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Some Hermite-Hadamard type integral inequalities for operator AG-preinvex functions

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Abstract. In this paper, we introduce the concept of operator AG-preinvex functions and prove some Hermite-Hadamard type inequalities for these functions. As application, we obtain some unitarily invariant norm inequalities for operators.

1 Introduction and preliminaries

The following Hermite-Hadamard inequality holds for any convex function f defined on $\mathbb{R}$

$$(b-a)f\left(\frac{a+b}{2}\right) \leq \int_a^b f(x)dx \leq (b-a)\frac{f(a)+f(b)}{2}, \quad a, b \in \mathbb{R}. \qquad (1)$$

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It was firstly discovered by Hermite in 1881 in the journal Mathesis (see [8]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result [10].

Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by Hadamard in 1893 [2]. In 1974, Mitrinovič found Hermites note in Mathesis [8]. Since (1) was known as Hadamards inequality, the inequality is now commonly referred as the Hermite-Hadamard inequality [10].

Definition 1 [13] *A continuous function* $f : I \subset \mathbb{R} \to \mathbb{R}^+$ *is said to be an AG-convex function (arithmetic-geometrically or log-convex function) on the interval* I *if*

$$f(\lambda a + (1-\lambda)b) \leq f(a)^{\lambda} f(b)^{1-\lambda}. \tag{2}$$

for $a, b \in I$ *and* $\lambda \in [0, 1]$, *i.e.,* $\log f$ *is convex.*

Theorem 1 [13] *Let* f *be an AG-convex function defined on* $[a, b]$. *Then, we have*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \sqrt{f\left(\frac{3a+b}{4}\right) f\left(\frac{a+3b}{4}\right)} \\ &\leq \exp\left(\frac{1}{b-a}\int_a^b \log(f(u))du\right) \\ &\leq \sqrt{f\left(\frac{a+b}{2}\right)}.\sqrt[4]{f(a)}.\sqrt[4]{f(b)} \\ &\leq \sqrt{f(a)f(b)}, \end{aligned} \tag{3}$$

where $u = \log t$.

Let $B(H)$ stands for the C^*-algebra of all bounded linear operators on a complex Hilbert space H with inner product $\langle\cdot,\cdot\rangle$. An operator $A \in B(H)$ is positive and write $A \geq 0$ if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. Let $B(H)_{sa}$ stand for the set of all self-adjoint elements of $B(H)$.

Let A be a self-adjoint operator in $B(H)$. The Gelfand map establishes a $*$-isometrically isomorphism Φ between the set $C(Sp(A))$ of all continuous functions defined on the spectrum of A, denoted by $Sp(A)$, and the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows:

for any $f, g \in C(Sp(A)))$ and any $\alpha, \beta \in \mathbb{C}$ we have:

- $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$;

- $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(\bar{f}) = \Phi(f)^*$;
- $\|\Phi(f)\| = \|f\| := \sup_{t\in \mathrm{Sp}(A)} |f(t)|$;
- $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in \mathrm{Sp}(A)$.

With this notation we define

$$f(A) = \Phi(f) \text{ for all } \ f \in C(\mathrm{Sp}(A))$$

and we call it the continuous functional calculus for a self-adjoint operator A.

If A is a self-adjoint operator and f is a real valued continuous function on $\mathrm{Sp}(A)$, then $f(t) \geq 0$ for any $t \in \mathrm{Sp}(A)$ implies that $f(A) \geq 0$, i.e. $f(A)$ is a positive operator on H. Moreover, if both f and g are real valued functions on $\mathrm{Sp}(A)$ then the following important property holds:

$$f(t) \geq g(t) \ \text{ for any } \ t \in \mathrm{Sp}(A) \ \text{ implies that } \ f(A) \geq g(A), \tag{4}$$

in the operator order of $B(H)$, see [14].

