Acta Univ. Sapientiae, Mathematica, **8**, 2 (2016) 255–270

DOI: 10.1515/ausm-2016-0017

Study of periodic and nonnegative periodic solutions of nonlinear neutral functional differential equations via fixed points

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Abstract. In this paper, we study the existence of periodic and nonnegative periodic solutions of the nonlinear neutral differential equation

$$\frac{d}{dt}x(t) = -a(t)h(x(t)) + \frac{d}{dt}Q(t, x(t-\tau(t))) + G(t, x(t), x(t-\tau(t))).$$

We invert this equation to construct a sum of a completely continuous map and a large contraction which is suitable for applying the modification of Krasnoselskii's theorem. The Caratheodory condition is used for the functions Q and G.

2010 Mathematics Subject Classification: 34A37, 34A12, 35B09, 35B10, 45J05
Key words and phrases: Krasnoselskii-Burton's theorem, large contraction, neutral differential equation, integral equation, periodic solution, positive solution

1 Introduction

Theory of functional differential equations with delay has undergone a rapid development in the previous fifty years. We refer the readers to [1]-[6], [8]-[15] and references therein for a wealth of reference materials on the subject. More recently researchers have given special attentions to the study of equations in which the delay argument occurs in the derivative of the state variable as well as in the independent variable, so-called neutral differential equations. In particular, qualitative analysis such as periodicity and positivity of solutions of neutral differential equations has been studied extensively by many authors.

Recently, in [1], the authors discussed the existence and positivity of periodic solutions for the first-order delay differential equation

$$x'(t) = -a(t)\,h(x(t)) + G(t, x(t-\tau(t))), \tag{1}$$

by employing the Krasnoselskii-Burton's fixed point theorem, the authors obtained existence results for periodic and positive periodic solutions.

In [14], the Krasnoselskii-Burton's fixed point theorem was used to establish the existence of periodic solutions for the first-order nonlinear neutral differential equation

$$\frac{d}{dt}x(t) = -a(t)\,h(x(t)) + c(t)\,x'(t-\tau(t)) + G(t, x(t), x(t-\tau(t))). \tag{2}$$

In [8], the authors used Krasnoselskii's fixed point theorem to establish the existence of periodic solutions for the nonlinear neutral differential equation

$$\frac{d}{dt}x(t) = -a(t)\,x(t) + \frac{d}{dt}Q(t, x(t-\tau(t))) + G(t, x(t), x(t-\tau(t))). \tag{3}$$

Also, the authors used the contraction mapping principle to show the uniqueness of periodic solutions and stability of the zero solutions of (3).

In the current paper, we are interested in the analysis of qualitative theory of periodic and nonnegative periodic solutions of neutral differential equations. Inspired and motivated by the works mentioned above and the papers [1]-[6], [8]-[15] and the references therein, we study the existence of periodic and nonnegative periodic solutions of the nonlinear neutral differential equation

$$\frac{d}{dt}x(t) = -a(t)\,h(x(t)) + \frac{d}{dt}Q(t, x(t-\tau(t))) + G(t, x(t), x(t-\tau(t))), \tag{4}$$

where a is a positive continuous real-valued function. The function $h : \mathbb{R} \to \mathbb{R}$ is continuous, $Q : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $G : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfying the

Caratheodory condition. Our purpose here is to use a modification of Krasnoselskii's fixed point theorem due to Burton (see [7], Theorem 3) to show the existence and nonnegativity of periodic solutions for equation (4). Clearly, the present problem is totally nonlinear so that the variation of parameters can not be applied directly. Then, we resort to the idea of adding and subtracting a linear term. As noted by Burton in [7], the added term destroys the contraction but it replaces with a so called large contraction which is suitable for fixed point theory. During the process we have to transform (4) into an integral equation written as a sum of two mappings, one is a large contraction and the other is completely continuous. After that, we use a variant of Krasnoselskii's fixed point theorem, to show the existence and nonnegativity of a periodic solution.

Note that in our consideration the neutral term $\frac{d}{dt}Q\left(t, x\left(t-\tau\left(t\right)\right)\right)$ of (4) produces nonlinearity in the derivative term $\frac{d}{dt}x\left(t-\tau\left(t\right)\right)$. The neutral term $\frac{d}{dt}x\left(t-\tau\left(t\right)\right)$ of (2) in [14] enters linearly. As a consequence, our analysis is different from that in [14].

