_t$ *almost surely for all* $t \in [0, T]$.

Lemma 1 [14] *Under the assumption of theorem 3.1 there is a sequence of stopping times* $(\tau_n)_{n\ge1}$*, a sequence of random variables* $(X_n)_{n\ge0}$ *and an equivalenty* Q *such that*

1. $\tau_0 = 0$*,* (τ_n) *is increasing and* $\bigcup_n\{\tau_n = T\}$ *has full probability,*
2. $(X_n)_{n\ge0}$ *is a* Q *martingale in the discrete time filtration* $(\mathfrak{g}_n = \mathfrak{F}_{\tau_n})_{n\ge0}$*, bounded in* $L^2(Q)$*,*
3. *if* $\tau_n \le t \le \tau_{n+1}$ *then* $|S_t - X_{n+1}| \le \varepsilon_t$.

Corollary 1 [14] *Assume that the continuous adapted process* S *evoling in* $\mathcal{O}$ *has conditional full support in* $\mathcal{O}$*. Let* τ *be a stopping time and denote by* $Q_{S|\mathfrak{F}_\tau}$ *the regular version of the conditional distribution of S given* $\mathfrak{F}_\tau$*.*

Then the support of the random measure $Q_{S|\mathfrak{F}_\tau}$ *is*

$$\operatorname{supp} Q_{S|\mathfrak{F}_\tau} = \left\{ g \in C([0,T], \mathcal{O}) : g|_{[0,\tau]} = S|_{[0,\tau]} \right\}, \qquad \text{almost surely.}$$

3 Conditional full support for stochastic integrals

We shall establish the CFS for processes of the form

$$Z_t := H_t + \int_0^t k_s dW_s, \qquad t \in [0,T],$$

where H is a continuous process, the integrator W is a Brownian motion, and the integrand k satisfies some varying assumptions (to be clarified below). We focus on three cases, each of which requires a separate treatment (see [7]).

First, we study the case:

(1) **Independent integrands and Brownian integrators**

Theorem 4 [7] *Let us define*

$$Z_t := H_t + \int_0^t k_s dW_s, \qquad t \in [0, T]$$

Suppose that

- $(H_t)_{t \in [0,T]}$ *is a continuous process*
- $(k_t)_{t \in [0,T]}$ *is a measurable process s.t.* $\int_0^T K_s^2 ds < \infty$
- $(W_t)_{t \in [0,T]}$ *is a standard Brownian motion independent of* H *and* k.

If we have

$$meas(t \in [0, T] : k_t = 0) = 0 \qquad \mathbf{P} - a.s$$

then Z *has CFS.*

As an application of this result, we show that several popular stochastic volatility models have the CFS property.

Application to stochastic volatility model:

Let us consider the price process $(P_t)_{t \in [0,T]}$ in $\mathbb{R}_+$ given by :

$$dP_t = P_t(f(t, V_t)dt + \rho g(t, V_t)dB_t + \sqrt{1 - \rho^2} g(t, V_t)dW_t,$$

$P_0 = p_0 \in \mathbb{R}_+$, where

(a) $f, g \in C([0, T] \times \mathbb{R}^d, \mathbb{R})$,

(b) (B, W) is a planar Brownian motion,

(c) $\rho \in (-1, 1)$,

(d) V is a (measurable) process in $\mathbb{R}^d$ s.t. $g(t, V_t) \neq 0$ a.s. for all $t \in [0, T]$,

(e) (B, V) is independent of W.

Write using Itô's formula:

$$\log P_t = \underbrace{\log P_0 + \int_0^t (f(s,V_s) - \frac{1}{2} g(s,V_s)^2) ds + \rho \int_0^t g(s,V_s) dB_s}_{=H_t}$$

$$\underbrace{+\sqrt{1-\rho^2} \int_0^t g(s,V_s) dW_s}_{=K_s}.$$

Since W is independent from B and V, the previous Theorem implies that $\log P$ has CFS, and from the next remark, it follows that P has CFS.

