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Approximation properties of (p, q)-Bernstein type operators

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Abstract. We introduce a new generalization of the q-Bernstein operators involving (p, q)-integers, and we establish some direct approximation results. Further, we define the limit (p, q)-Bernstein operator, and we obtain its estimation for the rate of convergence. Finally, we introduce the (p, q)-Kantorovich type operators, and we give a quantitative estimation.

1 Introduction

The applications of q-calculus in the field of approximation theory have led to the discovery of new generalizations of the Bernstein operators. The first generalization involving q-integers was obtained by Lupaş [7] in 1987. Ten years later Phillips [12] gave another generalization of the Bernstein operators introducing the so-called q-Bernstein operators. In comparison with Phillips' generalization, the Lupaş' generalization gives rational functions rather than polynomials. Nowadays, q-Bernstein operators form an area of an intensive research. A survey of the obtained results and references in this area during the first decade of study can be found in [11]. After that several well-known positive linear operators and other new operators have been generalized to their q-variants, and their approximation behavior have been studied (see e.g. [1] and [3]).

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The (p, q)-calculus is a further new generalization of the q-calculus, its basic definitions and some properties may be found in the papers [6], [13], [14], [15]. The (p, q)-integers $[n]_{p,q}$ are defined by

$$[n]_{p,q} = \frac{p^n - q^n}{p - q},$$

where $n = 0, 1, 2, \ldots$ and $0 < q < p \leq 1$. For $p = 1$, we recover the well-known q-integers (see [5]). Obviously

$$[n]_{p,q} = p^{n-1}[n]_{q/p}. \tag{1}$$

The (p, q)-factorials $[n]_{p,q}!$ are defined by

$$[n]_{p,q}! = \begin{cases} [1]_{p,q}[2]_{p,q}\ldots[n]_{p,q}, & \text{if} \quad n \geq 1 \\ 1, & \text{if} \quad n = 0, \end{cases}$$

and the (p, q)-binomial coefficients are given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}, \quad 0 \leq k \leq n.$$

Further, we set

$$(a - b)^n_{p,q} = \begin{cases} (a-b)(pa-qb)\ldots(p^{n-1}a - q^{n-1}b), & \text{if} \quad n \geq 1 \\ 1, & \text{if} \quad n = 0. \end{cases}$$

By simple computations, using (1), we get

$$[n]_{p,q}! = p^{n(n-1)/2}[n]_{q/p}!, \tag{2}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = p^{\{n(n-1)-k(k-1)-(n-k)(n-k-1)\}/2} \begin{bmatrix} n \\ k \end{bmatrix}_{q/p} \tag{3}$$

and

$$(a - b)^n_{p,q} = p^{n(n-1)/2}(a - b)^n_{q/p}, \tag{4}$$

where

$$(a - b)^n_q = \begin{cases} (a-b)(a-qb)\ldots(a - q^{n-1}b), & \text{if} \quad n \geq 1 \\ 1, & \text{if} \quad n = 0 \end{cases}$$

in the case when $0 < q < 1$.

The goal of the paper is to introduce a new generalization of the q-Bernstein operators involving (p, q)-integers. These (p, q)-Bernstein operators approximate each continuous function uniformly on $[0, 1]$, and some direct approximation results are established with the aid of the modulus of continuity given by

$$\omega(f;\delta) = \sup\{|f(x) - f(y)| : x, y \in [0, 1], |x - y| \le \delta\}, \quad \delta > 0, \tag{5}$$

where $f \in C[0, 1]$. Further, we define the limit (p, q)-Bernstein operator and we estimate the rate of convergence by the modulus of continuiuty (5). The concept of limit q-Bernstein operator was introduced by Il'inskii and Ostrovska [4], and its rate of convergence was established by Wang and Meng in [16]. Finally, we define a (p, q)-Kantorovich variant of the (p, q)-Bernstein operators, and we give a quantitative estimation using (5).