The organization of this paper is as follows. In Section 2, we present the inversion of totally nonlinear neutral differential equation (4), some definitions and Krasnoselskii-Burton's fixed point theorem. For details on Krasnoselskii-Burton's theorem we refer the reader to [7]. In Sections 3 and 4, we present our main results on existence of periodic and nonnegative periodic solutions of (4).

2 Preliminaries

For $T > 0$ define $P_T = \{\phi : \phi \in C\left(\mathbb{R}, \mathbb{R}\right),\ \phi\left(t+T\right) = \phi\left(t\right)\}$ where $C\left(\mathbb{R}, \mathbb{R}\right)$ is the space of all real valued continuous functions. Then P_T is a Banach space when it is endowed with the supremum norm

$$\|x\| = \max_{t\in[0,T]} |x\left(t\right)|.$$

In this paper we assume that

$$a\left(t-T\right) = a\left(t\right),\ \tau\left(t-T\right) = \tau\left(t\right),\ \tau\left(t\right) \geq \tau^* > 0, \tag{5}$$

with τ continuously and τ^* is constant, a is positive and

$$1 - e^{-\int_{t-T}^{t} a(s)ds} \equiv \frac{1}{\eta} \neq 0. \tag{6}$$

The functions $Q(t,x)$ and $G(t,x,y)$ are periodic in t of period T. That is

$$Q(t-T,x) = Q(t,x), \; G(t-T,x,y) = G(t,x,y). \tag{7}$$

The following lemma is fundamental to our results.

Lemma 1 *Suppose (5)–(7) hold. If $x \in P_T$, then x is a solution of equation (4) if and only if*

$$\begin{aligned} x(t) &= \eta \int_{t-T}^{t} \kappa(t,u)\, a(u) [x(u) - h(x(u))]\, du + Q(t, x(t-\tau(t))) \\ &+ \eta \int_{t-T}^{t} \kappa(t,u) [-a(u) Q(u, x(u-\tau(u))) + G(u, x(u), x(u-\tau(u)))]\, du, \end{aligned} \tag{8}$$

where

$$\kappa(t,u) = e^{-\int_u^t a(s)ds}. \tag{9}$$

Proof. Let $x \in P_T$ be a solution of (4). Rewrite the equation (4) as

$$\begin{aligned} &\frac{d}{dt}[x(t) - Q(t, x(t-\tau(t)))] + a(t)[x(t) - Q(t, x(t-\tau(t)))] \\ &= a(t)x(t) - a(t)h(x(t)) - a(t)Q(t, x(t-\tau(t))) + G(t, x(t), x(t-\tau(t))) \\ &= a(t)[x(t) - h(x(t))] - a(t)Q(t, x(t-\tau(t))) + G(t, x(t), x(t-\tau(t))). \end{aligned}$$

Multiply both sides of the above equation by $e^{\int_0^t a(s)ds}$ and then integrate from $t-T$ to t to obtain

$$\begin{aligned} &\int_{t-T}^{t} \left[(x(u) - Q(u, x(u-\tau(u))))\, e^{\int_0^u a(s)ds}\right]' du \\ &= \int_{t-T}^{t} a(u)[x(u) - h(x(u))]\, e^{\int_0^u a(s)ds} du \\ &+ \int_{t-T}^{t} [-a(u)Q(u, x(u-\tau(u))) + G(u, x(u), x(u-\tau(u)))]\, e^{\int_0^u a(s)ds} du. \end{aligned}$$

As a consequence, we arrive at

$$\begin{aligned}
&(x(t) - Q(t, x(t - \tau(t))))\, e^{\int_0^t a(s)ds} \\
&- (x(t-T) - Q(t-T, x(t-T-\tau(t-T))))\, e^{\int_0^{t-T} a(s)ds} \\
&= \int_{t-T}^{t} a(u)\,[x(u) - h(x(u))]\, e^{\int_0^u a(s)ds} du \\
&+ \int_{t-T}^{t} [G(u, x(u), x(u-\tau(u))) - a(u)\, Q(u, x(u-\tau(u)))]\, e^{\int_0^u a(s)ds} du.
\end{aligned}$$

By dividing both sides of the above equation by $\exp(\int_0^t a(s)\,ds)$ and using the fact that $x(t) = x(t-T)$, we obtain

$$\begin{aligned}
&x(t) - Q(t, x(t-\tau(t))) \\
&= \eta \int_{t-T}^{t} a(u)\,[x(u) - h(x(u))]\, e^{-\int_u^t a(s)ds} du \\
&+ \eta \int_{t-T}^{t} [G(u, x(u), x(u-\tau(u))) - a(u)\, Q(u, x(u-\tau(u)))]\, e^{\int_0^u a(s)ds} du.
\end{aligned} \tag{10}$$

The converse implication is easily obtained and the proof is complete. □

Now, we give some definitions which we are going to use in what follows.