Remark 1 *If $I \subset \mathbb{R}$ is an open interval and $f : \mathbb{R} \longrightarrow I$ is a homeomorphism, then $g \longmapsto f \circ g$ is a homeomorphism between $C_x([0,T])$ and $C_{f(x)}([0,T], I)$.*

Hence, for $f(X)$, understood as a process on I, we have

$$f(X) \quad \text{has} \quad \mathbb{F} - \text{CFS} \Longleftrightarrow X \quad \text{has} \quad \mathbb{F} - \text{CFS}. \tag{5}$$

Next, we weaken the assumption about independence and consider the second case:

(2) **Progressive integrands and Brownian integrators**

Remark 2 *In general, the assumption about independence between W and (H,k) is necessary.*

Namely, if e.g.

$$H_t = 1; k_t := e^{W_t - \frac{1}{2}t}; t \in [0,T],$$

then $Z = k = \xi(W)$, the Dolans exponential of W, which is strictly positive and thus does not have CFS if the process is considered in $\mathbb{R}$.

Theorem 5 [7] *Suppose that*

- $(X_t)_{t\in[0,T]}$ *and* $(W_t)_{t\in[0,T]}$ *are continuous process,*
- h *and* k *are progressive* $[0,T] \times C([0,T])^2 \longrightarrow \mathbb{R}$,
- ε *is a random variable,*
- *and* $\mathcal{F}_t = \sigma\{\varepsilon, X_s, W_s : s \in [0,t]\}, t \in [0,T]$

If W *is an* $(\mathcal{F}_t)_{t\in[0,T]}$*-Brownian motion and*

- $E[e^{\lambda\int_0^T k_s^{-2}ds}] < \infty$ *for all* $\lambda > 0$,
- $E[e^{2\int_0^T k_s^{-2}h_s^2 ds}] < \infty$ *and*
- $\int_0^T k_s^2 ds \leq \overline{K}$ *a.s for some constant* $\overline{K} \in (0, \infty)$,

then the process

$$Z_t = \varepsilon + \int_0^t h_s ds + \int_0^t k_s dw_s, \qquad t \in [0, T]$$

has CFS.

(3) **Independent integrands and general integrators**

Since the Brownian motion has CFS, one might wonder if the previous results can be generalized to the case where the integrator is merely a continuous process with CFS. While the proofs of these results use quite heavily methods specific to Brownian motion (martingales, time changes), so in the case of independent integrands of finite variations, we are able to prove this conjecture.

Theorem 6 [7] *Suppose that*

- $(H_t)_{t\in[0,T]}$ *is a continuous process,*
- $(k_t)_{t\in[0,T]}$ *is a process of finite variation, and*
- $X = (X_t)_{t\in[0,T]}$ *is a continuous process independent of* H *and* k.

Let us define

$$Z_t := H_t + \int_0^t k_s dX_s, \qquad t \in [0, T].$$

If X *has CFS and*

$$\inf_{t\in[0,T]} |k_t| > 0 \qquad \mathbf{P} - \text{a.s.},$$

then Z *has CFS.*

4 Main result

In this part, we will use the following theorem to demonstrate the absence of arbitration without calculating the risk-neutral probability for the two models below.

Theorem 7 [4, theorem 1.2] *Let* (X_t) *be an* $\mathbb{R}^d_+$*-valued, continuous adapted process satisfying (CFS); then* X *admits an* ε*-consistent pricing system for all* $\varepsilon > 0$.

4.1 Ornstein-Uhlenbeck process driven by Brownian motion

The (one-dimensional) Gaussian Ornstein-Uhlenbeck process $X = (X_t)_t \geq 0$ can be defined as the solution to the stochastic differential equation (SDE)

$$dX_t = \theta(\mu - X_t)dt + \sigma dW_t \qquad t > 0.$$

Where we see

$$X_t = X_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \int_0^t \sigma e^{\theta(s-t)}\, dW_s. \qquad t \geq 0.$$

It is readily seen that X_t is normally distributed. We have

$$X_t = \underbrace{X_0 e^{-\theta t} + \mu(1 - e^{-\theta t})}_{H_t} + \int_0^t \underbrace{\sigma e^{\theta(s-t)}}_{K_s} dW_s. \qquad t \geq 0. \tag{6}$$

To establish the property of CFS for this process, the conditions of theorem 3.1 will be applied.