Definition 2 *A real valued continuous function* f *on an interval* I *is said to be operator convex function if*

$$f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda)f(B),$$

in the operator order, for all $\lambda \in [0,1]$ *and self-adjoint operators* A *and* B *in* $B(H)$ *whose spectra are contained in* I*.*

In [4] Dragomir investigated the operator version of the Hermite-Hadamard inequality for operator convex functions. Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I then, for any self-adjoint operators A and B with spectra in I, the following inequalities holds

$$\begin{aligned} f\left(\frac{A+B}{2}\right) &\leq 2\int_{\frac{1}{4}}^{\frac{3}{4}} f(tA + (1-t)B)dt \\ &\leq \frac{1}{2}\left[f\left(\frac{3A+B}{4}\right) + f\left(\frac{A+3B}{4}\right)\right] \\ &\leq \int_0^1 f\left((1-t)A + tB\right) dt \\ &\leq \frac{1}{2}\left[f\left(\frac{A+B}{2}\right) + \frac{f(A)+f(B)}{2}\right] \\ &\leq \frac{f(A)+f(B)}{2}, \end{aligned}$$

for the first inequality in above, see [12].

In [5], Ghazanfari et al. gave the concept of operator preinvex function and obtained Hermite-Hadamard type inequality for operator preinvex function.

Definition 3 [5] *Let* X *be a real vector space, a set* $S \subseteq X$ *is said to be invex with respect to the map* $\eta : S \times S \to X$*, if for every* $x, y \in S$ *and* $t \in [0,1]$*,*

$$x + t\eta(x,y) \in S.$$

It is obvious that every convex set is invex with respect to the map $\eta(x,y) = x - y$, but there exist invex sets which are not convex (see [1]).

Let $S \subseteq X$ be an invex set with respect to η. For every $x, y \in S$. the η-path P_{xv} joining the points x and $v := x + \eta(y,x)$ is defined as follows

$$P_{xv} := \{z : z = x + t\eta(y,x), t \in [0,1]\}.$$

The mapping η is said to satisfy the condition (C) if for every $x, y \in S$ and $t \in [0,1]$,

$$\eta(y, y + t\eta(y,x)) = -t\eta(x,y), \quad \eta(x, y + t\eta(x,y)) = (1-t)\eta(x,y).$$

Note that for every $x, y \in S$ and every $t_1, t_2 \in [0,1]$, from conditions in (C), we have

$$\eta(y + t_2\eta(x,y), y + t_1\eta(x,y)) = (t_2 - t_1)\eta(x,y), \tag{5}$$

see [9] for details.

Definition 4 *Let* $S \subseteq B(H)_{sa}$ *be an invex set with respect to* $\eta : S \times S \to B(H)_{sa}$*. Then, the continuous* $f : \mathbb{R} \to \mathbb{R}$ *is said to be operator preinvex with respect to* η *on* S*, if for every* $A, B \in S$ *and* $t \in [0,1]$*,*

$$f(A + t\eta(B,A)) \le (1-t)f(A) + tf(B), \tag{6}$$

in the operator order in $B(H)$*.*

Every operator convex function is operator preinvex with respect to the map $\eta(A,B) = A - B$, but the converse does not hold (see [5]).

Theorem 2 [5] *Let* $S \subseteq B(H)_{sa}$ *be an invex set with respect to* $\eta : S \times S \to B(H)_{sa}$ *and* η *satisfies condition* (C)*. If for every* $A, B \in S$ *and* $V = A + \eta(B,A)$ *the function* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *is operator preinvex with respect to* η *on* η*-path* P_{AV} *with spectra of* A *and spectra of* V *in the interval* I*. Then we have the inequality*

$$f\left(\frac{A+V}{2}\right) \le \int_0^1 f((A + t\eta(B,A))dt \le \frac{f(A) + f(B))}{2}.$$

Throughout this paper, we introduce the concept of operator AG-preinvex functions and obtain some Hermite-Hadamard type inequalities for these class of functions. These results lead us to obtain some inequalities unitarily invariant norm inequalities for operators.

2 Some inequalities for operator AG-preinvex functions

In this section, we prove some Hermite-Hadamard type inequalities for operator AG-preinvex functions.

