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An extension of a variant of d'Alemberts functional equation on compact groups

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Abstract. All paper is related with the non-zero continuous solutions $f : G \to \mathbb{C}$ of the functional equation

$$f(x\sigma(y)) + f(\tau(y)x) = 2f(x)f(y), \quad x, y \in G,$$

where σ, τ are continuous automorphism or continuous anti-automorphism defined on a compact group G and possibly non-abelian, such that $\sigma^2 = \tau^2 = id$. The solutions are given in terms of unitary characters of G.

1 Introduction

Let G be a compact group, let σ, τ be continuous automorphism or continuous anti-automorphism such that $\sigma^2 = \tau^2 = id$. We consider the functional equation

$$f(x\sigma(y)) + f(\tau(y)x) = 2f(x)f(y), \quad x, y \in G, \tag{1}$$

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where $f : G \to \mathbb{C}$ is the function to determine. This equation, in the case where G is abelian, has been studied by many authors (see, e.g., Shin'ya [7, Corollary 3.12], and Stetkær [8, Theorem 14.9]). Eq. (1) is a generalization of the following variant of d'Alembert's functional equation

$$f(xy) + f(\sigma(y)x) = 2f(x)f(y), \quad x, y \in G, \tag{2}$$

which was introduced and solved on semi-groups by Stetkær in [9]. Some information, applications and numerous references concerning (2), d'Alembert's functional equation

$$f(x+y) + f(x-y) = 2f(x)f(y), \quad x, y \in \mathbb{R}, \tag{3}$$

and their further generalizations can be found e.g. in ([5, 3, 4, 1, 8, 9, 10, 11]).

The purpose of the present paper is to solve the functional equation (1) in the case where G is a compact group and possibly non-abelian. Our approach uses the harmonic analysis and the representation theory on compact groups. We note that the idea of using Fourier analysis for solving (1) goes back to [2].

Throughout the rest of this paper, G is a compact group with identity element e. By solutions (resp. representations), we always mean continuous solutions (resp. continuous representations).

2 Preliminaries

In this section, we set up some notation and conventions and briefly review some fundamental facts in Fourier analysis which will be used later.

Let dx denotes the normalized Haar measure on G. Let $\hat{G}$ stand for the set of equivalence classes irreducible unitary representations of G. For $[\pi] \in \hat{G}$, the notation d_π denotes the dimension of the representation space of π and $\mathcal{E}_\pi = \mathrm{span}\{\sqrt{\pi_{ij}} : i, j = 1, \ldots, d_\pi\}$ the linear span of the matrix elements of π. For $f \in L^2(G)$, the Fourier transform of f is defined by

$$\hat{f}(\pi) = \int_G f(x)\pi(x)^{-1}dx \in M_{d_\pi}(\mathbb{C}) \text{ for all } [\pi] \in \hat{G},$$

where $M_{d_\pi}(\mathbb{C})$ is the space of all $d_\pi \times d_\pi$ complex matrix.

As usual, the left and right regular representations of G in $L^2(G)$ are defined by

$$(L_y f)(x) = f(y^{-1}x) \quad \text{and} \quad (R_y f)(x) = f(yx),$$

respectively, where $f \in L^2(G)$ and $x, y \in G$.

The following properties will be useful later

$$\widehat{(L_y f)}(\pi) = \hat{f}(\pi)\pi(y)^{-1} \text{ and } \widehat{(R_y f)}(\pi) = \pi(y)\hat{f}(\pi)$$

for all $y \in G$, and $\pi \in \hat{G}$.

3 Main result

The following Lemmas will be used in the proof of Theorem 1.

Lemma 1 *Let* G *be a compact group and* π *be a unitary irreducible representation of* G*. Suppose every* $x \in G$*, there is* $c_x \in \mathbb{C}$ *such that*

$$\pi(\sigma(x)) + \pi(\tau(x)) = c_x I_{d_\pi}, \tag{4}$$

then $d_\pi = 1$.

Proof. Let $(\mathcal{H}; \langle\ ,\ \rangle)$ denote the complex Hilbert space on which the representation π acts. We will consider two cases, $\pi \circ \sigma \simeq \pi \circ \tau$ or not.

In the first case. From (4) we get that

$$\pi(\sigma(x))_{ij} + \pi(\tau(x))_{ij} = 0 \quad \text{for } i \neq j, \ 1 \leq i, j \leq d_\pi, \quad x \in G.$$

Since $\pi \circ \sigma \not\simeq \pi \circ \tau$ we have $\mathcal{E}_{\pi\circ\sigma} \perp \mathcal{E}_{\pi\circ\tau}$. Hence $(\pi \circ \sigma)_{ij} = 0$ for $i \neq j$, so $\pi \circ \sigma$ is a diagonal matrix. Since $\pi \circ \sigma$ is irreducible we have $d_\pi = 1$.