The processes (H_s) and (K_s) in (6) satisfy

1. Process (H_s) is a continuous process,
2. (K_s) is a measurable process such that $\int_0^T K_s^2 ds < \infty$, and
3. (W_t) is a standard Brownian motion independent of H and K.

Consequently, the process (X_t) has the property of CFS and there are the consistent price systems which can be seen as generalization of equivalent martingale measures.

This shows that this price process doesn't admit arbitrage opportunities under arbitrary small transaction and using it, we guarantee no-arbitrage without calculating the risk-neutral probability.

4.2 Independent integrands and Brownian Bridge integrators.

To state our main result for the application of CFC in which the Brownian Bridge is the integrator, we need to recall some facts of Brownian bridge.

Let us start with a Brownian motion $B = (B_t, t \geq 0)$ and its natural filtration $\mathbb{F}^B$. Define a new filtration as $\mathbb{G} = (\mathcal{G}_t, t \geq 0)$ with $\mathcal{G}_t = \mathcal{F}_t^{(B_1)} = \mathcal{F}_t^B \vee \sigma(B_1)$. In this filtration, the process $(B_t, t \geq 0)$ is no longer a martingale. It is easy to be convinced of this by looking at the process $(E(B_1 \mid \mathcal{F}_t^{(B_1)}), t \leq 1)$: this process is identically equal to B_1, not to B_t, hence $(B_t; t \geq 0)$ is not a $\mathbb{G}$-martingale. However, $(B_t, t \geq 0)$ is a $\mathbb{G}$-semi-martingale, as follows from the next proposition 1.

In general, if $\mathbb{H} = (\mathcal{H}_t, t \geq 0)$ is a filtration larger than $\mathbb{F} = (\mathcal{F}_t, t \geq 0)$, i.e., $\mathcal{F}_t \subset \mathcal{H}_t, \forall t \geq 0$ (we shall write $\mathbb{F} \subset \mathbb{H}$), it is not true that an $\mathbb{F}$-martingale remains a martingale in the filtration $\mathbb{H}$. It is not even true that $\mathbb{F}$-martingales remain $\mathbb{H}$-semi-martingales.

Before giving this proposition, we recall the definition of Brownian bridge.

Definition 4 *The Brownian bridge* $(b_t; 0 \leq t \leq 1)$ *is defined as the conditioned process* $(B_t; t \leq 1 | B_1 = 0)$.

Note that $B_t = (B_t - tB_1) + tB_1$ *where, from the Gaussian property, the process* $(B_t - tB_1; t \leq 1)$ *and the random variable* B_1 *are independent. Hence*

$$(b_t; 0 \leq t \leq 1) \stackrel{law}{=} (B_t - tB_1;\ 0 \leq t \leq 1).$$

The Brownian bridge process is a Gaussian process, with zero mean and covariance function $s(1-t); s \leq t$. *Moreover, it satisfies* $b_0 = b_1 = 0$.

Proposition 1 [15] *Let* $\mathcal{F}_t^{(B_1)} = \cap_{\epsilon > 0} \mathcal{F}_{t+\epsilon} \vee \sigma(B_1)$. *The process*

$$\beta_t = B_t - \int_0^{t \wedge 1} \frac{B_1 - B_s}{1 - s} ds$$

is an $\mathbb{F}^{(B_1)}$*-martingale, and an* $\mathbb{F}^{(B_1)}$ *Brownian motion. In other words,*

$$B_t = \beta_t - \int_0^{t \wedge 1} \frac{B_1 - B_s}{1 - s} ds$$

is the decomposition of B *as an* $\mathbb{F}^{(B_1)}$*-semi-martingale.*

Example of application: The following example was studied by Monique Jeanblanc et al. [15], we will later introduce our approach to this application, this approach is based on the conditional full support property. M.Jeanblanc et al. study within the problem occurring in insider trading: existence of arbitrage using strategies adapted w.r.t. the large filtration.