Definition 1 *The map* $f : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ *is said to satisfy Carathéodory conditions with respect to* $L^1[0,T]$ *if the following conditions hold.*

(i) *For each* $z \in \mathbb{R}^n$*, the mapping* $t \mapsto f(t,z)$ *is Lebesgue measurable.*

(ii) *For almost all* $t \in [0,T]$*, the mapping* $z \mapsto f(t,z)$ *is continuous on* $\mathbb{R}^n$*.*

(iii) *For each* $r > 0$*, there exists* $\alpha_r \in L^1([0,T], \mathbb{R})$ *such that for almost all* $t \in [0,T]$ *and for all* z *such that* $|z| < r$*, we have* $|f(t,z)| \le \alpha_r(t)$*.*

T. A. Burton observed that Krasnoselskii's result (see [12]) can be more attractive in applications with certain changes and formulated Theorem 1 below (see [7] for the proof).

Definition 2 *Let* $(\mathcal{M}, d)$ *be a metric space and assume that* $B : \mathcal{M} \to \mathcal{M}$*.* B *is said to be a large contraction, if for* $\varphi, \psi \in \mathcal{M}$*, with* $\varphi \ne \psi$*, we have* $d(B\varphi, B\psi) < d(\varphi, \psi)$*, and if* $\forall \epsilon > 0$*,* $\exists \delta < 1$ *such that*

$$[\varphi, \psi \in \mathcal{M},\ d(\varphi,\psi) \ge \epsilon] \Longrightarrow d(B\varphi, B\psi) < \delta d(\varphi, \psi).$$

It is proved in [7] that a large contraction defined on a closed bounded and complete metric space has a unique fixed point.

Theorem 1 (Krasnoselskii-Burton) *Let $\mathcal{M}$ be a closed bounded convex nonempty subset of a Banach space $(\mathcal{B}, \|.\|)$. Suppose that A and B map $\mathcal{M}$ into $\mathcal{M}$ such that*
(i) A *is completely continuous,*
(ii) B *is large contraction,*
(ii) $x, y \in \mathcal{M}$, *implies* $Ax + By \in \mathcal{M}$.
Then there exists $z \in \mathcal{M}$ *with* $z = Az + Bz$.

3 Existence of periodic solutions

To apply Theorem 1, we need to define a Banach space $\mathcal{B}$, a closed bounded convex subset $\mathcal{M}$ of $\mathcal{B}$ and construct two mappings; one is a completely continuous and the other is large contraction. So, we let $(\mathcal{B}, \|.\|) = (P_T, \|.\|)$ and

$$\mathcal{M} = \{\varphi \in P_T,\ \|\varphi\| \le L\} \tag{11}$$

with $L \in (0, 1]$. For $x \in \mathcal{M}$, let the mapping H be defined by

$$H(x) = x - h(x), \tag{12}$$

and by (8), define the mapping $S : P_T \to P_T$ by

$$\begin{aligned}(S\varphi)(t) &\\ = \eta \int_{t-T}^{t} &\kappa(t,u)\, a(u)\, H(\varphi(u))\, du + Q(t, \varphi(t-\tau(t))) \\ + \eta \int_{t-T}^{t} &\kappa(t,u) \left[-a(u)\, Q(u, \varphi(u-\tau(u))) + G(u, \varphi(u), \varphi(u-\tau(u)))\right] du.\end{aligned} \tag{13}$$

Therefore, we express the above equation as

$$(S\varphi)(t) = (A\varphi)(t) + (B\varphi)(t),$$

where $A, B : P_T \to P_T$ are given by

$$\begin{aligned}(A\varphi)(t) &\\ = Q(t, \varphi&(t-\tau(t))) \\ + \eta \int_{t-T}^{t} &\kappa(t,u) \left[-a(u)\, Q(u, \varphi(u-\tau(u))) + G(u, \varphi(u), \varphi(u-\tau(u)))\right] du.\end{aligned} \tag{14}$$

and

$$(B\varphi)(t) = \eta \int_{t-T}^{t} \kappa(t,u)\, a(u)\, H(\varphi(u))\, du. \tag{15}$$

We will assume that the following conditions hold.

