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# Trace inequalities of Cassels and Grüss type for operators in Hilbert spaces

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**Abstract.** Some trace inequalities of Cassels type for operators in Hilbert spaces are provided. Applications in connection to Grüss inequality and for convex functions of selfadjoint operators are also given.

## 1 Introduction

Let $\bar{\mathbf{a}} = (a_1, \ldots, a_n)$ and $\bar{\mathbf{b}} = (b_1, \ldots, b_n)$ be two positive $n$-*tuples* with

$$0 < m_1 \le a_i \le M_1 < \infty \text{ and } 0 < m_2 \le b_i \le M_2 < \infty; \tag{1}$$

for each $i \in \{1, \ldots, n\}$, and some constants $m_1$, $m_2$, $M_1$, $M_2$.

The following reverses of the Cauchy-Bunyakovsky-Schwarz inequality for positive sequences of real numbers are well known:

a) *Pólya-Szegö's inequality* [44]:

$$\frac{\sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2}{\left(\sum_{k=1}^{n} a_k b_k\right)^2} \le \frac{1}{4}\left(\sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}}\right)^2.$$

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b) *Shisha-Mond's inequality* [48]:

$$\frac{\sum_{k=1}^{n} a_k^2}{\sum_{k=1}^{n} a_k b_k} - \frac{\sum_{k=1}^{n} a_k b_k}{\sum_{k=1}^{n} b_k^2} \leq \left[ \left( \frac{M_1}{m_2} \right)^{\frac{1}{2}} - \left( \frac{m_1}{M_2} \right)^{\frac{1}{2}} \right]^2 .$$

If $\overline{\mathbf{w}} = (w_1, \ldots, w_n)$ is a positive sequence, then the following weighted inequalities also hold:

c) *Cassels' inequality* [15]. If the positive real sequences $\overline{\mathbf{a}} = (a_1, \ldots, a_n)$ and $\overline{\mathbf{b}} = (b_1, \ldots, b_n)$ satisfy the condition

$$0 < m \leq \frac{a_k}{b_k} \leq M < \infty \text{ for each } k \in \{1, \ldots, n\}, \tag{2}$$

then

$$\frac{\left( \sum_{k=1}^{n} w_k a_k^2 \right) \left( \sum_{k=1}^{n} w_k b_k^2 \right)}{\left( \sum_{k=1}^{n} w_k a_k b_k \right)^2} \leq \frac{(M+m)^2}{4mM}.$$

d) *Greub-Reinboldt's inequality* [34]. We have

$$\left( \sum_{k=1}^{n} w_k a_k^2 \right) \left( \sum_{k=1}^{n} w_k b_k^2 \right) \leq \frac{(M_1 M_2 + m_1 m_2)^2}{4 m_1 m_2 M_1 M_2} \left( \sum_{k=1}^{n} w_k a_k b_k \right)^2 ,$$

provided $\overline{\mathbf{a}} = (a_1, \ldots, a_n)$ and $\overline{\mathbf{b}} = (b_1, \ldots, b_n)$ satisfy the condition (1).

For other recent results providing discrete reverse inequalities, see the monograph online [15].

The following reverse of Schwarz's inequality in inner product spaces holds [16].

**Theorem 1 (Dragomir, 2003, [16])** *Let* $A, a \in \mathbb{C}$ *and* $x, y \in H$, *a complex inner product space with the inner product* $\langle \cdot, \cdot \rangle$. *If*

$$\operatorname{Re} \langle Ay - x, x - ay \rangle \geq 0, \tag{3}$$

*or equivalently,*

$$\left\| x - \frac{a + A}{2} \cdot y \right\| \leq \frac{1}{2} |A - a| \, \|y\| , \tag{4}$$

*holds, then we have the inequality*

$$0 \leq \|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2 \leq \frac{1}{4} |A - a|^2 \|y\|^4 . \tag{5}$$

*The constant* $\frac{1}{4}$ *is sharp in* (5).

In 1935, G. Grüss [35] proved the following integral inequality which gives an approximation of the integral mean of the product in terms of the product of the integrals means as follows:

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx \cdot \frac{1}{b-a}\int_a^b g(x)\,dx\right| \tag{6}$$
$$\leq \frac{1}{4}(\Phi-\phi)(\Gamma-\gamma),$$

where $f, g : [a,b] \to \mathbb{R}$ are integrable on $[a,b]$ and satisfy the condition

$$\phi \leq f(x) \leq \Phi,\ \gamma \leq g(x) \leq \Gamma \tag{7}$$

for each $x \in [a,b]$, where $\phi$, $\Phi$, $\gamma$, $\Gamma$ are given real constants.

