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Ricci solitons on QR-hypersurfaces of a quaternionic space form $\mathbb{Q}^n$

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Abstract. The purpose of this paper is to study Ricci solitons on QR-hypersurfaces M of a quaternionic space form $\mathbb{Q}^n$ such that the shape operator A with respect to N has one eigenvalue. We prove that Ricci soliton on QR- hypersurfaces M with eigenvalue zero is steady and for eigenvalue nonzero is shrinking.

1 Introduction

A Ricci soliton is defined on a Riemannian manifold (M, g) by

$$\frac{1}{2}\mathcal{L}_V g + \mathrm{Ric} - \lambda g = 0 \tag{1}$$

where $L_V g$ is the Lie-derivative of the metric tensor g with respect to V and λ is a constant on M. The Ricci soliton is a natural generalization of an Einstein metric. The Ricci soliton is said to be shrinking, steady and expanding according as $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$, respectively. Compact Ricci solitons are the fixed points of the Ricci flow:

$$\frac{\partial}{\partial t} g(t) = -2\mathrm{Ric}(g(t)) \tag{2}$$

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projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings and often arise as blow-up limits for the Ricci flow on compact manifolds. We denote a Ricci soliton by $(M, g, V; \lambda)$ and call the vector field V the potential vector field of the Ricci soliton. A trivial Ricci soliton is one for which V is Killing or zero. If its potential vector field $V = \nabla f$ such that f is some smooth function on M then a Ricci soliton $(M, g, V; \lambda)$ is called a gradient Ricci soliton and the smooth function f is called the potential function. It was proved by Grigory Perelman in [15] that any compact Ricci soliton is the sum of a gradient of some smooth function f up to the addition of a Killing field. Thus compact Ricci solitons are gradient Ricci solitons. In particular, Perelman applied Ricci solitons to solve the long standing Poincare conjecture posed in 1904.

Hamilton [7] and Ivey [10] proved that a Ricci soliton on a compact manifold has constant curvature in dimension 2 and 3, respectively. In [11], Ki proved that there are no real hypersurfaces with parallel Ricci tensor in a complex space form $\widetilde{M}^n(c)$ with $c \neq 0$ when $n \geq 3$. Kim [12] proved that when $n = 2$, this is also true. In particular, these results give that there is not any Einstein real hypersurfaces in a non-flat complex space form.

In [13], Chen studied important results on Ricci solitons which occur obviously on some Riemannian submanifolds. He presented several recent new criterions of trivial compact shrinking Ricci solitons.

Cho and Kimura [3] studied on Ricci solitons of real hypersurfaces in a non-flat complex space form and showed that a real hypersurface M in a non-flat complex space form $\widetilde{M}^n(c \neq 0)$ does not admit a Ricci soliton such that the Reeb vector field ξ is potential vector field. They defined so called η-Ricci soliton, such that satisfies

$$\frac{1}{2}\mathcal{L}_V g + \mathrm{Ric} - \lambda g - \mu \eta \otimes \eta = 0 \tag{3}$$

where λ, μ are constants. They first proved that a real hypersurface M of a non-flat complex space form $\widetilde{M}^n(c)$ which accepts an η-Ricci soliton is a Hopf-hypersurface and classified that η-Ricci soliton real hypersurfaces in a non-flat complex space form.

We study Ricci solitons on QR-hypersurfaces M of a quaternionic space form $\mathbb{Q}^n$ such that the shape operator A with respect to N has one eigenvalues . We prove that Ricci soliton on QR- hypersurfaces M with eigenvalue zero is steady and for eigenvalue nonzero is shrinking.

