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New characterization of symmetric groups of prime degree

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Abstract. In this paper, will show that a symmetric group of prime degree $p \geq 5$ is recognizable by its order and a special conjugacy class size of $(p-1)!$.

1 Introduction

There are a few finite groups that are determined up to isomorphism solely by their order, such as $\mathbb{Z}_2$ or $\mathbb{Z}_{15}$. Still other finite groups are determined by their order together with other data, such as the number of elements of each order, the structure of the prime graph, the number of order components, the number of Sylow p-subgroups for each prime p, etc. In this paper, we investigate the possibility of characterizing Sym_p by simple conditions when p is prime number.

Our result states that: if p is a prime number, then the groups Sym_p are determined up to isomorphism by their order and a special conjugacy class size of $(p-1)!$. In fact, the main theorem of our paper is as follows.

Theorem A *Let* G *be a group. Then* $G \cong \mathrm{Sym}_p$ *if and only if* $|G| = p!$ *and* G *has a special conjugacy class size of* $(p-1)!$*, where* $p \geq 5$ *is a prime number.*

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For related results, Chen et al. in [14] shows that the projective special linear groups $L_2(p)$ recognizable by their order and a special conjugacy class size, where p is a prime number. As a consequence of their result, they showed that Thompson's conjecture is valid for $L_2(p)$.

Put $N(G) = \{n : G \text{ has a conjugacy class of size } n\}$. By Thompson's conjecture if L is a finite non-abelian simple group, G is a finite group with a trivial center, and $N(G) = N(L)$, then $L \cong G$.

Similar characterizations have been found in [9], [13], [2], and [3] for the groups: sporadic simple groups, simple K_3-groups (a finite simple group is called a simple K_n-group if its order is divisible by exactly n distinct primes), ${}^2D_n(2)$, ${}^2D_{n+1}(2)$ and alternating group of degree p, $p+1$, $p+2$, where p is a prime number.

The *prime graph* of a finite group G, denoted by $\Gamma(G)$, is the graph whose vertices are the prime divisors of G and where prime p is defined to be adjacent to prime q ($\neq p$) if and only if G contains an element of order pq.

We denote by $\pi(G)$ the set of prime divisors of $|G|$. Let $t(G)$ be the number of connected components of $\Gamma(G)$ and let $\pi_1, \pi_2, \ldots, \pi_{t(G)}$ be the connected components of $\Gamma(G)$. If $2 \in \pi(G)$, then we always suppose $2 \in \pi_1$.

We can express $|G|$ as a product of integers $m_1, m_2, \ldots, m_{t(G)}$, where $\pi(m_i) = \pi_i$ for each i. The numbers m_i are called the order components of G. In particular, if m_i is odd, then we call it an odd component of G. Write $OC(G)$ for the set $\{m_1, m_2, \ldots, m_{t(G)}\}$ of order components of G and $T(G)$ for the set of connected components of G.

Williams and Kondrat'ev (see [11, 4, 5]), proved a series of important results on prime graphs. We apply these results to prove Theorem A.

Suppose G and S are two finite groups satisfying $t(\Gamma(S)) \geq 2$, $N(G) = N(S)$, and $Z(G) = 1$. Then it is proved in [7, Lemma 1.4] that $|G| = |S|$. Therefore, if $N(G) = N(\mathrm{Sym}_p)$, and $Z(G) = 1$, then $|G| = |\mathrm{Sym}_p|$. Now, Theorem A follows that $G \cong \mathrm{Sym}_p$. Hence, Thompson's conjecture is valid under a weak condition for the symmetric groups of prime degree.

According to the classification theorem of finite simple groups and [4, 5, 11, 12], we can list the order components of finite simple groups with disconnected prime graphs as in Tables 1-3 in [1].

We say $p^k \parallel m$ if $p^k \mid m$ and $p^{k+1} \nmid m$. The other notation and terminology in this paper are standard, and the reader is referred to [10] if necessary.

2 Preliminary results

Definition 1 *A group* G *is a finite Frobenius group, if it contains a subgroup* H *such that* $1 \neq H \neq G$ *and* $H \cap H^g = 1$ $H \cap H^g = 1$ *for all* $g \in G - H$. *A subgroup with these properties is called a Frobenius complement of* G. *The Frobenius kernel of* G*, with respect to* H*, is defined by* $K = (G - \bigcup_{g \in G} H^g) \bigcup \{1\}$.