Definition 5 [13] *A continuous function* $f : I \subseteq \mathbb{R} \to \mathbb{R}^+$ *is said to be operator AG-convex (concave) if*

$$f(\lambda A + (1-\lambda)B) \leq \ (\geq) \ f(A)^{\lambda} f(B)^{1-\lambda}$$

for $0 \leq \lambda \leq 1$ *and self-adjoint operators* A *and* B *in* $B(H)$ *whose spectra are contained in* I.

Example 1 [6, Corollary 7.6.8] *Let* A *and* B *be to positive definite* $n \times n$ *complex matrices. For* $0 < \alpha < 1$, *we have*

$$|\alpha A + (1-\alpha)B| \geq |A|^{\alpha}|B|^{1-\alpha} \tag{7}$$

where $|\cdot|$ *denotes determinant of a matrix.*

Let f be an operator AG-convex function, for commutative positive operators $A, B \in B(H)$ whose spectra are contained in I, then we have

$$\begin{aligned} f\left(\frac{A+B}{2}\right) &\leq \int_0^1 \sqrt{f(\alpha A + (1-\alpha)B) f((1-\alpha)A + \alpha B)} d\alpha \\ &\leq \sqrt{f(A)f(B)}, \end{aligned} \tag{8}$$

(see [13] for more inequalities).

Definition 6 *Let* $S \subseteq B(H)_{sa}$ *be an invex set with respect to* $\eta : S \times S \to B(H)_{sa}$. *A continuous function* $f : I \subseteq \mathbb{R} \to \mathbb{R}^+$ *is called operator AG-preinvex with respect to* η *on* S *if*

$$f(A + t\eta(B, A)) \leq f(A)^{1-t} f(B)^t$$

for $t \in [0, 1]$ *such that spectra of* A *and* B *are contained in* I.

Remark 1 *Let* f *be an operator AG-preinvex function, in a commutative case, we then get*

$$\begin{aligned} f(A + t\eta(B, A)) &\leq f(A)^{1-t} f(B)^t \\ &\leq (1-t)f(A) + tf(B) \\ &\leq \max\{f(A), f(B)\} \end{aligned}$$

It means that f *is operator quasi preinvex i.e.,* $f(A+t\eta(B,A)) \leq \max\{f(A), f(B)\}$.

We need the following lemma for giving Hermite-Hadamard type inequalities for operator preinvex function.

Lemma 1 *Let* $S \subseteq B(H)_{sa}$ *be an invex set with respect to* $\eta : S \times S \to B(H)_{sa}$ *and* $f : I \subseteq \mathbb{R} \to \mathbb{R}^+$ *be a continuous function on the interval* I*. Suppose that* η *satisfies condition (C). Then for every* $A, B \in S$ *and* $V = A + \eta(B, A)$ *and for some fixed* $s \in (0, 1]$ *the function* f *is operator AG-preinvex with respect to* η *on* η*-path* P_{AV} *with spectra of* A *and* V *in the interval* I *if and only if the function* $\varphi_{A,B}$ *defined by*

$$\varphi_{A,B}(t) = f(A + t\eta(B, A)) \tag{9}$$

is a log-convex function on $[0, 1]$.

Proof. Let φ be a log-convex function on $[0, 1]$, we should prove that f is operator AG-preinvex with respect to η.
For every $C_1 := A + t_1\eta(B, A) \in P_{AV}$, $C_2 := A + t_2\eta(B, A) \in P_{AV}$, fixed $\lambda \in [0, 1]$, by (9) we have

$$\begin{aligned} f(C_1 + \lambda\eta(C_2, C_1)) &= f(A + t_1\eta(B, A) + \lambda\eta(A + t_2\eta(B, A), A + t_1\eta(B, A))) \\ &= f(A + t_1\eta(B, A) + \lambda(t_2 - t_1)\eta(B, A)) \\ &= f(A + (t_1 + \lambda t_2 - \lambda t_1)\eta(B, A)) \\ &= f(A + ((1-\lambda)t_1 + \lambda t_2)\eta(B, A)) \\ &= \varphi((1-\lambda)t_1 + \lambda t_2) \\ &\leq \varphi(t_1)^{1-\lambda}\varphi(t_2)^{\lambda} \\ &= (f(A + t_1\eta(B, A)))^{1-\lambda} (f(A + t_2\eta(B, A)))^{\lambda}. \end{aligned}$$