2 Preliminaries

Let $\overline{M}$ be a real $(n+p)$-dimensional quaternionic Kähler manifold. Then, by definition, there is a 3-dimensional vector bundle V consisting with tensor fields of type $(1,1)$ over $\overline{M}$ satisfying the following conditions $(a), (b)$ and (c):
(a) In any coordinate neighborhood $\overline{\mathcal{U}}$, there is a local basis $\{F, G, H\}$ of V such that

$$\begin{aligned} &F^2 = -I, && G^2 = -I, && H^2 = -I, \\ &FG = -GF = H, && MGH = -HG = F, && HF = -FH = G. \end{aligned} \tag{4}$$

(b) There is a Riemannian metric g which is hermite with respect to all of F, G and H.
(c) For the Riemannian connection $\overline{\nabla}$ with respect to g

$$\begin{pmatrix} \overline{\nabla}F \\ \overline{\nabla}G \\ \overline{\nabla}H \end{pmatrix} = \begin{pmatrix} 0 & r & -q \\ -r & 0 & p \\ q & -p & 0 \end{pmatrix} \begin{pmatrix} F \\ G \\ H \end{pmatrix} \tag{5}$$

where p, q and r are local 1-forms defined in $\overline{\mathcal{U}}$. Such a local basis $\{F, G, H\}$ is called a canonical local basis of the bundle V in $\overline{\mathcal{U}}$ [9].

For canonical local basis $\{F, G, H\}$ and $\{F', G', H'\}$ of V in coordinate neighborhoods of $\overline{\mathcal{U}}$ and $\overline{\mathcal{U}}'$, it follows that in $\overline{\mathcal{U}} \cap \overline{\mathcal{U}}'$

$$\begin{pmatrix} F' \\ G' \\ H' \end{pmatrix} = \begin{pmatrix} s_{xy} \end{pmatrix} \begin{pmatrix} F \\ G \\ H \end{pmatrix} \qquad (x, y = 1, 2, 3)$$

where s_{xy} are local differentiable functions with $(s_{xy}) \in SO(3)$ as a consequence of (4). As is well known [9], every quaternionic Kähler manifold is orientable. Let $\overline{M}$ be quaternion Kaehler manifold and M be a real submanifold of $\overline{M}$. Then, M is said QR-submanifold if there exists a vector subbundle ν of the normal bundle such that we have

$$\begin{aligned} &F\nu_x = \nu_x, && G\nu_x = \nu_x, && H\nu_x = \nu_x, \\ &F\nu_x^{\perp}, && G\nu_x^{\perp}, && H\nu_x^{\perp} \subset T_xM, \end{aligned}$$

for $x \in M$, where $\nu^{\perp}$ is the complementary orthogonal bundle to ν in $TM^{\perp}$. We denote by D the complementary orthogonal distribution to $D^{\perp}$ in TM. Then D is invariant with respect to the action of $\{F, G, H\}$ i.e. we have

$$\begin{aligned} &FD_x = D_x, && GD_x = D_x, && HD_x = D_x, \\ &FD_x^{\perp}, && GD_x^{\perp}, && HD_x^{\perp} \subset T_x^{\perp}M, \end{aligned}$$

for any $x \in M$, where $TM = D \oplus D^{\perp}$ and $TM^{\perp} = \nu \oplus \nu^{\perp}$. D is called quaternion distribution.

Now let M be an n-dimensional QR-submanifold of maximal QR-dimension, that is, of $(p-1)$ QR-dimension isometrically immersed in $\overline{M}$. Then by definition there is a unit normal vector field N such that $\nu_x^{\perp} = span\{N\}$ at each point x in M. We set

$$FN = -U, \quad GN = -V, \quad HN = -W. \tag{6}$$

Denoting by D_x the maximal quaternionic invariant subspace

$$T_xM \cap FT_xM \cap GT_xM \cap HT_xM,$$

of T_xM, we have $D_x^{\perp} \supset \mathrm{Span}\{U, V, W\}$, where $D_x^{\perp}$ means the complementary orthogonal subspace to D_x in T_xM. But, using (4), we can prove that $D_x^{\perp} = \mathrm{Span}\{U, V, W\}$ [13]. Thus we have

$$T_xM = D_x \oplus \mathrm{Span}\{U, V, W\}, \quad \forall x \in M,$$

which together with (4) and (6) imply

$$FT_xM, GT_xM, HT_xM \subset T_xM \oplus \mathrm{Span}\{\xi\}.$$

Therefore, for any tangent vector field X and for a local orthonormal basis $\{N_\alpha\}_{\alpha=1,...,p}$ $(N_1 := N)$ of normal vectors to M, we have

$$FX = \varphi X + u(X)N, \quad GX = \psi X + v(X)N, \quad HX = \theta X + \omega(X)N, \tag{7}$$

$$\begin{aligned} FN_\alpha &= -U_\alpha + P_1N_\alpha, \quad GN_\alpha = -V_\alpha + P_2N_\alpha, \\ HN_\alpha &= -W_\alpha + P_3N_\alpha, \quad (\alpha = 1, ..., p). \end{aligned} \tag{8}$$