Moreover, the constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller one.

In [18], in order to generalize the Grüss integral inequality in abstract structures the author has proved the following inequality in inner product spaces.

**Theorem 2 (Dragomir, 1999, [18])** *Let $(H, \langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$ $(\mathbb{K} = \mathbb{R},\mathbb{C})$ and $e \in H$, $\|e\| = 1$. If $\varphi$, $\gamma$, $\Phi$, $\Gamma$ are real or complex numbers and $x$, $y$ are vectors in $H$ such that the conditions*

$$\operatorname{Re}\langle \Phi e - x, x - \varphi e\rangle \geq 0 \text{ and } \operatorname{Re}\langle \Gamma e - y, y - \gamma e\rangle \geq 0 \tag{8}$$

*hold, then we have the inequality*

$$|\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle| \leq \frac{1}{4}|\Phi-\varphi|\,|\Gamma-\gamma|. \tag{9}$$

*The constant $\frac{1}{4}$ is best possible in the sense that it can not be replaced by a smaller constant.*

For other results of this type, see the recent monograph [21] and the references therein.

For other Grüss type results for integral and sums see the papers [1]-[3], [8]-[10], [17]-[24], [31], and the references therein.

In order to state some reverses of Schwarz and Grüss type inequalities for trace operators on complex Hilbert spaces we need some preparations as follows.

## 2 Some facts on trace of operators

Let $(H, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and $\{e_i\}_{i \in I}$ an *orthonormal basis* of $H$. We say that $A \in \mathcal{B}(H)$ is a *Hilbert-Schmidt operator* if

$$\sum_{i \in I} \|Ae_i\|^2 < \infty. \tag{10}$$

It is well know that, if $\{e_i\}_{i \in I}$ and $\{f_j\}_{j \in J}$ are orthonormal bases for $H$ and $A \in \mathcal{B}(H)$ then

$$\sum_{i \in I} \|Ae_i\|^2 = \sum_{j \in I} \|Af_j\|^2 = \sum_{j \in I} \|A^* f_j\|^2 \tag{11}$$

showing that the definition (10) is independent of the orthonormal basis and $A$ is a Hilbert-Schmidt operator iff $A^*$ is a Hilbert-Schmidt operator.

Let $\mathcal{B}_2(H)$ the set of Hilbert-Schmidt operators in $\mathcal{B}(H)$. For $A \in \mathcal{B}_2(H)$ we define

$$\|A\|_2 := \left( \sum_{i \in I} \|Ae_i\|^2 \right)^{1/2} \tag{12}$$

for $\{e_i\}_{i \in I}$ an orthonormal basis of $H$. This definition does not depend on the choice of the orthonormal basis.

Using the triangle inequality in $l^2(I)$, one checks that $\mathcal{B}_2(H)$ is a *vector space* and that $\|\cdot\|_2$ is a norm on $\mathcal{B}_2(H)$, which is usually called in the literature as the *Hilbert-Schmidt norm.*

Denote *the modulus* of an operator $A \in \mathcal{B}(H)$ by $|A| := (A^*A)^{1/2}$.

Because $\||A|x\| = \|Ax\|$ for all $x \in H$, $A$ is Hilbert-Schmidt iff $|A|$ is Hilbert-Schmidt and $\|A\|_2 = \||A|\|_2$. From (11) we have that if $A \in \mathcal{B}_2(H)$, then $A^* \in \mathcal{B}_2(H)$ and $\|A\|_2 = \|A^*\|_2$.

If $\{e_i\}_{i \in I}$ an orthonormal basis of $H$, we say that $A \in \mathcal{B}(H)$ is *trace class* if

$$\|A\|_1 := \sum_{i \in I} \langle |A| e_i, e_i \rangle < \infty. \tag{13}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i \in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i \in I} \langle Ae_i, e_i \rangle, \tag{14}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of $H$. Note that this coincides with the usual definition of the trace if $H$ is finite-dimensional. We observe that the series (14) converges absolutely and it is independent from the choice of basis.

Utilising the trace notation we obviously have that

$$\langle A, B\rangle_2 = \operatorname{tr}(B^*A) = \operatorname{tr}(AB^*) \text{ and } \|A\|_2^2 = \operatorname{tr}(A^*A) = \operatorname{tr}\left(|A|^2\right)$$

for any $A, B \in \mathcal{B}_2(H)$.