Our approach is to prove the existence of no arbitrage in the case $0 \le t < 1$ without calculating the dynamics of wealth and risk neutral probability.

Let
$$dS_t = S_t(\mu dt + \sigma db_t),$$
where μ and σ are constants and S_t defines the price of a risky asset. Assume that the riskless asset has a constant interest rate r.

The wealth of an agent is
$$dX_t = rX_t dt + \widehat{\pi}_t(dS_t - rS_t dt) = rX_t dt + \pi_t \sigma X_t(dW_t + \theta dt); \quad X_0 = x,$$
where $\theta = \frac{\mu - r}{\sigma}$ and $\pi = (\widehat{\pi} S_t / X_t)$ assumed to be an $\mathbb{F}^B$-adapted process.

Here, $\widehat{\pi}$ is the number of shares of the risky asset, and π the proportion of wealth invested in the risky asset. It follows that
$$\ln(X_T^{\pi,x}) = \ln x + \int_0^T (r - \frac{1}{2}\pi_s^2\sigma^2 + \theta\pi_s\sigma) ds + \int_0^T \sigma\pi_s dW_s$$
Then,
$$E(\ln(X_T^{\pi,x})) = \ln x + \int_0^T E\left(r - \frac{1}{2}\pi_s^2\sigma^2 + \theta\pi_s\sigma\right) ds$$
The solution of $\max E(\ln(X_T^{\pi,x}))$ is $\pi_s = \frac{\theta}{\sigma}$ and
$$\sup E(\ln(X_T^{\pi,x}) = \ln x + T\left(r + \frac{1}{2}\theta^2\right)$$
Note that, if the coefficients r, μ and σ are $\mathbb{F}$-adapted, the same computation leads to
$$\sup E(\ln(X_T^{\pi,x}) = \ln x + \int_0^T E\left(r_t + \frac{1}{2}\theta_t^2\right) dt,$$
where $\theta_t = \frac{\mu_t - r_t}{\sigma_t}$.

We now enlarge the filtration with S_1.

In the enlarged filtration, setting, for $t < 1, \alpha_t = \frac{B_1 - B_t}{1-t}$, the dynamics of S are
$$dS_t = S_t((\mu + \sigma\alpha_t)dt + \sigma d\beta_t),$$
and the dynamics of the wealth are
$$dX_t = rX_t dt + \pi_t\sigma X_t(d\beta_t + \widetilde{\theta}_t dt), \quad X_0 = x$$

with $\widetilde{\theta}_t = \frac{\mu-r}{\sigma} + \alpha_t$.
The solution of $\max \mathsf{E}(\ln(X_T^{\pi,x}))$ is $\pi_s = \frac{\widetilde{\theta}_s}{\sigma}$.

Then, for $T < 1$,

$$\ln(X_T^{\pi,x,*}) = \ln x + \int_0^T (r + \frac{1}{2}\widetilde{\theta}_s^2) ds + \int_0^T \sigma\pi_s d\beta_s$$

$$\begin{aligned} \mathsf{E}(\ln(X_T^{\pi,x,*})) &= \ln x + \int_0^T (r + \frac{1}{2}(\theta^2 + \mathsf{E}(\alpha_s^2) + 2\theta \mathsf{E}(\alpha_s)) ds \\ &= \ln x + (r + \frac{1}{2}\theta^2)T + \frac{1}{2}\int_0^T \mathsf{E}(\alpha_s^2) ds, \end{aligned}$$

where we have used the fact that $\mathsf{E}(\alpha_t) = 0$ (if the coefficients r, μ and σ are $\mathbb{F}$-adapted, α is orthogonal to $\mathcal{F}_t$, hence $\mathsf{E}(\alpha_t\theta_t) = 0$).