Then it is easily seen that $\{\varphi, \psi, \theta\}$ and $\{P_1, P_2, P_3\}$ are skew-symmetric endomorphisms acting on T_xM and $T_xM^{\perp}$, respectively.

Moreover, the hermitian property of $[F, G, H\}$ implies

$$\begin{aligned} g(X, \varphi U_\alpha) &= -u(X)g(N_1, P_1N_\alpha), \\ g(X, \psi V_\alpha) &= -v(X)g(N_1, P_2N_\alpha), \\ g(X, \theta W_\alpha) &= -w(X)g(N_1, P_3N_\alpha), \quad (\alpha = 1, ..., p). \end{aligned} \tag{9}$$

Also, from the hermitian properties

$$\begin{aligned} g(FX, N_\alpha) &= -g(X, FN_\alpha), \quad g(GX, N_\alpha) = -g(X, GN_\alpha), \\ g(HX, N_\alpha) &= -g(X, HN_\alpha), \quad (\alpha = 1, ..., p). \end{aligned}$$

It follows that

$$g(X,U_\alpha)=u(X)\delta_{1\alpha},\quad g(X,V_\alpha)=v(X)\delta_{1\alpha},\quad g(X,W_\alpha)=w(X)\delta_{1\alpha},$$

and hence

$$\begin{aligned}&g(X,U_1)=u(X),\quad g(X,V_1)=v(X),\quad g(X,W_1)=w(X),\\&U_\alpha=0,\quad V_\alpha=0,\quad W_\alpha=0,\quad (\alpha=2,...p).\end{aligned}\tag{10}$$

On the other hand, comparing (6) and (8) with $\alpha=1$, we have $U_1=U, V_1=V, W_1=W$, which together with (6) and (10) imply

$$\begin{array}{lll}g(X,U)=u(X), & g(X,V)=v(X), & g(X,W)=w(X),\\ u(U)=1, & v(V)=1, & w(W)=1,\\ FN=-U, & GN=-V, & HN=-W\\ FN_{\alpha=P_1N_\alpha}, & GN_{\alpha=P_2N_\alpha} & HN_{\alpha=P_3N_\alpha},\quad (\alpha=2,...,p).\end{array}$$

from which, taking account of the skew-symmetry of P_1, P_2 and P_3 and using (9), we also have

$$\begin{array}{lll}u(\varphi X)=0, & v(\psi X)=0, & w(\theta X)=0,\\ \varphi U=0, & \psi V=0, & \theta W=0,\\ P_1N=0, & P_2N=0, & P_3N=0,\end{array}\tag{11}$$

From the equations of (6), we also have

$$\begin{array}{llll}\psi U=-W, & v(U)=0, & \theta U=V, & w(U)=0,\\ \varphi V=W, & u(V)=0, & \theta V=-U, & w(V)=0,\\ \varphi W=-V, & u(W)=0, & \psi W=U, & v(W)=0.\end{array}\tag{12}$$

Now, let ∇ be the Levi-Civita connection on M and $\nabla^\perp$ the normal connection induced from $\overline{\nabla}$ in the normal bundle $TM^\perp$ of M. The Gauss and Weingarten formula are given by

$$\begin{aligned}&\overline{\nabla}_XY=\nabla_XY+h(X,Y),\\&\overline{\nabla}_XN_\alpha=-A_\alpha X+\nabla^\perp_XN_\alpha,\quad (\alpha=1,\dots,p),\end{aligned}\tag{13}$$

for any $X,Y\in\chi(M)$ and $N_\alpha\in\Gamma^\infty(T(M)^\perp)$, $(\alpha=1,\dots,p)$. h is the second fundamental form and A_α are shape operator corresponding to N_α.