The following Hölder's type inequality has been obtained by Ruskai in [45]

$$|\operatorname{tr}(AB)| \leq \operatorname{tr}(|AB|) \leq \left[\operatorname{tr}\left(|A|^{1/\alpha}\right)\right]^{\alpha}\left[\operatorname{tr}\left(|B|^{1/(1-\alpha)}\right)\right]^{1-\alpha} \tag{15}$$

where $\alpha \in (0,1)$ and $A, B \in \mathcal{B}(H)$ with $|A|^{1/\alpha}, |B|^{1/(1-\alpha)} \in \mathcal{B}_1(H)$.

In particular, for $\alpha = \frac{1}{2}$ we get the Schwarz inequality

$$|\operatorname{tr}(AB)| \leq \operatorname{tr}(|AB|) \leq \left[\operatorname{tr}\left(|A|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(|B|^2\right)\right]^{1/2} \tag{16}$$

with $A, B \in \mathcal{B}_2(H)$.

For the theory of trace functionals and their applications the reader is referred to [49].

For some classical trace inequalities see [11], [13], [42] and [53], which are continuations of the work of Bellman [5]. For related works the reader can refer to [4], [6], [11], [32], [36], [37], [39], [46] and [50].

We denote by

$$\mathcal{B}_1^+(H) := \{P : P \in \mathcal{B}_1(H), \ P \text{ and is selfadjoint and } P \geq 0\}.$$

We obtained recently the following result [29]:

**Theorem 3** *For any $A, C \in \mathcal{B}(H)$ and $P \in \mathcal{B}_1^+(H)\setminus\{0\}$ we have the inequality*

$$\begin{aligned}
&\left|\frac{\operatorname{tr}(PAC)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| \\
&\leq \inf_{\lambda\in\mathbb{C}}\|A - \lambda\cdot 1_H\|\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right) \\
&\leq \inf_{\lambda\in\mathbb{C}}\|A - \lambda\cdot 1_H\|\left[\frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)} - \left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right]^{1/2},
\end{aligned} \tag{17}$$

*where $\|\cdot\|$ is the operator norm.*

In the following we establish other similar results for trace that generalize the classical Cassels' inequality stated in the introduction.

## 3 Cassels type trace inequalities

For two given operators $T, U \in B(H)$ and two given scalars $\alpha, \beta \in \mathbb{C}$ consider the transform

$$\mathcal{C}_{\alpha,\beta}(T,U) = (T^* - \bar{\alpha}U^*)(\beta U - T).$$

This transform generalizes the transform

$$\mathcal{C}_{\alpha,\beta}(T) := (T^* - \bar{\alpha}1_H)(\beta 1_H - T) = \mathcal{C}_{\alpha,\beta}(T, 1_H),$$

where $1_H$ is the identity operator, which has been introduced in [27] in order to provide some generalizations of the well known Kantorovich inequality for operators in Hilbert spaces.

We recall that a bounded linear operator $T$ on the complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$ is called *accretive* if $\operatorname{Re}\langle Ty, y\rangle \geq 0$ for any $y \in H$.

Utilizing the following identity

$$\begin{aligned}\operatorname{Re}\langle \mathcal{C}_{\alpha,\beta}(T,U)x, x\rangle &= \operatorname{Re}\langle \mathcal{C}_{\beta,\alpha}(T,U)x, x\rangle \\ &= \frac{1}{4}|\beta-\alpha|^2\|Ux\|^2 - \left\|Tx - \frac{\alpha+\beta}{2}\cdot Ux\right\|^2 \\ &= \frac{1}{4}|\beta-\alpha|^2\left\langle |U|^2 x, x\right\rangle - \left\langle \left|T - \frac{\alpha+\beta}{2}\cdot U\right|^2 x, x\right\rangle\end{aligned} \tag{18}$$

that holds for any scalars $\alpha$, $\beta$ and any vector $x \in H$, we can give a simple characterization result that is useful in the following:

**Lemma 1** *For $\alpha, \beta \in \mathbb{C}$ and $T, U \in B(H)$ the following statements are equivalent:*

(i) *The transform $\mathcal{C}_{\alpha,\beta}(T,U)$ (or, equivalently, $\mathcal{C}_{\beta,\alpha}(T,U)$) is accretive;*

(ii) *We have the norm inequality*

$$\left\|Tx - \frac{\alpha+\beta}{2}\cdot Ux\right\| \leq \frac{1}{2}|\beta-\alpha|\,\|Ux\| \tag{19}$$