In the second case, i.e., $\pi \circ \sigma \simeq \pi \circ \tau$, there exists a unitary operator T on $\mathcal{H}$ such that

$$\pi \circ \sigma(x) = T^* \pi \circ \tau(x) T, \quad x \in G.$$

Since T is a unitary matrix, by the spectral theorem for normal operators applied to T, we infer that T is diagonalizable. Then $\mathcal{H}$ has an orthonormal basis $(e_1, e_2, \ldots, e_{d_\pi})$ consisting of eigenvectors of T. We write $Te_i = \lambda_i e_i$ where $\lambda_i \in \mathbb{C}$ for $i = 1, 2, \ldots, d_\pi$. Actually $|\lambda_i| = 1$, because T is unitary. For any $i = 1, 2, \ldots, d_\pi$, we compute that

$$\begin{aligned}
(\pi \circ \sigma(x))_{ii} &= \langle \pi \circ \sigma(x) e_i, e_i \rangle = \langle (T)^* \pi(\tau(x)) e_i, e_i \rangle \\
&= \langle \pi(\tau(x)) T e_i, T e_i \rangle = \langle \lambda_i \pi(\tau(x)) e_i, \lambda_i e_i \rangle \\
&= \lambda_i \overline{\lambda_i} \langle \pi(\tau(x)) e_i, e_i \rangle = |\lambda_i|^2 (\pi \circ \tau(x))_{ii} = (\pi \circ \tau(x))_{ii},
\end{aligned}$$

for all $x \in G$. From (4), we infer that

$$2(\pi \circ \tau(x))_{ii} = 2f(x), \tag{5}$$

for all $i = 1, \ldots, d_\pi$ and $x \in G$. Then $d_\pi = 1$. Indeed, if $d_\pi > 1$, then (5) implies that $(\pi \circ \tau)_{ii} = (\pi \circ \tau)_{11}$ for all $i = 2 \ldots, d_\pi$. But if you use Schur's orthogonality relations which say $\frac{1}{d_\pi}(\pi \circ \tau)_{ii}$ is an orthonormal basis, we get a contradiction. Then $d_\pi = 1$. □

Lemma 2 *Let* $f : G \to \mathbb{C}$ *be a non-zero solution of* (1). *Then exists* $[\pi] \in \widehat{G}$ *such that* $\hat{f}(\pi)$ *is invertible.*

Proof. Reformulate (3) to

$$2f(x)f = R_{\sigma(x)}f + L_{\tau(x^{-1})}f, \quad x \in G.$$

Taking the Fourier transform to the last equation and using the identities given in section 2, we have

$$\hat{f}(\pi)\pi(\tau(x)) + \pi(\sigma(x))\hat{f}(\pi) = 2f(x)\hat{f}(\pi), \quad x \in G. \tag{6}$$

Since $f \not\equiv 0$, there exists $[\pi] \in \widehat{G}$ with $\hat{f}(\pi) \neq 0$. Now, let ν be a vector in $\ker \hat{f}(\pi)$. From (6), we infer that $\hat{f}(\pi)\pi(\tau(x))\nu = 0$ for all $x \in G$, this implies that $\hat{f}(\pi)\pi(x)\nu = 0$ for all $x \in G$. So $\pi(x) \ker \hat{f}(\pi) \subset \ker \hat{f}(\pi)$ for all $x \in G$. Since π is irreducible and $\hat{f}(\pi) \neq 0$, we have $\ker \hat{f}(\pi) = \{0\}$. This implies that $\hat{f}(\pi)$ is bijective, thus invertible as a matrix. □

Lemma 3 *Let* $f : G \to \mathbb{C}$ *be a non-zero solution of* (1). *Then* f *is central.*

Proof. Using Lemma 2 and equality (6), we see that there exists $[\pi] \in \widehat{G}$ such that

$$\pi(\sigma(x)) + \hat{f}(\pi)^{-1}\pi(\tau(x))\hat{f}(\pi) = 2f(x)I_{d_\pi}, \quad x \in G. \tag{7}$$

Taking the trace on both sides of (7) we obtain that

$$tr(\pi(\sigma(x))) + tr(\pi(\tau(x))) = 2d_\pi f(x), \quad x \in G,$$

which abbreviates to

$$f(x) = \frac{1}{2d_\pi}(tr(\pi(\sigma(x))) + tr(\pi(\tau(x)))), \quad x \in G. \tag{8}$$

Each terms on the right hand side of (8) is a central function, because trace is a central function. Hence f is central. □

By help of the previous lemmas, we now describe the complete solution of (1) on an arbitrary compact group. It is clear that $f \equiv 0$ is a solution of (1), so in the following theorem we are only concerned with the non-zero solutions.