2 (p, q)-Bernstein operators

For $0 < q < p \le 1$, $f \in C[0, 1]$, $x \in [0, 1]$ and $n = 1, 2, \ldots$, we define the (p, q)-Bernstein polynomials as follows:

$$B_{n,p,q}(f;x) = \sum_{k=0}^{n} p^{\{k(k-1)-n(n-1)\}/2} \left[\begin{array}{c} n \\ k \end{array} \right]_{p,q} x^k (1-x)_{p,q}^{n-k} f\left(p^n \frac{[k]_{p,q}}{[n]_{p,q}}\right). \tag{6}$$

For $p = 1$ and $0 < q < 1$, we recover the q-Bernstein polynomials (see [12]):

$$B_{n,q}(f;x) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_{q} x^k (1-x)_{q}^{n-k} f\left(\frac{[k]_{q}}{[n]_{q}}\right). \tag{7}$$

Theorem 1 *If the sequences* (p_n) *and* (q_n) *satisfy* $0 < q_n < p_n \le 1$ *for* $n = 1, 2, \ldots,$ *and* $p_n \to 1$, $q_n \to 1$, $p_n^n \to 1$ *as* $n \to \infty$, *then*

$$|B_{n,p_n,q_n}(f;x) - f(x)| \le 2\omega\left(f; \left(2(1-p_n^n)x^2 + \frac{x(1-x)}{[n]_{q_n/p_n}}\right)^{1/2}\right)$$

for all $f \in C[0, 1]$ *and* $x \in [0, 1]$.

Proof. By (6), (3)-(4) and (1), we have

$$B_{n,p,q}(f;x) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_{q/p} x^k (1-x)_{q/p}^{n-k} f\left(p^k \frac{[k]_{q/p}}{[n]_{q/p}}\right). \tag{8}$$

Hence, in view of [12, (13)], we obtain

$$B_{n,p,q}(1;x) = B_{n,q/p}(1;x) = 1. \tag{9}$$

By (8) and [12, (14)], we get

$$p^n x = p^n B_{n,q/p}(t;x) \le B_{n,p,q}(t;x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q/p} x^k (1-x)_{q/p}^{n-k} p^k \frac{[k]_{q/p}}{[n]_{q/p}}. \tag{10}$$

Analogously, by (8) and [12, (15)], we get

$$\begin{aligned} B_{n,p,q}(t^2;x) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q/p} x^k (1-x)_{q/p}^{n-k} p^{2k} \frac{[k]_{q/p}^2}{[n]_{q/p}^2} \\ &\le B_{n,q/p}(t^2;x) = x^2 + \frac{x(1-x)}{[n]_{q/p}}. \end{aligned} \tag{11}$$

On the other hand, it is known for (5) that

$$\omega(f;\lambda\delta) \le (1+\lambda)\omega(f;\delta), \tag{12}$$

where $\lambda \ge 0$ and $\delta > 0$. Then, by (8), [12, (13)], Hölder's inequality and (9)-(11), we obtain

$$\begin{aligned} &|B_{n,p_n,q_n}(f;x) - f(x)| \\ &\le \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q_n/p_n} x^k (1-x)_{q_n/p_n}^{n-k} \left| f\left(p_n^k \frac{[k]_{q_n/p_n}}{[n]_{q_n/p_n}}\right) - f(x) \right| \\ &\le \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q_n/p_n} x^k (1-x)_{q_n/p_n}^{n-k} \omega\left(f; \left| p_n^k \frac{[k]_{q_n/p_n}}{[n]_{q_n/p_n}} - x \right| \right) \\ &\le \omega(f;\delta) \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q_n/p_n} x^k (1-x)_{q_n/p_n}^{n-k} \left(1 + \delta^{-1} \left| p_n^k \frac{[k]_{q_n/p_n}}{[n]_{q_n/p_n}} - x \right| \right) \\ &\le \omega(f;\delta) \left\{ 1 + \delta^{-1} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q_n/p_n} x^k (1-x)_{q_n/p_n}^{n-k} \right.\right. \\ &\quad \left.\left. \times \left(p_n^k \frac{[k]_{q_n/p_n}}{[n]_{q_n/p_n}} - x \right)^2 \right)^{1/2} \right\} \\ &= \omega(f;\delta) \left\{ 1 + \delta^{-1} (B_{n,p_n,q_n}(t^2;x) - 2x B_{n,p_n,q_n}(t;x) \right. \\ &\quad \left. + x^2 B_{n,p_n,q_n}(1;x))^{1/2} \right\} \end{aligned}$$

$$\leq \omega(f;\delta)\left\{1+\delta^{-1}\left(x^2+\frac{x(1-x)}{[n]_{q_n/p_n}}-2p_n^n x^2+x^2\right)^{1/2}\right\}$$
$$= \omega(f;\delta)\left\{1+\delta^{-1}\left(2(1-p_n^n)x^2+\frac{x(1-x)}{[n]_{q_n/p_n}}\right)^{1/2}\right\}.$$