(H1) $a \in L^1[0,T]$ is bounded.

(H2) Q, G satisfies Carathéodory conditions with respect to $L^1[0,T]$.

(H3) There exists periodic functions q_1, $q_2 \in L^1[0,T]$, with period T, such that

$$|Q(t,x)| \leq q_1(t)|x| + q_2(t).$$

(H4) There exists periodic functions g_1, g_2, $g_3 \in L^1[0,T]$, with period T, such that

$$|G(t,x,y)| \leq g_1(t)|x| + g_2(t)|y| + g_3(t).$$

Now, we need the following assumptions

$$q_1(t)L + q_2(t) \leq \frac{\gamma_1}{2} L, \tag{16}$$

$$g_1(t)L + g_2(t)L + g_3(t) \leq \gamma_2 L a(t), \tag{17}$$

$$J(\gamma_1 + \gamma_2) \leq 1, \tag{18}$$

where γ_1, γ_2 and J are positive constants with $J \geq 3$.

Lemma 2 *For A defined in (14), suppose that (5)–(7), (16)–(18) and (H1)–(H4) hold. Then $A : \mathcal{M} \to \mathcal{M}$.*

Proof. Let A be defined by (14). Obviously, $A\varphi$ is continuous. First by (5) and (7), a change of variable in (14) shows that $(A\varphi)(t+T) = (A\varphi)(t)$. That is, if $\varphi \in P_T$ then $A\varphi$ is periodic with period T. Next, let $\varphi \in \mathcal{M}$, by (16)–(18)

and (H1)–(H4) we have

$$
\begin{aligned}
&|(A\varphi)(t)| \\
&\leq |Q(t, \varphi(t-\tau(t)))| \\
&+ \eta \int_{t-T}^{t} \kappa(t,u)\left(a(u)|Q(u, \varphi(u-\tau(u)))| + |G(u, \varphi u, \varphi(u-\tau(u)))|\right) du \\
&\leq q_1(t)|\varphi(t-\tau(t))| + q_2(t) \\
&+ \eta \int_{t-T}^{t} \kappa(t,u)\, a(u)\left[q_1(u)|\varphi(u-\tau(u))| + q_2(u)\right] du \\
&+ \eta \int_{t-T}^{t} \kappa(t,u)\left[g_1(u)|\varphi(u)| + g_2(u)|\varphi(u-\tau(u))| + g_3(u)\right] du \\
&\leq \gamma_1 L + \gamma_2 L \leq \frac{L}{J} \leq L.
\end{aligned}
$$

That is $A\varphi \in \mathcal{M}$. □

Lemma 3 *For $A : \mathcal{M} \to \mathcal{M}$ defined in (14), suppose that (5)–(7), (16)–(18) and (H1)–(H4) hold. Then A is completely continuous.*

Proof. We show that A is continuous in the supremum norm, Let $\varphi_n \in \mathcal{M}$ where n is a positive integer such that $\varphi_n \to \varphi$ as $n \to \infty$. Then

$$
\begin{aligned}
&|(A\varphi_n)(t) - (A\varphi)(t)| \\
&\leq |Q(t, \varphi_n(t-\tau(t))) - Q(t, \varphi(t-\tau(t)))| \\
&+ \eta \int_{t-T}^{t} \kappa(t,u)\, a(u)\, |Q(u, \varphi_n(u-\tau(u))) - Q(u, \varphi(u-\tau(u)))| du \\
&+ \eta \int_{t-T}^{t} \kappa(t,u)\, |G(u, \varphi_n(u), \varphi_n(u-\tau(u))) - G(u, \varphi(u), \varphi(u-\tau(u)))| du
\end{aligned}
$$

By the Dominated Convergence Theorem, $\lim_{n\to\infty} |(A\varphi_n)(t) - (A\varphi)(t)| = 0$. Then A is continuous.