Let

$$\begin{aligned} V^{\mathbb{F}}(x) &= \max \mathsf{E}(\ln(X_T^{\pi,x})); \pi \text{ is } \mathbb{F} \text{ admissible} \\ V^{\mathbb{G}}(x) &= \max \mathsf{E}(\ln(X_T^{\pi,x})); \pi \text{ is } \mathbb{G} \text{ admissible} \end{aligned}$$

Then $V^{\mathbb{G}}(x) = V^{\mathbb{F}}(x) + \frac{1}{2}\mathsf{E}\int_0^T \alpha_s^2 ds = V^{\mathbb{F}}(x) - \frac{1}{2}\ln(1-T)$.

If $T = 1$, the value function is infinite: there is an arbitrage opportunity and there exists no an e.m.m. such that the discounted price process $(e^{-rt}S_t, t \leq 1)$ is a $\mathbb{G}$-martingale. However, for any $\epsilon \in]0;1]$, there exists a uniformly integrable $\mathbb{G}$-martingale L defined as

$$dL_t = \frac{\mu - r + \sigma\sigma_t}{\sigma} L_t d\beta_t, t \leq 1-\epsilon, \quad L_0 = 1,$$

such that, setting $d\mathbb{Q}\,|_{\mathcal{G}_t} = L_t d\mathbb{P}\,|_{\mathcal{G}_t}$, the process $(e^{-rt}S_t; t \leq 1-\epsilon)$ is a $(\mathbb{Q}, \mathbb{G})$-martingale.

This is the main point in the theory of insider trading where the knowledge of the terminal value of the underlying asset creates an arbitrage opportunity and this is effective at time 1.

Our approach to this example: We consider the previous example. Let

$$dS_t = S_t(\mu dt + \sigma db_t),$$

The standard Brownian bridge $b(t)$ is a solution of the following stochastic equation.

$$\begin{aligned} db_t &= -\frac{b_t}{1-t} dt + dW_t; \qquad 0 \leq t < 1 \\ b_0 &= 0. \end{aligned} \tag{7}$$

The solution of the above equation is

$$b_t = (1-t)\int_0^t \frac{1}{1-s} dW_s,$$

We may now verify that S has CFS.

By positivity of S, Itô's formula yields

$$\log S_t = \log S_0 + \left\{ \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma\left(1-t\right) \int_0^t \frac{1}{1-s} dW_s \right\}, \qquad 0 \le t < 1.$$

We have

$$\log S_t = \underbrace{\log S_0 + \left(\mu - \frac{\sigma^2}{2}\right) t}_{=:H_t} + \int_0^t \underbrace{\sigma\left(1-t\right)\frac{1}{1-s}}_{=:K_s} dW_s, \qquad 0 \le t < 1.$$

1. (H_t) is a continuous process,
2. $(K_s) = \sigma(1-t)\frac{1}{1-s}$ is a measurable process s.t. $\int_0^t K_s^2 ds < \infty$,
3. (W_t) is a standard Brownian motion independent of H and K,

which clearly satisfy the assumptions of theorem (3.1) and $\log S_t$ has CFS, then S has CFS for $0 \le t < 1$ and there is the consistent price systems and this is a martingale. Using it, we guarantee no-arbitrage without calculating the risk-neutral probability.

5 Conclusion

In this paper, we have investigated the conditional full support for two processes, the Ornstein Uhlenbeck and the Stochastic integral in which the Brownian Bridge is the integrator, and we have also built the absence of arbitrage opportunities without calculating the risk-neutral probability in the existence of the consistent price systems which admit a martingale measure.

Prospects: In mathematical finance, the CoxIngersollRoss model (or CIR model) describes the evolution of interest rates. It is a type of "one factor model" (short rate model) as it describes interest rate movements as driven by only one source of market risk. The model can be used in the valuation of interest rate derivatives. It was introduced in 1985 by John C. Cox, Jonathan E. Ingersoll and Stephen A. Ross as an extension of the Vasicek model. The CIR model specifies that the instantaneous interest rate follows the stochastic differential equation, also named the CIR Process:

$$dX_t = \theta(\mu - X_t)dt + \sigma\sqrt{X_t}dW_t \qquad t > 0,$$

where (W_t) is a Wiener process and θ, μ and σ are the parameters. The parameter θ corresponds to the speed of adjustment, μ to the mean and σ to volatility. The drift factor $\theta(\mu - X_t)$ is exactly the same as in the Vasicek model. It ensures a mean reversion of the interest rate towards the long run value μ, with speed of adjustment governed by the strictly positive parameter θ.