Theorem 1 *The non-zero solutions* $f : G \to \mathbb{C}$ *of* (1) *are the functions of the form* $f = (\chi + \chi \circ \sigma \circ \tau)/2$, *where* $\chi : G \to \mathbb{C}$ *is a character such that:*

1. $\chi \circ \sigma \circ \tau = \chi \circ \tau \circ \sigma$, *and*

2. χ *is* σ*-even and/or* τ*-even (i.e.,* $\chi \circ \sigma = \chi$ *and/or* $\chi \circ \tau = \chi$*).*

Proof. We have f is central. This implies that $\widehat{f}(\pi)$ is an intertwining operator for π. But π is irreducible, so $\widehat{f}(\pi) = \mu I_{d_\pi}$ for some $\mu \in \mathbb{C}$ by Schur's lemma.

Actually $\mu \neq 0$, because $\widehat{f}(\pi) \neq 0$. Now Eq. (7) coalesce into

$$\pi(\sigma(x)) + \pi(\tau(x)) = 2f(x)I_{d_\pi}, \qquad x \in G. \tag{9}$$

From $d_\pi = 1$, we see that π is a unitary character, say $\pi = \chi$, so

$$f = \frac{\chi \circ \sigma + \chi \circ \tau}{2}.$$

If $\chi \circ \sigma = \chi \circ \tau$, then letting $\chi := \chi \circ \sigma$ we have $f = \chi$. Substituting $f = \chi$ into (1) we get that $\chi \circ \sigma + \chi \circ \tau = 2\chi$. So $\chi = \chi \circ \sigma = \chi \circ \tau$. Then f has the desired form.

If $\chi \circ \sigma \neq \chi \circ \tau$, substituting $f = (\chi \circ \sigma + \chi \circ \tau)/2$ into (1) we find after a reduction that

$$\begin{aligned}&\chi \circ \sigma(x)[\chi(y) + \chi \circ \sigma \circ \tau(y) - \chi \circ \sigma(y) - \chi \circ \tau(y)] + \chi \circ \tau(x)[\chi \circ \tau \circ \sigma(y)\\ &+\chi \circ \tau \circ \tau(y) - \chi \circ \sigma(y) - \chi \circ \tau(y)] = 0\end{aligned}$$

for all $x, y \in G$. Since $\chi \circ \sigma \neq \chi \circ \tau$ we get from the theory of multiplicative functions (see for instance [9, Theorem 3.18]) that both terms are 0, so

$$\begin{cases} \chi \circ \sigma(x)[\chi(y) + \chi \circ \sigma \circ \tau(y) - \chi \circ \sigma(y) - \chi \circ \tau(y)] = 0 \\ \chi \circ \tau(x)[\chi \circ \tau \circ \sigma(y) + \chi(y) - \chi \circ \sigma(y) - \chi \circ \tau(y)] = 0 \end{cases} \tag{10}$$

for all $x, y \in G$. Since $\chi \circ \sigma \neq \chi \circ \tau$ at least one of $\chi \circ \sigma$ and $\chi \circ \tau$ is not zero. We have $\chi \circ \sigma \neq 0$ and $\chi \circ \tau \neq 0$. From (1), we have

$$\chi \circ \sigma + \chi \circ \tau = \chi + \chi \circ \sigma \circ \tau = \chi \circ \tau \circ \sigma + \chi.$$

Using $\chi + \chi \circ \sigma \circ \tau = \chi \circ \tau \circ \sigma + \chi$ and the fact that $\chi \circ \sigma \neq \chi \circ \tau$, we see that $\chi = \chi$ and $\chi \circ \sigma \circ \tau = \chi \circ \tau \circ \sigma$. Thus

$$\chi \circ \tau = \chi \circ \sigma \circ \tau \circ \sigma.$$

We now use $\chi \circ \sigma + \chi \circ \tau = \chi + \chi \circ \sigma \circ \tau$, we get that χ is σ-even or τ-even.

Finally, in view of these cases we deduce that f has the form stated in Theorem 1. $\square$

Similarly to Theorem 1, we can get the solution of functional equation (1) when σ, τ are continuous anti-automorphism such that $\sigma^2 = \tau^2 = id$.