We next show that A is completely continuous. Let $\varphi \in \mathcal{M}$, then, by Lemma 2, we see that

$$\|A\varphi\| \leq L.$$

And so the family of functions $A\varphi$ is uniformly bounded. Again, let $\varphi \in \mathcal{M}$. Without loss of generality, we can pick $\omega < t$ such that $t - \omega < T$. Then

$$\begin{aligned}
&|(A\varphi)(t) - (A\varphi)(\omega)| \\
&\leq |Q(t, \varphi(t - \tau(t))) - Q(\omega, \varphi(\omega - \tau(\omega)))| \\
&+ \eta \left| \int_{t-T}^{t} \kappa(t,u)\, a(u)\, Q(u, \varphi(u - \tau(u)))\, du \right. \\
&\left. - \int_{\omega-T}^{\omega} \kappa(\omega,u)\, a(u)\, Q(u, \varphi(u - \tau(u)))\, du \right| \\
&+ \eta \left| \int_{t-T}^{t} \kappa(t,u)\, G(u, \varphi(u), \varphi(u - \tau(u)))\, du \right. \\
&\left. - \int_{\omega-T}^{\omega} \kappa(\omega,u)\, G(u, \varphi(u), \varphi(u - \tau(u)))\, du \right| \\
&\leq |Q(t, \varphi(t - \tau(t))) - Q(\omega, \varphi(\omega - \tau(\omega)))| \\
&+ 2\eta\kappa_0 \int_{\omega-T}^{t-T} \left[a(u)\, q_L(u) + g_{\sqrt{2}L}(u) \right] du \\
&+ \eta \int_{\omega-T}^{\omega} |\kappa(t,u) - \kappa(\omega,u)| \left[a(u)\, q_L(u) + g_{\sqrt{2}L}(u) \right] du \\
&\leq |Q(t, \varphi(t - \tau(t))) - Q(\omega, \varphi(\omega - \tau(\omega)))| \\
&+ 2\eta\kappa_0 \int_{\omega}^{t} \left[a(u)\, q_L(u) + g_{\sqrt{2}L}(u) \right] du \\
&+ \eta \int_{0}^{T} |\kappa(t,u) - \kappa(\omega,u)| \left[a(u)\, q_L(u) + g_{\sqrt{2}L}(u) \right] du,
\end{aligned}$$

where $\kappa_0 = \max_{u \in [t-T,t]} \{\kappa(t,u)\}$, then by the Dominated Convergence Theorem $|(A\varphi)(t) - (A\varphi)(\omega)| \to 0$ as $t - \omega \to 0$ independently of $\varphi \in \mathcal{M}$. Thus $(A\varphi)$ is equicontinuous. Hence by Ascoli-Arzela's theorem A is completely continuous. □

Now, we state an important result see [1, Theorem 3.4] and for convenience we present below its proof, we deduce by this theorem that the following are sufficient conditions implying that the mapping H given by (12) is a large contraction on the set $\mathcal{M}$.

(H5) $h : \mathbb{R} \to \mathbb{R}$ is continuous on $[-L, L]$ and differentiable on $(-L, L)$,

(H6) the function h is strictly increasing on $[-L, L]$,

(H7) $\sup_{t \in (-L,L)} h'(t) \leq 1$.

Theorem 2 *Let* $h : \mathbb{R} \to \mathbb{R}$ *be a function satisfying (H5)–(H7). Then the mapping* H *in (12) is a large contraction on the set* $\mathcal{M}$.

Proof. Let $\varphi, \psi \in \mathcal{M}$ with $\varphi \neq \psi$. Then $\varphi(t) \neq \psi(t)$ for some $t \in \mathbb{R}$. Let us denote the set of all such t by $D(\varphi, \psi)$, i.e.,

$$D(\varphi, \psi) = \{t \in \mathbb{R} : \varphi(t) \neq \psi(t)\}.$$

For all $t \in D(\varphi, \psi)$, we have

$$\begin{aligned} &|(H\varphi)(t) - (H\psi)(t)| \\ &\leq |\varphi(t) - \psi(t) - h(\varphi(t)) + h(\psi(t))| \\ &\leq |\varphi(t) - \psi(t)| \left| 1 - \frac{h(\varphi(t)) - h(\psi(t))}{\varphi(t) - \psi(t)} \right|. \end{aligned} \tag{19}$$

Since h is a strictly increasing function we have

$$\frac{h(\varphi(t)) - h(\psi(t))}{\varphi(t) - \psi(t)} > 0 \text{ for all } t \in D(\varphi, \psi). \tag{20}$$