As prospects, we establish the condition of CFS for the Cox-Ingersoll-Ross model.

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References

[1] R. Fernholz, On the diversity of equity markets, *J. Math. Econom.*, **31** (1999), 393417, doi: 10.1016/S0304-4068(97)00018-9. MR1684221 (2000e:91072).

[2] R. Fernholz, Equity portfolios generated by functions of ranked market weights, *Finance Stoch.*, **5** (2001), 469486, doi: 10.1007/s007800100044. MR1861997 (2002h:91052).

[3] R. Fernholz, I. Karatzas, C. Kardaras, Diversity and relative arbitrage in equity markets, *Finance Stoch.*, **9** (2005), 127. doi: 10.1007/s00780-004-0129-4. MR2210925 (2006k:60123).

[4] P. Guasoni, M. Rasonyi, W. Schachermaye, Consistent price systems and face-lifting pricing under transaction costs, *Ann. Appl. Probab.*, **18** (2)(2008), 491–520.

[5] D. Gasbarra, T. Sottinen, H. van Zanten, Conditional full support of Gaussian processes with stationary increments, Preprint 487, University of Helsinki, Department of Mathematics and Statistics, February 17, 2011.

[6] C. Bender, T. Sottinen, E. Valkeila, Pricing by hedging and no-arbitrage beyond semimartingales, *Finance Stoch.*, **12** (4) (2008), 441–468.

[7] M. S. Pakkanen, Stochastic integrals and conditional full support, arXiv:0811.1847 (2009).

[8] L. C. G. Rogers, Equivalent martingale measures and no-arbitrage, *Stochastics Rep.*, **51** (1994), 41-49.

[9] R. Fernholz, On the diversity of equity markets, *J. Math. Econom.*, **31** (1999), 393417, doi

[10] G. Kallianpur, Abstract Wiener processes and their reproducing kernel Hilbert spaces, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, 17 (1971), 113–123.

[11] D. W. Stroock, S. R. S. Varadhan, *On the support of diffusion processes with applications to the strong maximum principle*, in Proc. Sixth Berkeley Symp. Math. Statist. and Probab. III: Probability Theory, Berkeley, Calif., 1972, Univ. California Press, 333–359.

[12] A. Millet, M. Sanz-Sol, A simple proof of the support theorem for diffusion pro- cesses, in Sminaire de Probabilities, XXVIII, vol. 1583 of Lecture Notes in Math., Springer, Berlin, 1994, pp. 36–48.

[13] A. Cherny, Brownian moving averages have conditional full support, *Ann. Appl. Probab.*, **18** (2008), 1825–1830.

[14] A. Herczegh, V. Prokaj, M. Rasonyi, *Diversity and no arbitrage*, arXiv:1301.4173v1 [q-fin.PM] 17 Jan 2013.

[15] M. Jeanblanc, Enlargements of Filtrations, June 9, 2010.

[16] Guasoni, Paolo and Rásonyi, Miklós and Schachermayer, Walter, Consistent price systems and face-lifting pricing under transaction costs, *Ann. Appl. Probab.*, (2008).

[17] A. Cherny, Brownian moving average have conditional full support, *The Annals of Applied Probability*, (2008).

[18] D. Gasbarra, T. Sottinen, H. Van Zanten, Conditional full support of Gaussian processes with stationary increments. *J. Appl. Prob.*, (2011).

[19] A. Herczegh, V. Prokaj, M. Rasonyi, Diversity and no arbitrage, *Stochastic Analysis and Applications*, **32** (5) (2014), 876–888.

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