Theorem 2 *The non-zero solutions $f : G \to \mathbb{C}$ of (1) are the functions of the form $f = (\chi + \chi \circ \sigma \circ \tau)/2$, where $\chi : G \to \mathbb{C}$ is a character such that:*

1. *$\chi \circ \sigma \circ \tau = \chi \circ \tau \circ \sigma$, and*

2. *χ is σ-even and/or τ-even (i.e., $\chi \circ \sigma = \chi$ and/or $\chi \circ \tau = \chi$).*

Proof. The proof is similar to the proof of Theorem 1. $\square$

4 Some applications of the main result

As immediate consequences of Theorems 1 and 2, we have the following corollaries.

Corollary 1 *Let G be a compact group and σ be a continuous homomorphism or continuous anti-homomorphism such that $\sigma \circ \sigma = id$. The non-zero solutions $f : G \to \mathbb{C}$ of of the functional equation*

$$f(x\sigma(y)) + f(\sigma(y)x) = 2f(x)f(y), \quad x, y \in G,$$

are the functions of the form $f = \chi$, where $\chi : G \to \mathbb{C}$ is a character such that χ is σ-even.

Proof. It suffices to take $\sigma(x) = \tau(x)$ for all $x \in G$ in Theorem 1 or in Theorem 2. $\square$

Corollary 2 *Let G be a compact group and σ be a continuous homomorphism such that $\sigma \circ \sigma = id$. The non-zero solutions $f : G \to \mathbb{C}$ of of the functional equation*

$$f(x\sigma(y)) + f(yx) = 2f(x)f(y), \quad x, y \in G,$$

are the functions of the form $f = (\chi + \chi \circ \sigma)/2$, where $\chi : G \to \mathbb{C}$ is a character.

Proof. It suffices to take $\tau(x) = x$ for all $x \in G$ in Theorem 1. $\square$

Corollary 3 *Let G be a compact group and τ be a continuous homomorphism such that $\tau \circ \tau = id$. The non-zero solutions $f : G \to \mathbb{C}$ of of the functional equation*

$$f(xy) + f(\tau(y)x) = 2f(x)f(y), \quad x, y \in G,$$

are the functions of the form $f = (\chi + \chi \circ \tau)/2$, where $\chi : G \to \mathbb{C}$ is a character.

Proof. It suffices to take $\sigma(x) = x$ for all $x \in G$ in Theorem 1. □

Corollary 4 *Let G be a compact group and σ be a continuous anti-homomorphism such that $\sigma \circ \sigma = id$. The non-zero solutions $f : G \to \mathbb{C}$ of of the functional equation*

$$f(x\sigma(y)) + f(y^{-1}x) = 2f(x)f(y), \quad x, y \in G,$$

are the functions of the form $f = (\chi + \overline{\chi \circ \sigma})/2$, where $\chi : G \to \mathbb{C}$ is a character such that χ is σ-even and/or $\overline{\chi} = \chi$.

Proof. It suffices to take $\tau(x) = x^{-1}$ for all $x \in G$ in Theorem 2. □

Corollary 5 *Let G be a compact group and τ be a continuous anti-homomorphism such that $\tau \circ \tau = id$. The non-zero solutions $f : G \to \mathbb{C}$ of of the functional equation*

$$f(xy^{-1}) + f(\tau(y)x) = 2f(x)f(y), \quad x, y \in G,$$

are the functions of the form $f = (\chi + \overline{\chi \circ \tau})/2$, where $\chi : G \to \mathbb{C}$ is a character such that χ is τ-even and/or $\overline{\chi} = \chi$.

Proof. It suffices to take $\sigma(x) = x^{-1}$ for all $x \in G$ in Theorem 2. □

Corollary 6 *The non-zero solutions $f : G \to \mathbb{C}$ of the functional equation*

$$f(xy) + f(yx) = 2f(x)f(y), \; x, y \in G,$$

are the functions of the form $f = \chi$, where $\chi : G \to \mathbb{C}$ is a unitary character.

Proof. It suffices to take $\sigma(x) = \tau(x) = x$ for all $x \in G$ in Theorem 1. □

Corollary 7 *The non-zero solutions $f : G \to \mathbb{C}$ of the functional equation*

$$f(xy^{-1}) + f(y^{-1}x) = 2f(x)f(y), \; x, y \in G,$$

are the functions of the form $f = \chi$, where $\chi : G \to \mathbb{C}$ is a unitary character such that $\overline{\chi} = \chi$.

Proof. It suffices to take $\sigma(x) = \tau(x) = x^{-1}$ for all $x \in G$ in Theorem 2. □

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