For each fixed $t \in D(\varphi, \psi)$ define the interval $I_t \subset [-L, L]$ by

$$I_t = \begin{cases} (\varphi(t), \psi(t)) & \text{if } \varphi(t) < \psi(t), \\ (\psi(t), \varphi(t)) & \text{if } \psi(t) < \varphi(t). \end{cases}$$

The Mean Value Theorem implies that for each fixed $t \in D(\varphi, \psi)$ there exists a real number $c_t \in I_t$ such that

$$\frac{h(\varphi(t)) - h(\psi(t))}{\varphi(t) - \psi(t)} = h'(c_t).$$

By (H6) and (H7) we have

$$0 \leq \inf_{u \in (-L, L)} h'(u) \leq \inf_{u \in I_t} h'(u) \leq h'(c_t) \leq \sup_{u \in I_t} h'(u) \leq \sup_{u \in (-L, L)} h'(u) \leq 1. \tag{21}$$

Hence, by (19)–(21) we obtain

$$|(H\varphi)(t) - (H\psi)(t)| \leq |\varphi(t) - \psi(t)| \left| 1 - \inf_{u \in (-L, L)} h'(u) \right|, \tag{22}$$

for all $t \in D(\varphi, \psi)$. This implies a large contraction in the supremum norm. To see this, choose a fixed $\epsilon \in (0, 1)$ and assume that φ and ψ are two functions in $\mathcal{M}$ satisfying

$$\epsilon \leq \sup_{t \in (-L, L)} |\varphi(t) - \psi(t)| = \|\varphi - \psi\|.$$

If $|\varphi(t) - \psi(t)| \le \frac{\epsilon}{2}$ for some $t \in D(\varphi, \psi)$, then we get by (21) and (22) that

$$|(H\varphi)(t) - (H\psi)(t)| \le |\varphi(t) - \psi(t)| \le \frac{1}{2}\|\varphi - \psi\|. \tag{23}$$

Since h is continuous and strictly increasing, the function $h\left(u + \frac{\epsilon}{2}\right) - h(u)$ attains its minimum on the closed and bounded interval $[-L, L]$. Thus, if $\frac{\epsilon}{2} \le |\varphi(t) - \psi(t)|$ for some $t \in D(\varphi, \psi)$, then by (H6) and (H7) we conclude that

$$1 \ge \frac{h(\varphi(t)) - h(\psi(t))}{\varphi(t) - \psi(t)} > \lambda,$$

where

$$\lambda := \frac{1}{2L} \min\left\{h\left(u + \frac{\epsilon}{2}\right) - h(u) : u \in [-L, L]\right\} > 0.$$

Hence, (19) implies

$$|(H\varphi)(t) - (H\psi)(t)| \le (1 - \lambda)\|\varphi - \psi\|. \tag{24}$$

Consequently, combining (23) and (24) we obtain

$$|(H\varphi)(t) - (H\psi)(t)| \le \delta\|\varphi - \psi\|, \tag{25}$$

where

$$\delta = \max\left\{\frac{1}{2}, 1 - \lambda\right\}.$$

The proof is complete. □

The next result shows the relationship between the mappings H and B in the sense of large contractions. Assume that

$$\max\{|H(-L)|, |H(L)|\} \le \frac{2L}{J}. \tag{26}$$

Lemma 4 *Let B be defined by (15), suppose (H5)–(H6) hold. Then $B : \mathcal{M} \to \mathcal{M}$ is a large contraction.*

Proof. Let B be defined by (15). Obviously, $B\varphi$ is continuous and it is easy to show that $(B\varphi)(t + T) = (B\varphi)(t)$. Let $\varphi \in \mathcal{M}$

$$\begin{aligned}|(B\varphi)(t)| &\le \int_{t-T}^{t} \kappa(t, u)\, a(u) \max\{|H(-L)|, |H(L)|\}\, du \\ &\le \frac{2L}{J} < L,\end{aligned}$$

which implies $B : \mathcal{M} \to \mathcal{M}$.