Next, differentiating the equations of (6) covariantly and comparing the tangential and normal parts, we have

$$\begin{aligned} \nabla_Y U &= r(Y)V - q(Y)W + \varphi A_1 Y, \\ \nabla_Y V &= -r(Y)U + p(Y)W + \psi A_1 Y, \\ \nabla_Y W &= q(Y)U - p(Y)V + \theta A_1 Y, \end{aligned} \tag{14}$$

For QR-hypersurfaces M in a quaternionic space form $\overline{M}$ of quaternionic sectional curvature $4k$ the Gauss and Codazzi equations are written as follow:

$$\begin{aligned} g(R(X,Y)Z,W) = k\{&g(Y,Z)X - g(X,Z)Y \\ &+ g(\varphi Y,Z)\varphi X - g(\varphi X,Z)\varphi Y - 2g(\varphi X,Y)\varphi Z \\ &+ g(\psi Y,Z)\psi X - g(\psi X,Z)GY - 2g(\psi X,Y)\psi Z \\ &+ g(\theta Y,Z)\theta X - g(\theta X,Z)\theta Y - 2g(\theta X,Y)\theta Z\} \\ &+ g(AY,Z)AX - g(AX,Z)AY, \end{aligned} \tag{15}$$

$$\begin{aligned} (\nabla_X A)Y - (\nabla_Y A)X = k\{&u(X)\varphi Y - u(Y)\varphi X - 2g(\varphi X,Y)U \\ &+ v(X)\psi Y - v(Y)\psi X - 2g(\psi X,Y)V \\ &+ w(X)\theta Y - w(Y)\theta X - 2g(\theta X,Y)W\}, \end{aligned} \tag{16}$$

hence the Ricci tensor is obtained as

$$\begin{aligned} Ric(X,Y) = k\{&(4n+7)g(X,Y) - 3\{u(X)u(Y) + v(X)v(Y) + w(X)w(Y)\}\} \\ &+ (\mathrm{trace}\, A)g(AX,Y) - g(AX,AY). \end{aligned} \tag{17}$$

for any tangent vector fields X, Y, Z on M, where R and Ric are the curvature and Ricci tensors of M, respectively.

3 Ricci soliton on QR hypersurfaces

Let M be a QR-hypersurface of a quaternionic space form $\overline{M}$ such that the shape operator A for unit normal vector field N has only one eigenvalue and let $\{e_1, \ldots, e_{4n-4}, U, V, W\}$ be a local orthonormal fram field such that $D^\perp =$ span$\{U, V, W\}$ and $D =$ span$\{e_1, \ldots, e_{n-1}, e_n = \varphi e_1, \ldots, e_{2n-2} = \varphi e_{n-1}, e_{2n-1} = \psi e_1, \ldots, e_{3n-3} = \psi e_{n-1}, e_{3n-2} = \theta e_1, \ldots, e_{4n-4} = \theta e_{n-1}\}$.
We first prove

Theorem 1 *If the shape operator* A *with respect to unit normal vector field* N *of* M *has only one eigenvalue, then* $\overline{M}$ *is a quaternionic Euclidean space.*

Proof. According to the assumption, it follows that $A = 0$ or $AX = \alpha X$ for all $X \in T(M)$.
In both cases the Codazzi equation (16), we obtain

$$\begin{aligned}(X\alpha)Y - (Y\alpha)X = k\{&u(X)\varphi Y - u(Y)\varphi X - 2g(\varphi X, Y)U \\ &+ v(X)\psi Y - v(Y)\psi X - 2g(\psi X, Y)V \\ &+ w(X)\theta Y - w(Y)\theta X - 2g(\theta X, Y)W\},\end{aligned} \tag{18}$$

for all $X, Y \in TM$. Putting $Y = U$, the equation (21) reduces to

$$(X\alpha)U - (U\alpha)X \quad = \quad k\{-\varphi X + v(X)W - w(X)V\}, \tag{19}$$

also by putting $Y = V$ and $Y = W$, we have

$$\begin{aligned}(X\alpha)V - (V\alpha)X = k\{-\psi X + w(X)U - u(X)W\}, \\ (X\alpha)W - (W\alpha)X = k\{-\theta X + u(X)V - v(X)U\}.\end{aligned} \tag{20}$$

since dim $M \geq 7$, we can use $X, \varphi X, \psi X, \theta X, U, V$ and W in such a way that they are linearly independent and thus $k = 0$. □