Lemma 1 [6, Theorem 10.3.1] Let G be a Frobenius group with Frobenius kernel H and Frobenius complement K. Then $|K| \equiv 1 \pmod{|H|}$.

Lemma 2 [8, Lemma 8] Let G be a finite group with $t(G) \geq 2$ and N a normal subgroup of G. If N is a π_i-group for some prime graph component of G, and $\mu_1, \mu_2, \ldots, \mu_r$ are some of order components of G but not a π_i-number then $\mu_1 \mu_2 \ldots \mu_r$ is a divisor of $|N| - 1$.

Now we state the following lemma which is proved in [5, Lemma 6], with some differences and classify the simple groups of Lie type with prime odd order component by θ function, which is introduced later.

Lemma 3 *If* L *is a simple group of Lie type and has prime odd order component* $p \geq 17$ *and* $\pi(L)$ *has at most* $\theta(L)$ *prime numbers* t*, where* $\frac{p+1}{2} < t < p$. *Then* $\theta(L) \leq 3$.

Throughout the proof of the above Lemma, we can divide simple groups of Lie type, L, with prime odd order component $p \geq 17$, into the following cases.

(1) $\theta(L) = 0$ if L is isomorphic to $A_{p'-1}(q)$, $A_{p'}(q)$, where $q-1 \mid p'+1$, $A_2(2)$, ${}^2A_{p'-1}(q)$, ${}^2A_{p'}(q)$, where $q+1 \mid p'+1$, ${}^2A_3(2)$, $B_n(q)$, where $n = 2^{m'}$ and q is odd, $B_{p'}(3)$, $C_n(q)$, where $n = 2^{m'}$ or $(n, q) = (p', 3)$, $D_{p'+1}(3)$, $D_{p'}(q)$, for $q = 3, 5$, ${}^2D_n(q)$, for $(n, q) = (2^{m'}, q)$, $(p', 3)$, where $5 \leq p' \neq 2^{m'} + 1$ or $(2^{m'} + 1, 3)$, where $5 \leq p' \neq 2^{m'} + 1$, $G_2(q)$, where $q \equiv \epsilon \pmod 3$ for $\epsilon = \pm 1$, ${}^3D_4(q)$, $E_6(q)$ or ${}^2E_6(q)$;

(2) $\theta(L) = 1$ if L is isomorphic to one of the simple groups $A_1(q)$, where $2 \mid q$, $A_2(4)$, ${}^2A_5(2)$, $C_{p'}(2)$, $D_n(2)$, where $n = p'$ or $p' + 1$, ${}^2D_n(2)$, where $(n, q) = (2^{m'} + 1, 2)$ or $(p' = 2^{m'} + 1, 3)$, where $m' \geq 2$, $E_7(2)$, $E_7(3)$, $F_4(q)$, ${}^2F_4(q)$, where $q = 2^{2n+1} > 2$, or $G_2(q)$, where $3 \mid q$;

(3) $\theta(L) = 2$ if L is isomorphic to the simple groups $A_1(q)$, where $q \equiv \epsilon \pmod 4$ for $\epsilon = \pm 1$, ${}^2B_2(q)$, where $q = 2^{2m'+1} > 2$, or ${}^2G_2(q)$, where $q = 3^{2m'+1} > 3$;

(4) $\theta(L) = 3$ if L is isomorphic to the simple groups $E_8(q)$ or ${}^2E_6(2)$.

Lemma 4 [5, Lemma 1] If $n \geq 6$ is a natural number, then there are at least $s(n)$ prime numbers p_i such that $\frac{n+1}{2} < p_i < n$. Here

$$\begin{aligned} s(n) &= 6 \text{ for } n \geq 48; \\ s(n) &= 5 \text{ for } 42 \leq n \leq 47; \\ s(n) &= 4 \text{ for } 38 \leq n \leq 41; \\ s(n) &= 3 \text{ for } 18 \leq n \leq 37; \\ s(n) &= 2 \text{ for } 14 \leq n \leq 17; \\ s(n) &= 1 \text{ for } 6 \leq n \leq 13. \end{aligned}$$

In particular, for every natural number $n > 6$, there exists a prime p such that $\frac{n+1}{2} < p < n$, and for every natural number $n > 3$, there exists an odd prime number p such that $n - p < p < n$.