Choosing $\delta = \left(2(1-p_n^n)x^2+\frac{x(1-x)}{[n]_{q_n/p_n}}\right)^{1/2}$, we arrive at the statement of our theorem. □

Theorem 2 *If the sequences* (p_n) *and* (q_n) *satisfy* $0 < q_n < p_n \leq 1$ *for* $n = 1, 2, \dots,$ *and* $p_n \to 1$, $q_n \to 1$, $p_n^n \to 1$ *as* $n \to \infty$, *then*

$$|B_{n,p_n,q_n}(f;x) - B_{n,q_n/p_n}(f;x)| \leq \omega(f; 1-p_n^n)$$

for all $f \in C[0,1]$ *and* $x \in [0,1]$.

Proof. Because $\left| p^k \frac{[k]_{q/p}}{[n]_{q/p}} - \frac{[k]_{q/p}}{[n]_{q/p}} \right| \leq 1 - p^k \leq 1 - p^n$ for $k = 0, 1, \dots, n$, we find from (8), (7) and [12, (13)], that

$$\begin{aligned}
&|B_{n,p_n,q_n}(f;x) - B_{n,q_n/p_n}(f;x)| \\
&\leq \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_{q_n/p_n} x^k (1-x)_{q_n/p_n}^{n-k} \left| f\left(p_n^k \frac{[k]_{q_n/p_n}}{[n]_{q_n/p_n}}\right) - f\left(\frac{[k]_{q_n/p_n}}{[n]_{q_n/p_n}}\right) \right| \\
&\leq \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_{q_n/p_n} x^k (1-x)_{q_n/p_n}^{n-k} \omega\left(f; \left| p_n^k \frac{[k]_{q_n/p_n}}{[n]_{q_n/p_n}} - \frac{[k]_{q_n/p_n}}{[n]_{q_n/p_n}} \right|\right) \\
&\leq \omega(f; 1-p_n^n) B_{n,q_n/p_n}(1;x) = \omega(f; 1-p_n^n),
\end{aligned}$$

which is the required estimation. □

Remark 1 There exist sequences (p_n) and (q_n) with the properties enumerated in Theorem 1: $p_n = 1 - \frac{1}{(n+1)^2}$ and $q_n = 1 - \frac{1}{n+1}$, $n = 1, 2, \dots$

We also mention, if $0 < q_n < p_n \leq 1$ for $n = 1, 2, \dots$, $p_n \to 1$ and $q_n \to 1$ as $n \to \infty$, then $[n]_{q_n/p_n} \to \infty$ and $\frac{[n]_{q_n/p_n}}{[n+1]_{q_n/p_n}} \to 1$ as $n \to \infty$.

Remark 2 In [9] and [10] are introduced two different generalizations of the q-Bernstein polynomials (7) involving (p, q)-integers. The first one does not preserve even the constant functions, and the second one is a (q/p)-Bernstein polynomial. Our (p, q)-Bernstein polynomials defined by (6) are different from the above mentioned generalizations. The advantage of (6) is that it allows us to introduce the limit (p, q)-Bernstein operator.

3 Limit (p,q)-Bernstein operator

For $q \in (0,1)$, Il'inskii and Ostrovska proved in [4] that for each $f \in C[0,1]$, the sequence $(B_{n,q}(f;x))$ converges to $B_{\infty,q}(f;x)$ as $n \to \infty$ uniformly for $x \in [0,1]$, where

$$B_{\infty,q}(f;x) = \begin{cases} \sum_{k=0}^{\infty} f(1-q^k)\frac{x^k}{(1-q)^k[k]_q!}\prod_{s=0}^{\infty}(1-q^s x), & \text{if} \quad 0 \leq x < 1 \\ f(1), & \text{if} \quad x = 1 \end{cases}$$

is the limit q-Bernstein operator. Wang and Meng [16] proved for all $f \in C[0,1]$ and $x \in [0,1]$ that

$$|B_{n,q}(f;x) - B_{\infty,q}(f;x)| \leq \left(2 + \frac{4}{q(1-q)}\ln\frac{1}{1-q}\right)\omega(f;q^n).$$

For $0 < q < p \leq 1$, the limit (p,q)-Bernstein operator $B_{\infty,p,q} : C[0,1] \to C[0,1]$ is defined as follows:

$$B_{\infty,p,q}(f;x) = \begin{cases} \sum_{k=0}^{\infty} f(p^k - q^k)\frac{p^{(k+1)k/2}x^k}{(p-q)^k[k]_{p,q}!}\prod_{s=0}^{\infty}\frac{p^s - q^s x}{p^s}, & \text{if} \quad x \in [0,1) \\ f(1), & \text{if} \quad x = 1. \end{cases} \tag{13}$$