By Theorem 2, H is large contraction on $\mathcal{M}$, then for any $\varphi, \psi \in \mathcal{M}$, with $\varphi \neq \psi$ and for any $\epsilon > 0$, from the proof of that Theorem, we have found a $\delta < 1$, such that

$$\begin{aligned}\left|(B\varphi)(t) - (B\psi)(t)\right| &= \left|\eta \int_{t-T}^{t} \kappa(t,u)\, a(u) \left[H(\varphi(u)) - H(\psi(u))\right] du\right| \\ &\leq \|\varphi - \psi\| \eta \int_{t-T}^{t} \kappa(t,u)\, a(u)\, du \leq \delta \|\varphi - \psi\|.\end{aligned}$$

The proof is complete. □

Theorem 3 *Suppose the hypothesis of Lemmas 2, 3 and 4 hold. Let* $\mathcal{M}$ *defined by (11). Then the equation (4) has a* T*-periodic solution in* $\mathcal{M}$*.*

Proof. By Lemma 2, 3, A is continuous and $A(\mathcal{M})$ is contained in a compact set. Also, from Lemma 4, the mapping B is a large contraction. Next, we show that if $\varphi, \psi \in \mathcal{M}$, we have $\|A\psi + B\varphi\| \leq L$. Let $\varphi, \psi \in \mathcal{M}$ with $\|\varphi\|, \|\psi\| \leq L$. By (16)–(18)

$$\begin{aligned}\|A\psi + B\varphi\| &\leq (\gamma_1 + \gamma_2) L + \frac{2}{J} L \\ &\leq \frac{L}{J} + \frac{2L}{J} \leq L.\end{aligned}$$

Clearly, all the hypotheses of the Krasnoselskii-Burton's theorem are satisfied. Thus there exists a fixed point $z \in \mathcal{M}$ such that $z = Az + Bz$. By Lemma 1 this fixed point is a solution of (4). Hence (4) has a T-periodic solution. □

4 Existence of nonnegative periodic solutions

In this section we obtain the existence of a nonnegative periodic solution of (4). By applying Theorem 1, we need to define a closed, convex, and bounded subset $\mathbb{M}$ of P_T. So, let

$$\mathbb{M} = \{\phi \in P_T : 0 \leq \phi \leq K\}. \tag{27}$$

where K is positive constant. To simplify notation, we let

$$m = \min_{u \in [t-T,t]} e^{-\int_u^t a(s) ds}, \quad M = \max_{u \in [t-T,t]} e^{-\int_u^t a(s) ds}. \tag{28}$$

It is easy to see that for all $(t, u) \in [0, 2T]^2$,

$$m \leq \kappa(t, u) \leq M. \tag{29}$$

Then we obtain the existence of a nonnegative periodic solution of (4) by considering the two cases;

(1) $Q(t, y) \geq 0 \ \forall t \in [0, T], \ y \in \mathbb{M}$.

(2) $Q(t, y) \leq 0 \ \forall t \in [0, T], \ y \in \mathbb{M}$.

In the case one, we assume for all $t \in [0, T]$, $x, y \in \mathbb{M}$, that there exist a positive constant c_1 such that

$$0 \leq Q(t, y) \leq c_1 y, \tag{30}$$

$$c_1 < 1, \tag{31}$$

$$0 \leq -a(t) Q(t, y) + G(t, x, y) \tag{32}$$

$$a(t) H(\varphi(t)) - a(t) Q(t, y) + G(t, x, y) \leq \frac{K(1 - c_1)}{M\eta T}. \tag{33}$$

Lemma 5 *Let* A*,* B *given by (14), (15) respectively, assume (30)–(33) hold. Then* $A, B : \mathbb{M} \to \mathbb{M}$*.*

Proof. Let A defined by (15). So, for any $\varphi \in \mathbb{M}$, we have

$$\begin{aligned} 0 &\leq (A\varphi)(t) \leq Q(t, \varphi(t - \tau(t))) \\ &+ \eta \int_{t-T}^{t} \kappa(t, u) [-a(u) Q(u, \varphi(u - \tau(u))) + G(u, \varphi(u), \varphi(u - \tau(u)))] \, du \\ &\leq \eta \int_{t-T}^{t} M \frac{K(1 - c_1)}{M\eta T} du + c_1 K = K, \end{aligned}$$

That is $A\varphi \in \mathbb{M}$.

Now, let B defined by (15). So, for any $\varphi \in \mathbb{M}$, we have

$$0 \leq (B\varphi)(t) \leq \eta \int_{t-T}^{t} M \frac{K(1 - c_1)}{M\eta T} du \leq \eta M T \frac{K}{M\eta T} = K.$$

That is $B\varphi \in \mathbb{M}$. □

Theorem 4 *Suppose the hypothesis of Lemmas 3, 4 and 5 hold. Then equation (4) has a nonnegative* T*-periodic solution* x *in the subset* $\mathbb{M}$.