Let $AX = \alpha X$, therefore by the relation (17), we obtain

$$\begin{aligned}
&Ric(e_i, e_j) = \{(4n-2)\alpha^2\}\delta_{ij}, \qquad (i, j = 1, \ldots, 4n-4), \\
&Ric(U, U) = (4n-2)\alpha^2, \\
&Ric(V, V) = (4n-2)\alpha^2, \\
&Ric(W, W) = (4n-2)\alpha^2, \\
&Ric(U, V) = 0, \\
&Ric(U, W) = 0, \\
&Ric(W, V) = 0, \\
&Ric(e_i, U) = 0, \\
&Ric(e_i, V) = 0, \\
&Ric(e_i, W) = 0, \qquad (i = 1, ..., 4n-4).
\end{aligned} \tag{21}$$

We consider QR-hypersurface M of a quaternionic space form $\mathbb{Q}^n$ satisfying Ricci soliton equation

$$\frac{1}{2}\mathcal{L}_{\widetilde{V}}g + Ric - \lambda g = 0 \tag{22}$$

with respect to potential vector field $\widetilde{V}$ on M for constant λ.
First Put

$$\widetilde{V} := fU, \qquad (f : M \to \mathbb{R}, f \neq 0) \tag{23}$$

Then definition of the Lie derivative and the first relation (14) imply

$$\begin{aligned}(\mathcal{L}_{fU}g)(X,Y) = {} & df(X)u(Y) + df(Y)u(X) \\ & + f\{r(X)v(Y) - q(X)w(Y) + r(Y)v(X) - q(Y)w(X) \\ & + g((\varphi A - A\varphi)Y, X)\}\end{aligned} \tag{24}$$

We compute

$$\begin{aligned}
&(\mathcal{L}_{fU}g)(U,U) = 2df(U), \\
&(\mathcal{L}_{fU}g)(V,V) = 2fr(V), \\
&(\mathcal{L}_{fU}g)(W,W) = -2fq(W), \\
&(\mathcal{L}_{fU}g)(U,V) = df(V) + fr(U), \\
&(\mathcal{L}_{fU}g)(U,W) = df(W) - fq(U), \\
&(\mathcal{L}_{fU}g)(W,V) = f\{-q(V) + r(W)\}, \\
&(\mathcal{L}_{fU}g)(U,e_i) = df(e_i), \\
&(\mathcal{L}_{fU}g)(V,e_i) = fr(e_i), \\
&(\mathcal{L}_{fU}g)(W,e_i) = -fq(e_i), \qquad (i = 1, \dots, 4n-4), \\
&(\mathcal{L}_{f\xi}g)(e_i,e_j) = 0 \qquad (i,j = 1, \dots, 4n-4).
\end{aligned} \tag{25}$$

Using relations (21) and (25), Ricci soliton equation (22) is equivalent to

$$\begin{aligned}
&df(U) = \lambda - (4n-2)\alpha^2, \\
&fr(V) = \lambda - (4n-2)\alpha^2, \\
&fq(W) = -\lambda + (4n-2)\alpha^2, \\
&df(V) = -fr(U), \\
&df(W) = fq(U), \\
&q(V) = r(W), \\
&df(e_i) = 0, \qquad (i = 1, \dots, 4n-4), \\
&r(e_i) = 0, \\
&q(e_i) = 0, \\
&\{(4n-2)\alpha^2 - \lambda\}\delta_{ij} = 0, \qquad (i,j = 1, \dots, 4n-4).
\end{aligned} \tag{26}$$

By the last relation (26), we have $\lambda = (4n-2)\alpha^2$ and thus the following theorem holds:

Theorem 2 *Let* M *be a* QR*-hypersurface of quternionic space form* $\mathbb{Q}^n$ *with* $AX = \alpha X$*. Then a Ricci soliton* $(M, g, \widetilde{V}, \lambda)$ *with potential vector field* $\widetilde{V} := fU$ *is shrinking Ricci soliton.*