Theorem 3 *Let* $p, q \in (0,1)$ *be given such that* $p^2 < q < p$. *Then, for every* $f \in C[0,1]$, $x \in [0,1]$ *and* $n = 1,2,\ldots,$ *we have*

$$|B_{n,p,q}(f;x) - B_{\infty,p,q}(f;x)| \leq \left(4 + \frac{6p^2}{q(p-q)}\ln\frac{p}{p-q}\right)\omega\left(f;\left(\frac{q}{p}\right)^n\right).$$

Proof. Due to (13) and (2), we have

$$B_{\infty,p,q}(f;x) = \sum_{k=0}^{\infty} f(p^k - q^k)\frac{x^k}{\left(1-\frac{q}{p}\right)^k[k]_{q/p}!}\prod_{s=0}^{\infty}\left(1 - \left(\frac{q}{p}\right)^s x\right). \tag{14}$$

We set

$$w_{n,k}(q;x) = \begin{bmatrix} n \\ k \end{bmatrix}_q x^k(1-x)_q^{n-k} \text{ and } w_{\infty,k}(q;x) = \frac{x^k}{(1-q)^k[k]_q!}\prod_{s=0}^{\infty}(1-q^s x).$$

Then, in view of (9) and [16, p. 154, (2.3)], we obtain

$$\sum_{k=0}^{n} w_{n,k}\left(\frac{q}{p};x\right) = \sum_{k=0}^{\infty} w_{\infty,k}\left(\frac{q}{p};x\right) = 1. \tag{15}$$

Using (8), (14) and (15), we find

$$\begin{aligned}
&|B_{n,p,q}(f;x) - B_{\infty,p,q}(f;x)| \\
&= \Bigg| \sum_{k=0}^{n} w_{n,k}\left(\frac{q}{p};x\right)\left\{f\left(p^k\frac{[k]_{q/p}}{[n]_{q/p}}\right) - f(p^k-q^k)\right\} \\
&\quad + \sum_{k=0}^{n}\left\{w_{n,k}\left(\frac{q}{p};x\right) - w_{\infty,k}\left(\frac{q}{p};x\right)\right\}\{f(p^k-q^k)-f(p^n)\} \\
&\quad - \sum_{k=n+1}^{\infty} w_{\infty,k}\left(\frac{q}{p};x\right)\{f(p^k-q^k)-f(p^n)\}\Bigg| \\
&\le \sum_{k=0}^{n} w_{n,k}\left(\frac{q}{p};x\right)\left|f\left(p^k\frac{[k]_{q/p}}{[n]_{q/p}}\right) - f(p^k-q^k)\right| \\
&\quad + \sum_{k=0}^{n}\left|w_{n,k}\left(\frac{q}{p};x\right) - w_{\infty,k}\left(\frac{q}{p};x\right)\right| |f(p^k-q^k)-f(p^n)| \\
&\quad + \sum_{k=n+1}^{\infty} w_{\infty,k}\left(\frac{q}{p};x\right)|f(p^k-q^k)-f(p^n)| \\
&=: I_1 + I_2 + I_3.
\end{aligned} \tag{16}$$

The estimation of I_1 : by (1), we have

$$\begin{aligned}
\left| p^k\frac{[k]_{q/p}}{[n]_{q/p}} - (p^k-q^k)\right| &= \frac{[k]_{q/p}}{[n]_{q/p}}\left| p^k - (p^k-q^k)p^{k-n}\frac{[n]_{p,q}}{[k]_{p,q}}\right| \\
&\le \left| p^k - (p^k-q^k)p^{k-n}\frac{p^n-q^n}{p^k-q^k}\right| = p^k\left(\frac{q}{p}\right)^n \le \left(\frac{q}{p}\right)^n
\end{aligned}$$

for $k = 0, 1, \ldots, n$. Hence, by (15),

$$I_1 \le \sum_{k=0}^{n} w_{n,k}\left(\frac{q}{p};x\right)\omega\left(f; \left| p^k\frac{[k]_{q/p}}{[n]_{q/p}} - (p^k-q^k)\right|\right) \le \omega\left(f; \left(\frac{q}{p}\right)^n\right). \tag{17}$$