Definition 2 *A group G is called a 2-Frobenius group, if it has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, where K and G/H are Frobenius groups with kernels H and K/H, respectively.*

Lemma 5 [11, Theorem A] Let G be a finite group with more than one prime graph components. Then either G is a Frobenius or a 2-Frobenius group, or G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that such that H and G/K are π_1-groups, K/H is a nonabelian simple group with $\pi_i \subseteq \pi(K)$ for every $i > 1$, and H is a nilpotent group, especially, $K/H \trianglelefteq G/H \trianglelefteq \mathrm{Aut}(K/H)$.

3 Proofs

Proof of the Theorem A. Since the necessity of the theorem can be checked easily, we only need to prove the sufficiency.

By hypothesis, there exists an element x of order p in G such that $C_G(x) = \langle x \rangle$ and $C_G(x)$ is a Sylow p-subgroup of G. By the Sylow's theorem, we have that $C_G(y) = \langle y \rangle$ for any element y in G of order p. So, $\{p\}$ is a prime graph component of G and $t(G) \geq 2$. In addition, p is the maximal prime divisor of $|G|$ and an odd order component of G. The proof of the Theorem A follows from the following Lemmas.

Lemma 6 G *has a normal series* $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ *such that* H *and* G/K *are* π_1*-groups,* K/H *is a non-abelian simple group and* H *is a nilpotent group.*

Proof. First, we prove that G is neither Frobenius nor 2-Frobenius. Suppose that G is a Frobenius group with a Frobenius kernel H and a Frobenius complement K. Then $|K| \mid |H|-1$, by Lemma 2. If $p \in \pi(H)$, then by Lemma 1, $|H| = p$ and $|K| = (p-1)!$. It follows that $(p-1)! \mid p-1$, a contradiction. If $p \in \pi(K)$, then $|K| = p$ and $|H| = (p-1)!$ by Lemma 1, and so $p \mid (p-1)!-1$. By Wilson's theorem in number theory, we get a contradiction. Hence, G is not a Frobenius group.

Assume that G is a 2-Frobenius group. By Lemma 3, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(K/H) = \{p\} = \pi_2(G)$, $\pi(H) \cup \pi(G/K) = \pi_1(G)$, and $|G/K| \mid (p-1)$. Then we have that K/H is of order p. By Lemma 4 and $|G| = p!$, we deduce that there exists $r \in \pi(G)$ such that $\frac{p-1}{2} < r < p$. So, $r \nmid p-1$ and hence, $|G/K| \mid (p-1)$ implies that $r \in \pi(H)$.

Let R be an r-Sylow subgroup of H. Thus $|G| = p!$ shows that R is a cyclic subgroup of order r. On the other hand, the same reasoning as before shows that $R \rtimes P$ is a Frobenius group, where P is a p-Sylow subgroup of K, so Lemma 1 forces $p \mid r-1$ and hence, $p+1 < r$, which is a contradiction. Therefore, G is not a 2-Frobenius group either. Now Lemma 5 imply Lemma 6. $\square$

If K/H has an element of order rq where r and q are primes, then G has also such element. Hence by definition of order components, an odd order component of G must be an odd order component of K/H. Note that $t(K/H) \geq 2$.

Lemma 7 (a) *If* $t \in \pi(H)$, *then* $t \leq \frac{p+1}{2}$;

(b) K/H *can not be isomorphic to a sporadic simple group. Moreover, if* K/H *is isomorphic to an alternating simple group, then we must have* $G \cong \mathrm{Sym}_p$.

Proof. (a) If t divides $|H|$ where $\frac{p+1}{2} < t < p$, then since H is nilpotent subgroup of G and the order of T, the Sylow t-subgroup of H, is equal to t. By Lemma 2, we must have $p \mid t-1$, which is impossible. Thus $|H|$ is not divisible by the primes t with $\frac{p+1}{2} < t < p$.

(b) We note that if $H \neq 1$, by nilpotency of H, we may assume that H is a t-group for $t \in \pi_1(G)$.