Conversely, let f be operator AG-preinvex, then, by (6)

$$\begin{aligned}
\varphi((1-\lambda)t_1+\lambda t_2) &= f(A+((1-\lambda)t_1+\lambda t_2)\eta(B,A)) \\
&= f(A+t_1\eta(B,A)+\lambda(t_2-t_1)\eta(B,A)) \\
&= f(A+t_1\eta(B,A)+\lambda\eta(A+t_2\eta(B,A),A+t_1\eta(B,A))) \\
&\leq f(A+t_1\eta(B,A))^{1-\lambda}f(A+t_2\eta(B,A))^{\lambda} \\
&= \varphi(t_1)^{1-\lambda}\varphi(t_2)^{\lambda}.
\end{aligned}$$

□

Theorem 3 *Let* $S \subseteq B(H)_{sa}$ *be an invex set with respect to* $\eta : S\times S \to B(H)_{sa}$ *and* $f : I \subseteq \mathbb{R} \to \mathbb{R}^{+}$ *be a continuous function on the interval* I. *Suppose that* η *satisfies condition (C). Then for the operator AG-preinvex function* f *with respect to* η *on* η*-path* P_{AV} *such that spectra of* A *and* V *are in* I, *we have*

$$\begin{aligned}
f\left(\frac{A+V}{2}\right) &\leq \sqrt{f\left(\frac{3A+V}{4}\right)f\left(\frac{A+3V}{4}\right)} \\
&\leq \exp\left(\int_0^1 \log(f(A+t\eta(B,A)))dt\right) \\
&\leq \sqrt{f\left(\frac{A+V}{2}\right)}\sqrt[4]{f(A)}\sqrt[4]{f(V)} \\
&\leq \sqrt{f(A)f(V)} \\
&\leq \frac{f(A)+f(V)}{2}
\end{aligned}$$

where $A, B \in S$ *and* $V = A+\eta(B,A)$ *and for some fixed* $s \in (0,1]$

Proof. Since f is an operator AG-preinvex function, so by Lemma 1 we have $\varphi(t) = f(A+t\eta(B,A))$ is log-convex on $[0,1]$.

On the other hand, in [11] we obtained the following inequalities for log-convex function φ on $[0,1]$:

$$\begin{aligned}
\varphi\left(\frac{1}{2}\right) &\leq \sqrt{\varphi\left(\frac{1}{4}\right)\varphi\left(\frac{3}{4}\right)} \\
&\leq \exp\left(\int_0^1 \log(\varphi(u))du\right)
\end{aligned}$$

$$\leq \sqrt{\varphi\left(\frac{1}{2}\right)}.\sqrt[4]{\varphi(0)}.\sqrt[4]{\varphi(1)}$$
$$\leq \sqrt{\varphi(0)\varphi(1)}. \tag{10}$$

By knowing that

$$\begin{aligned}
\varphi(0) &= f(A) \\
\varphi\left(\frac{1}{4}\right) &= f\left(A+\frac{1}{4}\eta(B,A)\right) = f\left(\frac{3A+V}{4}\right) \\
\varphi\left(\frac{1}{2}\right) &= f\left(A+\frac{1}{2}\eta(B,A)\right) = f\left(\frac{A+V}{2}\right) \\
\varphi(1) &= f(V),
\end{aligned}$$

we obtain

$$\begin{aligned}
f\left(\frac{A+V}{2}\right) &\leq \sqrt{f\left(\frac{3A+V}{4}\right)f\left(\frac{A+3V}{4}\right)} \\
&\leq \exp\left(\int_0^1 \log(f(A+t\eta(B,A)))dt\right) \\
&\leq \sqrt{f\left(\frac{A+V}{2}\right)}\sqrt[4]{f(A)}\sqrt[4]{f(V)} \\
&\leq \sqrt{f(A)f(V)}.
\end{aligned}$$

□

3 Some unitarily invariant norm inequalities for operator AG-preinvex functions

In this section we prove some unitarily invariant norm inequalities for operators.