The estimation of I_2 : for $k=0,1,\dots,n$, we have $|p^k-q^k-p^n|\le p^k(1-p^{n-k})+q^k\le p^k+q^k$. Hence, by (12),

$$\begin{aligned}|f(p^k-q^k)-f(p^n)| &\le \omega(f;|p^k-q^k-p^n|)\le \omega(f;p^k+q^k)\\ &= \omega\left(f;\frac{p^k+q^k}{(q/p)^n}\left(\frac{q}{p}\right)^n\right)\le\left(1+\frac{p^k+q^k}{(q/p)^n}\right)\omega\left(f;\left(\frac{q}{p}\right)^n\right). \qquad (18)\end{aligned}$$

But

$$\begin{aligned}\left(1+\frac{p^k+q^k}{(q/p)^n}\right)\left(\frac{p}{q}\right)^k &= \left(\frac{q}{p}\right)^{-n}\left(\left(\frac{q}{p}\right)^n+p^k+q^k\right)\left(\frac{p}{q}\right)^k\\ &= \left(\frac{q}{p}\right)^{-n}\left(\left(\frac{q}{p}\right)^{n-k}+\left(\frac{p^2}{q}\right)^k+p^k\right)\le 3\left(\frac{q}{p}\right)^{-n},\end{aligned}$$

because $p^2<q<p$ and $k=0,1,\dots,n$. Then, by (18), we obtain

$$\begin{aligned}I_2 &\le \sum_{k=0}^{n}\left|w_{n,k}\left(\frac{q}{p};x\right)-w_{\infty,k}\left(\frac{q}{p};x\right)\right|3\left(\frac{q}{p}\right)^{k-n}\omega\left(f;\left(\frac{q}{p}\right)^n\right)\\ &= 3\left(\frac{q}{p}\right)^{-n}\omega\left(f;\left(\frac{q}{p}\right)^n\right)\sum_{k=0}^{n}\left(\frac{q}{p}\right)^k\left|w_{n,k}\left(\frac{q}{p};x\right)-w_{\infty,k}\left(\frac{q}{p};x\right)\right|.\end{aligned}$$

Taking into account the estimation

$$\sum_{k=0}^{n} q^k|w_{n,k}(q;x)-w_{\infty,k}(q;x)|\le\frac{2q^n}{q(1-q)}\ln\frac{1}{1-q},$$

where $0<q<1$ (see [16, p. 156, (2.9)]), we find that

$$I_2\le\frac{6p^2}{q(p-q)}\ln\frac{p}{p-q}\;\omega\left(f;\left(\frac{q}{p}\right)^n\right). \qquad (19)$$

The estimation of I_3 : for $k\ge n+1$, we have $|p^k-q^k-p^n|\le p^n(1-p^{k-n})+q^k\le p^n+q^n$. Hence, by (12) and $p^2<q<p$, we get

$$\begin{aligned}|f(p^k-q^k)-f(p^n)| &\le \omega(f;|p^k-q^k-p^n|)\le\omega(f;p^n+q^n)\\ &\le \left(1+\frac{p^n+q^n}{(q/p)^n}\right)\omega\left(f;\left(\frac{q}{p}\right)^n\right)=\left(1+\left(\frac{p^2}{q}\right)^n+p^n\right)\omega\left(f;\left(\frac{q}{p}\right)^n\right)\\ &\le 3\omega\left(f;\left(\frac{q}{p}\right)^n\right).\end{aligned}$$

Then, by (15),

$$I_3 \leq 3\omega\left(f;\left(\frac{q}{p}\right)^n\right)\sum_{k=n+1}^{\infty} w_{\infty,k}\left(\frac{q}{p};x\right) \leq 3\omega\left(f;\left(\frac{q}{p}\right)^n\right). \tag{20}$$

Combining (16)-(17) and (19)-(20), we obtain the statement of the theorem. □

4 (p, q)-Kantorovich operators

Our (p,q)-Kantorovich operators are defined as follows:

$$\begin{aligned} K_{n,p,q}(f;x) &= \frac{[n+1]_{p,q}}{p^n}\sum_{k=0}^{n} p^{\{k(k-1)-n(n-1)\}/2}\left[\begin{matrix} n \\ k \end{matrix}\right]_{p,q} x^k(1-x)_{p,q}^{n-k} \\ &\quad \times q^{-k}\int_{p^{n+1}\frac{[k]_{p,q}}{[n+1]_{p,q}}}^{p^n\frac{[k+1]_{p,q}}{[n+1]_{p,q}}} f(u)\, d_{q/p}^R u, \end{aligned} \tag{21}$$

where $f \in C[0,1]$, $x \in [0,1]$, $n = 1, 2, \ldots$, and the Riemann type q-integral of f over the interval $[a,b]$ $(0 \leq a < b;\ 0 < q < 1)$ is given by (see [2], [8])

$$\int_a^b f(u)\, d_q^R u = (1-q)(b-a)\sum_{j=0}^{\infty} q^j f(a+(b-a)q^j). \tag{22}$$