If $K/H \cong J_4$, then $p = 43$. Since $19 \in \pi(G) \backslash \pi(\mathrm{Aut}(J_4))$, then $19 \in \pi(H)$. By Lemma 2, $43 \mid 19^i - 1$ for $i = 1$ or 2, which is impossible.

If $K/H \cong M_{22}$, then $p = 11$. Since $5^2 \mid |G|$ and $5 \parallel |\mathrm{Aut}(M_{22})|$, $5 \in \pi(H)$. So by Lemma 2, we get a contradiction.

If $K/H \cong J_2$, then $p = 7$, but $2^4 \parallel |G|$ and $2^7 \parallel |K/H|$, which is impossible. If K/H is isomorphic to other sporadic simple groups we can view a contradiction similarly.

Now let K/H be isomorphic to an alternating group. By Tables 1 and 2 in [1], K/H must be isomorphic to Alt_n, where $n = p$, $p+1$ or $p+2$. If $n > p$, then $n! > p!$ and it contradicts $|K/H| \mid |G|$. Hence $K/H \cong \mathrm{Alt}_p$. Since $\mathrm{Aut}(K/H) \simeq \mathrm{Sym}_p$, G/H is isomorphic to Alt_p or Sym_p. If $G/H \simeq \mathrm{Sym}_p$, then $H = 1$. Therefore, $G \cong \mathrm{Sym}_p$

If $G/H \simeq \mathrm{Alt}_p$, then $H \cong Z_2$, by Lemma 2, we get a contradiction. □

Lemma 8 *If* $t \in \pi(G/H)$ *and* $\frac{p+1}{2} < t < p$, *then* $t \in \pi(K/H)$.

Proof. It follows from Lemma 7(a), and the proof of Lemma 6(d) in [5]. □

Lemma 9 K/H *can not be isomorphic to a simple group of Lie type.*

Proof.. By Lemmas 8 and 4, we must have $17 \leq p \leq 37$ and $\theta(K/H) \geq 2$. Therefore K/H is isomorphic to one of the following simple groups.

(1) $L_2(q)$, where $q \equiv \epsilon \pmod 4$ for $\epsilon = \pm 1$;

(2) ${}^2B_2(q)$, where $q = 2^{2m'+1} > 2$;

(3) ${}^2G_2(q)$, where $q = 3^{2m'+1} > 3$;

(4) $E_8(q)$ or ${}^2E_6(2)$.

Since one of the odd order components of K/H is equal to p, by Tables 2 and 3 in [1], we must have:

(1) $K/H \cong L_2(17)$, for $p = 17$;

(2) $K/H \cong L_2(19)$, ${}^2G_2(27)$ or ${}^2E_6(2)$, for $p = 19$;

(3) $K/H \cong L_2(q)$, where $p = q$ is equal to 23, 29 or 31;

(4) $K/H \cong L_2(37)$, ${}^2G_2(27)$ or ${}^2E_6(2)$, for $p = 37$.

Let $p = 17$. Then $K/H \cong L_2(17)$. Since $13 \notin \pi(L_2(17))$, we get a contradiction by Lemma 8.

Let $p = 19$. If $K/H \cong L_2(19)$, then since $17 \notin \pi(L_2(19))$, we get a contradiction by Lemma 8. Since $37 \mid |{}^2G_2(27)|$, $K/H \ncong {}^2G_2(27)$. For the case $K/H \cong {}^2E_6(2)$, we view a contradiction by $2^{36} \mid |{}^2E_6(2)|$.

Let $K/H \cong L_2(q)$, where $p = q$ is equal to 23, 29, 31, or 37. If $p = 37$, then since $23 \notin \pi(L_2(37))$, we get a contradiction by Lemma 8. Similarly, we view a contradiction when $p = 23$, 29, or 31.

Let $p = 37$. If $K/H \cong L_2(37)$, then since $23 \notin \pi(L_2(37))$, we get a contradiction. Since $23 \notin \pi({}^2G_2(27))$, $K/H \ncong {}^2G_2(27)$. If $K/H \cong {}^2E_6(2)$, we view a contradiction by $2^{36} \mid |{}^2E_6(2)|$. □

Now Lemmas 7(b) and 9 imply Theorem A.

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