We consider the wide class of unitarily invariant norms $|||\cdot|||$. Each of these norms is defined on an ideal in $B(H)$ and it will be implicitly understood that when we talk of $|||T|||$, then the operator T belongs to the norm ideal associated with $|||\cdot|||$. Each unitarily invariant norm $|||\cdot|||$ is characterized by the invariance property $|||UTV||| = |||T|||$ for all operators T in the norm ideal associated with $|||\cdot|||$ and for all unitary operators U and V in $B(H)$.

For $1 \leq p < \infty$, the Schatten p-norm of a compact operator A is defined by $\|A\|_p = (\mathrm{Tr}\,|A|^p)^{1/p}$, where Tr is the usual trace functional. Note that for compact operator A we have, $\|A\| = s_1(A)$, and if A is a Hilbert-Schmidt operator, then $\|A\|_2 = (\sum_{j=1}^{\infty} s_j^2(A))^{1/2}$. These norms are special examples of the more general class of the Schatten p-norms which are unitarily invariant [3].

Remark 2 *The author of [7] proved that if $A, B, X \in B(H)$ such that A, B are positive operators, then for $0 \leq \nu \leq 1$ we have*

$$|||A^{\nu}XB^{1-\nu}||| \leq |||AX|||^{\nu}|||XB|||^{1-\nu}. \tag{11}$$

Let $X = I$ in above inequality, we then get

$$|||A^{\nu}B^{1-\nu}||| \leq |||A|||^{\nu}|||B|||^{1-\nu}. \tag{12}$$

Lemma 2 *Let f be an operator AG-preinvex function and η satisfies the condition (C). Then the function $\varphi_{A,B} : [0, 1] \to \mathbb{R}$ defined as follows*

$$\varphi(t) = |||f(A + t\eta(B, A))|||$$

is log-convex.

Proof. Let $t_1, t_2 \in [0, 1]$, we have

$$\begin{aligned}
\varphi((1-\lambda)t_1 + \lambda t_2) &= |||f(A + ((1-\lambda)t_1 + \lambda t_2)\eta(B, A))||| \\
&= |||f\left(A + t_1\eta(B, A) + \lambda(t_2 - t_1)\eta(B, A)\right)||| \\
&= |||f\left(A + t_1\eta(B, A) + \lambda\eta(A + t_2\eta(B, A), A + t_1\eta(B, A))\right)||| \\
&\leq |||f(A + t_1\eta(B, A))^{1-\lambda}f(A + t_2\eta(B, A))^{\lambda}||| \\
&\leq |||f(A + t_1\eta(B, A))|||^{1-\lambda}|||f(A + t_2\eta(B, A))|||^{\lambda} \quad \text{by (12)} \\
&= \varphi(t_1)^{1-\lambda}\varphi(t_2)^{\lambda}.
\end{aligned}$$

□

Theorem 4 *Let $S \subseteq B(H)_{sa}$ be an invex set with respect to $\eta : S \times S \to B(H)_{sa}$ and $f : I \subseteq \mathbb{R} \to \mathbb{R}^+$ be a continuous function on the interval I. Suppose that η satisfies condition (C). Then for the operator AG-preinvex function f with* *respect to* η *on* η*-path* P_{AV} *such that spectra of* A *and* V *are in* I*, we have*

