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On packing density of growing size circles

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Abstract. For any given natural number an arrangement of growing size circles, a packing of the plane will be constructed such that its packing density coincides – in the asymptotical sense – with that of the 'classical' hexagonal circle packing!

1 The construction

For a natural $n \geq 3$ let us define a special circle packing as follows. First, we circumscribe the unit circle with n circles of the same radius $r_{n,1}$ such that they also touch their both neighbours. Thus we get zone one, $Z_{n,1}$.

Then we draw n circles of the same radius $r_{n,2}$ such that they touch two circles from $Z_{n,1}$ and also their both neighbours with radius $r_{n,2}$, getting this way $Z_{n,2}$, etc.

Denote by $S_{n,k}$ the set of circles of the first k zones:

$$S_{n,k} = \cup_{i=1}^{k} Z_{n,i},$$

and let

$$S_n = \cup_{k=1}^{\infty} S_{n,k}.$$

Then S_n is a packing of the plane, an infinite set of circles with pairwise disjoint interiors, and a natural problem is to find the fraction of the plane filled by the circles making up this packing.

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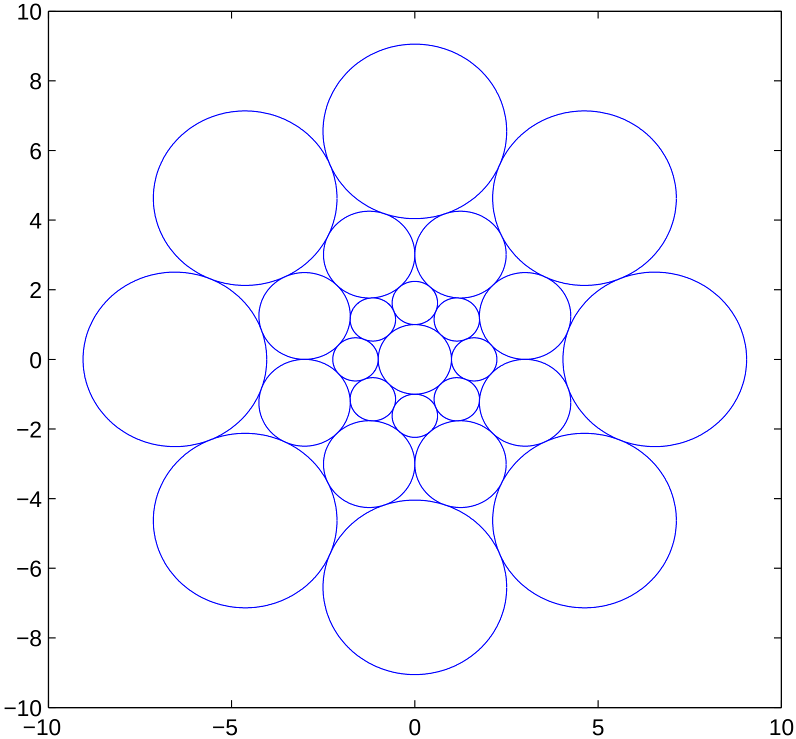

Figure 1. The set $S_{8,3}$, i.e. the first three zones for $n = 8$.

2 The main theorem

The packing density of the arrangement S_n related to a bounded domain $D \subset \mathbb{R}^2$ is the ratio

$$\frac{\sum |C \cap D|}{|D|}, \ C \in S_n,$$

where $|\cdot|$ denotes the area of its argument. It is customary to define (see e.g. Kuperberg [1]) the packing density in an Euclidean space by means of a limit, taking e.g. balls B_r of radius r centered at the origin:

$$\lim_{r\to\infty} \frac{\sum |C \cap B_r|}{|B_r|}, \ C \in S_n.$$

However, in our case taking polygons is more capable. Denote by $P_{n,k}$ the regular $n-$gon with vertices at the centres of circles in $Z_{n,k}$, and let

$$\delta_{n,k} = \frac{|S_n \cap P_{n,k}|}{|P_{n,k}|} \equiv \frac{|S_{n,k} \cap P_{n,k}|}{|P_{n,k}|}.$$

Then,

$$\delta_n = \lim_{k\to\infty} \delta_{n,k}$$

is the packing density of S_n, and

$$\delta^* = \lim_{n\to\infty} \delta_n$$

is the quantity we are interested in.

Theorem 1 *With the notations above we have*

$$\delta^* = \frac{\pi}{2\sqrt{3}}.$$

Remark 1 *As is known (see e.g. the survey on the first page in [2], showing the contributions of Lagrange, A. Thue, L. Fejes Tóth to the subject), the optimal packing density for circles is just this quantity - a curious coincidence!*

Remark 2 *The interested reader should also consult [3] and [4] for further information.*

3 The proof

Let $n, k \in \mathbb{N},\ n \geq 3$ be given. Assume that the centre of one of the circles belonging to $Z_{n,1}$ lies on the x-axis, i.e. at $A_{n,1} := (1 + r_{n,1}, 0)$.

Denote by $B_{n,1}$ the point, where the half-line $y = \tan(\frac{\pi}{n})\,x$ is tangent to the circle chosen. It suffices to consider the 'basic' sector $B_{n,1}OA_{n,1}$, as is seen on Figure 2 for the case $n = 8,\ k = 3$.