Now, let $A = 0$, using relation (17),it follows that

$$\mathrm{Ric} = 0 \tag{27}$$

QR-hypersurface M $(n \geq 2)$ is considered in a quaternionic space form $\mathbb{Q}^n$ satisfying Ricci soliton equation.
By relations (27) and (25), Ricci soliton equation (22) is equivalent to

$$\begin{aligned}
&df(U) = \lambda,\\
&fr(V) = \lambda,\\
&fq(W) = -\lambda,\\
&df(V) = -fr(U),\\
&df(W) = fq(U),\\
&q(V) = r(W),\\
&df(e_i) = 0, \qquad (i = 1, \ldots, 4n-4),\\
&r(e_i) = 0,\\
&q(e_i) = 0,\\
&\lambda\delta_{ij} = 0, \qquad (i, j = 1, \ldots, 4n-4).
\end{aligned} \tag{28}$$

Using the last relation (28), it follows $\lambda = 0$ and hence

Theorem 3 *Let* M *be a* QR*-hypersurface of quaternionic space form* $\mathbb{Q}^n$ *with* $A = 0$*. Then a Ricci soliton* $(M, g, \widetilde{V}, \lambda)$ *with potential vector field* $V := fU$ *is steady Ricci soliton.*

hence, similar results were obtained when each structural vector fields $\{V, W\}$ of structure quaternionic $\{U, V, W\}$ be the potential vector field.

References

[1] C. Boyer, K. Galicki, *Sasakian geometry*, Oxford Mathematical Monographs. Oxford University Press, Oxford, 2008.

[2] B.Y. Chen, *Ricci solitons on Riemannian submanifolds*,Riemannian Geometry and Applications to Engineering and Economics Bucharest, Romania, Proceedings of the Conference RIGA 2014 (May 19-21, 2014, Bucuresti).

[3] J. T. Cho, M. Kimura, Ricci solitons and real hypersurfaces in a complex space form, *Tohoku Math. J.*, **61** (2009), 205–212.

[4] J. T. Cho, M. Kimura, Ricci solitons on locally conformally flat hypersurfaces in space forms, *Journal of Geometry and Physics*, **62** (2012), 1882–1891.

[5] B. Chow, D. Knopf, *The Ricci flow: An introduction* , Math. Surveys Monogr. 110, American Mathematical Society, Providence, RI, 2004.

[6] M. Djorić, M. Okumura, CR *submanifolds of complex projective space*, Springer, 2010.

[7] R. S. Hamilton, *The Ricci flow on surfaces Mathematics and general relativity* , (Santa Cruz, CA, 1986), 237–262, Contemp. Math. 71, American Math. Soc., Providence, RI, 1988.

[8] C. He, M. Zhu, *Ricci solitons on Sasakian manifolds*, arXiv:1103.4407v2 [math.DG], 26 sep (2011),

[9] S. Ishihara, Quaternion Kaehlerian manifolds, *J. Differential Geom.*, **9** (1974), 483–500.

[10] T. Ivey, Ricci solitons on compact 3-manifolds, *Differential Geom. Appl.*, **3** (1993), 301–307.

[11] U. H. Ki, Real hypersurfaces with parallel Ricci tensor of a complex space form, *Tsukuba J. Math.*, **13** (1989), 73–81.

[12] U. K. Kim, Nonexistence of Ricci-parallel real hypersurfaces in P_2C or H_2C, *Bull. Korean Math. Soc.*, **41** (2004), 699–708.

[13] J. H. Kwon, J. S. Pak, QR-submanifolds of $(p-1)$ QR-dimension in a quaternionic projective space $QP^{\frac{(n+p)}{4}}$, *Acta Math. Hungar.*, **86** (2000), 89–116.

[14] J. Morgan, G. Tian, *Ricci Flow and the Poincare Conjecture*, Clay Mathematics Monographs, 5, Cambridge, MA, 2014.

[15] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, http://arXiv.org/ abs/math.DG/02111159, preprint.

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