$$\begin{aligned}\left|\left|\left|f\left(\frac{A+V}{2}\right)\right|\right|\right| &\leq \sqrt{\left|\left|\left|f\left(\frac{3A+V}{4}\right)\right|\right|\right|\left|\left|\left|f\left(\frac{A+3V}{4}\right)\right|\right|\right|} \\ &\leq \exp\left(\int_0^1 \log(|||f(A+t\eta(B,A))|||)dt\right) \\ &\leq \sqrt{\left|\left|\left|f\left(\frac{A+V}{2}\right)\right|\right|\right|}\sqrt[4]{|||f(A)|||}\sqrt[4]{|||f(V)|||} \\ &\leq \sqrt{|||f(A)|||\,|||f(V)|||} \\ &\leq \frac{|||f(A)|||+|||f(V)|||}{2}.\end{aligned}$$

where $A, B \in S$ *and* $V = A + \eta(B, A)$ *and for some fixed* $s \in (0, 1]$

Proof. Since f is an operator AG-preinvex function, so by Lemma 2 we have $\varphi(t) = |||f(A + t\eta(B, A))|||$ is log-convex on $[0, 1]$.

On the other hand, in [11] we obtained the following inequalities for log-convex function φ on $[0, 1]$:

$$\begin{aligned}\varphi\left(\frac{1}{2}\right) &\leq \sqrt{\varphi\left(\frac{1}{4}\right)\varphi\left(\frac{3}{4}\right)} \\ &\leq \exp\left(\int_0^1 \log(\varphi(u))du\right) \\ &\leq \sqrt{\varphi\left(\frac{1}{2}\right)}.\sqrt[4]{\varphi(0)}.\sqrt[4]{\varphi(1)} \\ &\leq \sqrt{\varphi(0)\varphi(1)}. \end{aligned} \tag{13}$$

By knowing that

$$\begin{aligned}\varphi(0) &= |||f(A)||| \\ \varphi\left(\frac{1}{4}\right) &= \left|\left|\left|f\left(A+\frac{1}{4}\eta(B,A)\right)\right|\right|\right| = \left|\left|\left|f\left(\frac{3A+V}{4}\right)\right|\right|\right| \\ \varphi\left(\frac{1}{2}\right) &= \left|\left|\left|f\left(A+\frac{1}{2}\eta(B,A)\right)\right|\right|\right| = \left|\left|\left|f\left(\frac{A+V}{2}\right)\right|\right|\right| \\ \varphi(1) &= |||f(V)|||,\end{aligned}$$

we obtain

$$\begin{aligned}
\left|\left|\left| f\left(\frac{A+V}{2}\right)\right|\right|\right| &\leq \sqrt{\left|\left|\left| f\left(\frac{3A+V}{4}\right)\right|\right|\right| \left|\left|\left| f\left(\frac{A+3V}{4}\right)\right|\right|\right|} \\
&\leq \exp\left(\int_0^1 \log(|||f(A+t\eta(B,A))|||)dt\right) \\
&\leq \sqrt{\left|\left|\left| f\left(\frac{A+V}{2}\right)\right|\right|\right|}\sqrt[4]{|||f(A)|||}\sqrt[4]{|||f(V)|||} \\
&\leq \sqrt{|||f(A)|||\,|||f(V)|||}.
\end{aligned}$$

□

Let $\eta(B,A) = B - A$ in the above theorem, then we obtain the following inequalities:

$$\begin{aligned}
\left|\left|\left| f\left(\frac{A+B}{2}\right)\right|\right|\right| &\leq \sqrt{\left|\left|\left| f\left(\frac{3A+B}{4}\right)\right|\right|\right| \left|\left|\left| f\left(\frac{A+3B}{4}\right)\right|\right|\right|} \\
&\leq \exp\left(\int_0^1 \log(|||f((1-t)A+tB)|||\, dt\right) \\
&\leq \sqrt{\left|\left|\left| f\left(\frac{A+B}{2}\right)\right|\right|\right|}\sqrt[4]{|||f(A)|||}\sqrt[4]{|||f(B)|||} \\
&\leq \sqrt{|||f(A)|||\,|||f(B)|||} \\
&\leq \frac{|||f(A)|||+|||f(B)|||}{2}. \qquad (14)
\end{aligned}$$

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