It is easy to see that the centre $A_{n,i}$ of the $i-th$ circle in the basic sector lies on the x-axis for i odd, and on the line $y = \tan(\frac{\pi}{n})\,x$ for i even, and just reversely for the $B'_{n,i}s$. Also note that the angles at the $B'_{n,i}s$ are rectangles. Introduce now the notations

$$s_n = \sin\left(\frac{\pi}{n}\right), \quad t_n = \tan\left(\frac{\pi}{n}\right).$$

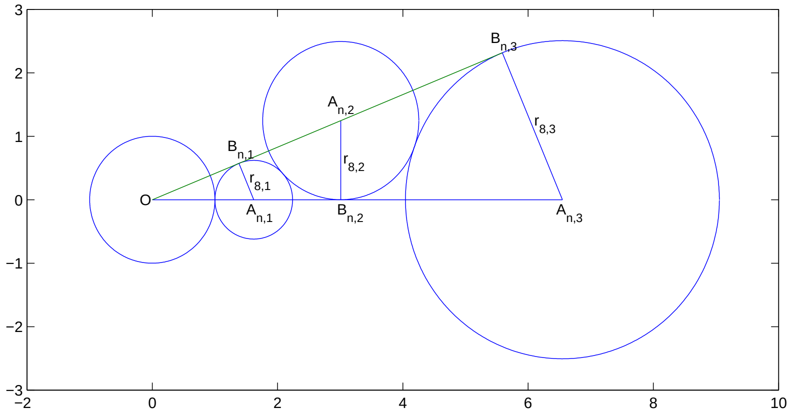

Figure 2. The basic sector for $S_{8,3}$

The radius $r_{n,1}$ can be obtained from the triangle $B_{n,1}OA_{n,1}$:

$$s_n = \frac{r_{n,1}}{1 + r_{n,1}} \quad \Rightarrow \quad r_{n,1} = \frac{s_n}{1 - s_n}$$

Since $A_{n,1}B_{n,2}A_{n,2}B_{n,1}$ and $A_{n,2}B_{n,3}A_{n,3}B_{n,2}$ are similar quadrilaterals (in fact, both are inscribed quadrilaterals with two rectangles), it follows that $r_{n,2}/r_{n,1} = r_{n,3}/r_{n,2}$, giving in general

$$r_{n,k} = r_{n,1}\, q_n^{k-1}, \quad q_n = \frac{r_{n,2}}{r_{n,1}}.$$

The area of the polygon $P_{n,k}$ is $2n$ times the area of triangle $OA_{n,k}B_{n,k}$, i.e.

$$|P_{n,k}| = \frac{n}{t_n} r_{n,k}^2 = \frac{n r_{n,1}^2}{t_n} q_n^{2k-2}.$$

The sum of areas of the circles in $S_{n,k}$ is

$$n\pi \sum_{i=1}^{k} r_{n,i}^2 = n\pi r_{n,1}^2 \sum_{i=1}^{k} q_n^{2k-2} = n\pi r_{n,1}^2 \frac{q_n^{2k} - 1}{q_n^2 - 1}.$$

However, the contribution of the k−th zone to $|Z_{n,k} \cap P_{n,k}|$ is only $\frac{n-2}{2}\pi r_{n,k}^2$, instead of $n\pi r_{n,k}^2$. Consequently we have

$$|S_{n,k} \cap P_{n,k}| = \pi r_{n,1}^2 q_n^{2k-2}\Big(\frac{n}{q_n^2 - 1} + \frac{n-2}{2}\Big) - \frac{\pi n r_{n,1}^2}{q_n^2 - 1}.$$

When calculating the limit of $\delta_{n,k}$ for $k \to \infty$, the magnitude of the quotient $q_n = \frac{r_{n,2}}{r_{n,1}}$ is decisive. Introducing the new variable

$$t = \tan\left(\frac{\pi}{2n}\right)$$

(cf. the standard trigonometric substitution $t = \tan(\frac{x}{2})$ in calculus) we have

$$s_n = \frac{2t}{1+t^2}, \quad t_n = \frac{2t}{1-t^2},$$

and also

$$r_{n,1} = \frac{2t}{(1-t)^2}, \quad r_{n,2} = \frac{2t(u+2t\sqrt{v})}{(1+t)^2(1-t)^4},$$

where

$$u = 1 + 4t^2 - t^4, \quad v = (3-t^2)(1+t^2).$$

The radius $r_{n,2}$ can be calculated by considering the triangles $A_{n,1}B_{n,2}A_{n,2}$ and $OB_{n,2}A_{n,2}$. Analysing the function $t \to \frac{r_{n,2}}{r_{n,1}}$ we see that it is greater than one for $0 < t < 1$, or equivalently, that the relation $q_n > 1$ holds for $n \geq 3$. Therefore, in case of $k \to \infty$ the second (negative) term in $|S_{n,k} \cap P_{n,k}|$ can be omitted to get

$$\delta_n = \frac{\pi t_n}{r_{n,2}^2 - r_{n,1}^2}\Big(r_{n,1}^2 + \frac{n-2}{2n}(r_{n,2}^2 - r_{n,1}^2)\Big).$$

With the notation $\omega = \frac{n-2}{2n}$ we obviously have $0 < \omega < 1$ for $n \geq 3$, which yields in a natural way the lower and upper bounds

$$\delta_n^0 < \delta_n < \delta_n^1.$$

Since the difference of these bounds is simply

$$\delta_n^1 - \delta_n^0 = \pi t_n = O(t) \quad (t \to 0),$$

we can replace δ_n e.g. by its lower bound

$$\delta_n^0 = \frac{\pi t_n r_{n,1}^2}{r_{n,2}^2 - r_{n,1}^2} = \frac{\pi(1-t^2)^3}{2(2tv + u\sqrt{v})}.$$

Having gotten rid of the singularity, we can immediately substitute $t = 0$ (corresponding to $n \to \infty$) to get the desired result

$$\delta^* = \lim_{n\to\infty} \delta_n.$$

□

Considering this surprising coincidence, one puts the question: is there a more general principle, this conclusion can be drawn from?

References

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