Remark 3 In [15] the (p,q)-integral of f over the interval $[0,a]$ is defined as

$$\int_0^a f(u)\, d_{p,q}u = (p-q)a\sum_{j=0}^{\infty}\frac{q^j}{p^{j+1}} f\left(a\frac{q^j}{p^{j+1}}\right),$$

where $0 < q < p \leq 1$. But $\frac{1}{p}a \notin [0,a]$ for $0 < p < 1$ (in the sum the case $j = 0$), thus the function f is not defined at $\frac{1}{p}a$. For this reason we use the Riemann type (q/p)-integral in (21).

Theorem 4 *If the sequences* (p_n) *and* (q_n) *satisfy* $0 < q_n < p_n \leq 1$ *for* $n = 1, 2, \ldots$, *and* $p_n \to 1$, $q_n \to 1$, $p_n^n \to 1$ *as* $n \to \infty$, *then*

$$|K_{n,p_n,q_n}(f;x) - f(x)| \leq 2\omega(f;\sqrt{\delta_n(x)})$$

for all $f \in C[0,1]$ *and* $x \in [0,1]$, *where*

$$\delta_n(x) = \left\{ 2(1-p_n^n)\frac{[n]_{q_n/p_n}}{[n+1]_{q_n/p_n}} + \left(1 - \frac{[n]_{q_n/p_n}}{[n+1]_{q_n/p_n}}\right)^2 - \frac{[n]_{q_n/p_n}}{[n+1]^2_{q_n/p_n}} \right\} \times x^2 + 3\frac{[n]_{q_n/p_n}}{[n+1]_{q_n/p_n}} x + \frac{1}{[n+1]^2_{q_n/p_n}}.$$

Proof. By (21), (3)-(4) and (1), we have

$$K_{n,p,q}(f;x) = [n+1]_{q/p} \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_{q/p} x^k (1-x)^{n-k}_{q/p} q^{-k} \int_{p^k \frac{[k]_{q/p}}{[n+1]_{q/p}}}^{p^k \frac{[k+1]_{q/p}}{[n+1]_{q/p}}} f(u)\, {}^{R}_{q/p}u. \tag{23}$$

By simple computations, using (22), we obtain

$$\int_{p^k \frac{[k]_{q/p}}{[n+1]_{q/p}}}^{p^k \frac{[k+1]_{q/p}}{[n+1]_{q/p}}} 1\, d^R_{q/p}u = p^k \frac{[k+1]_{q/p} - [k]_{q/p}}{[n+1]_{q/p}} = \frac{q^k}{[n+1]_{q/p}}, \tag{24}$$

$$\int_{p^k \frac{[k]_{q/p}}{[n+1]_{q/p}}}^{p^k \frac{[k+1]_{q/p}}{[n+1]_{q/p}}} u\, d^R_{q/p}u = \frac{q^k}{[n+1]_{q/p}} \left(p^k \frac{[k]_{q/p}}{[n+1]_{q/p}} + \frac{p}{p+q} \frac{q^k}{[n+1]_{q/p}} \right) \tag{25}$$

and

$$\int_{p^k \frac{[k]_{q/p}}{[n+1]_{q/p}}}^{p^k \frac{[k+1]_{q/p}}{[n+1]_{q/p}}} u^2\, d^R_{q/p}u = \frac{q^k}{[n+1]_{q/p}} \left(p^{2k} \frac{[k]^2_{q/p}}{[n+1]^2_{q/p}} + \frac{2p}{p+q} p^k \frac{[k]_{q/p}}{[n+1]_{q/p}} \times \frac{q^k}{[n+1]_{q/p}} + \frac{p^2}{p^2+pq+q^2} \frac{q^{2k}}{[n+1]^2_{q/p}} \right). \tag{26}$$

In what follows, taking into account (23)-(26), the proof is similar to the proof of Theorem 1, therefore we omit the details. □

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