Proof. By Lemma 3, A is completely continuous. Also, from Lemma 4, the mapping B is a large contraction. By Lemma 5, $A, B : \mathbb{M} \to \mathbb{M}$. Next, we show that if $\varphi, \psi \in \mathbb{M}$, we have $0 \le A\psi + B\varphi \le K$. Let $\varphi, \psi \in \mathbb{M}$ with $0 \le \varphi, \psi \le K$. By (30)–(33)

$$\begin{aligned}
&(A\psi)(t) + (B\varphi)(t) \\
&= \eta \int_{t-T}^{t} \kappa(t,u)\, a(u)\, H(\varphi(u))\, du + Q(t, \psi(t - \tau(t))) \\
&+ \eta \int_{t-T}^{t} \kappa(t,u) \left[-a(u)\, Q(u, \psi(u - \tau(u))) + G(u, \psi(u), \psi(u - \tau(u)))\right] du \\
&\le \eta \int_{t-T}^{t} \kappa(t,u) \frac{K(1 - c_1)}{M\eta T} du + c_1 K \\
&\le \eta \int_{t-T}^{t} M \frac{K(1 - c_1)}{M\eta T} du + c_1 K = K.
\end{aligned}$$

On the other hand,

$$(A\psi)(t) + (B\varphi)(t) \ge 0.$$

Clearly, all the hypotheses of the Krasnoselskii-Burton's theorem are satisfied. Thus there exists a fixed point $z \in \mathbb{M}$ such that $z = Az + Bz$. By Lemma 1 this fixed point is a solution of (4) and the proof is complete. □

In the case two, we substitute conditions (30)–(33) with the following conditions respectively. We assume that there exist a negative constant c_2 such that

$$c_2 y \le Q(t, y) \le 0, \tag{34}$$

$$-c_2 < 1, \tag{35}$$

$$\frac{-c_2 K}{M\eta T} \le a(t) H(\varphi(t)) - a(t)\, Q(t, y) + G(t, x, y). \tag{36}$$

$$a(t) H(\varphi(t)) - a(t)\, Q(t, y) + G(t, x, y) \le \frac{K}{M\eta T}. \tag{37}$$

Theorem 5 *Suppose (34)–(37) and the hypothesis of Lemmas 2, 3 and 4 hold. Then equation (4) has a nonnegative* T*-periodic solution* x *in the subset* $\mathbb{M}$.

Proof. By Lemma 2, 3, A is completely continuous. Also, from Lemma 4, the mapping B is a large contraction. To see that, it is easy to show as in Lemma 5 $A, B : \mathbb{M} \to \mathbb{M}$. Next, we show that if $\varphi, \psi \in \mathbb{M}$, we have $0 \leq A\psi + B\varphi \leq K$. Let $\varphi, \psi \in \mathbb{M}$ with $0 \leq \varphi, \psi \leq K$. By (34)–(37)

$$\begin{aligned}
&(A\psi)(t) + (B\varphi)(t) \\
&= \eta \int_{t-T}^{t} \kappa(t,u)\, a(u)\, H(\varphi(u))\, du + Q(t, \psi(t-\tau(t))) \\
&+ \eta \int_{t-T}^{t} \kappa(t,u) \left[-a(u)\, Q(u, \psi(u-\tau(u))) + G(u, \psi(u), \psi(u-\tau(u)))\right] du \\
&\leq \eta \int_{t-T}^{t} \kappa(t,u) \frac{K}{M\eta T} du = \eta \int_{t-T}^{t} M \frac{K}{M\eta T} du = K.
\end{aligned}$$

On the other hand,

$$(A\psi)(t) + (B\varphi)(t) \geq \eta \int_{t-T}^{t} M \frac{-c_2 K}{M\eta T} du + c_2 K = 0.$$

Clearly, all the hypotheses of the Krasnoselskii-Burton's theorem are satisfied. Thus there exists a fixed point $z \in \mathbb{M}$ such that $z = Az + Bz$. By Lemma 1 this fixed point is a solution of (4) and the proof is complete. □

Acknowledgements

The authors would like to thank the anonymous referee for his/her valuable comments and good advice.

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Received: December 10, 2014