*for any $x \in H$;*

(iii) *We have the following inequality in the operator order*

$$\left|T - \frac{\alpha+\beta}{2}\cdot U\right|^2 \leq \frac{1}{4}|\beta-\alpha|^2|U|^2.$$

As a consequence of the above lemma we can state:

**Corollary 1** *Let* $\alpha,\ \beta \in \mathbb{C}$ *and* $T,\ U \in B(H)$. *If* $\mathcal{C}_{\alpha,\beta}(T,U)$ *is accretive, then*

$$\left\| T - \frac{\alpha+\beta}{2} \cdot U \right\| \le \frac{1}{2} |\beta - \alpha| \|U\| . \tag{20}$$

**Remark 1** *In order to give examples of linear operators* $T,\ U \in B(H)$ *and numbers* $\alpha,\ \beta \in \mathbb{C}$ *such that the transform* $\mathcal{C}_{\alpha,\beta}(T,U)$ *is accretive, it suffices to select two bounded linear operator* $S$ *and* $V$ *and the complex numbers* $z,\ w$ $(w \neq 0)$ *with the property that* $\|Sx - zVx\| \le |w| \|Vx\|$ *for any* $x \in H$, *and, by choosing* $T = S,\ U = V,\ \alpha = \frac{1}{2}(z+w)$ *and* $\beta = \frac{1}{2}(z-w)$ *we observe that* $T$ *and* $U$ *satisfy* (19), *i.e.,* $\mathcal{C}_{\alpha,\beta}(T,U)$ *is accretive.*

The following result also holds:

**Lemma 2** *Let, either* $P \in \mathcal{B}_+(H),\ A,\ B \in \mathcal{B}_2(H)$ *or* $P \in \mathcal{B}_1^+(H),\ A,\ B \in \mathcal{B}(H)$ *and* $\gamma,\ \Gamma \in \mathbb{C}$. *Then*

$$\mathrm{Re}\left(\mathrm{tr}\left[P\left(A^* - \overline{\gamma}B^*\right)(\Gamma B - A)\right]\right) \ge 0 \tag{21}$$

*if and only if*

$$\mathrm{tr}\left(P\left|A - \frac{\gamma+\Gamma}{2}B\right|^2\right) \le \frac{1}{4}|\Gamma-\gamma|^2 \mathrm{tr}\left(P|B|^2\right). \tag{22}$$

*To simplify the writing, we the say that* $(A, B)$ *satisfies the* $P$-$(\gamma,\Gamma)$-*trace property.*

**Proof.** We have the equalities

$$\begin{aligned} &\frac{1}{4}|\Gamma-\gamma|^2 P|B|^2 - P\left|A - \frac{\gamma+\Gamma}{2}B\right|^2 \\ &= P\left[\frac{1}{4}|\Gamma-\gamma|^2|B|^2 - \left|A - \frac{\gamma+\Gamma}{2}B\right|^2\right] \\ &= P\left[\frac{1}{4}|\Gamma-\gamma|^2|B|^2 - \left(A - \frac{\gamma+\Gamma}{2}B\right)^*\left(A - \frac{\gamma+\Gamma}{2}B\right)\right] \end{aligned} \tag{23}$$

$$
\begin{aligned}
&= P\left[\frac{1}{4}|\Gamma-\gamma|^2|B|^2-|A|^2+\frac{\overline{\gamma+\Gamma}}{2}B^*A+\frac{\gamma+\Gamma}{2}A^*B-\left|\frac{\gamma+\Gamma}{2}\right|^2|B|^2\right]\\
&= P\left[-|A|^2+\frac{\overline{\gamma+\Gamma}}{2}B^*A+\frac{\gamma+\Gamma}{2}A^*B+\left(\frac{1}{4}|\Gamma-\gamma|^2-\left|\frac{\gamma+\Gamma}{2}\right|^2\right)|B|^2\right]\\
&= P\left[-|A|^2+\frac{\overline{\gamma+\Gamma}}{2}B^*A+\frac{\gamma+\Gamma}{2}A^*B-\operatorname{Re}(\Gamma\overline{\gamma})|B|^2\right]
\end{aligned}
$$

for any bounded operators $A$, $B$, $P$ and the complex numbers $\gamma, \Gamma\in\mathbb{C}$.

Taking the trace in (23) we get

$$
\begin{aligned}
&\frac{1}{4}|\Gamma-\gamma|^2\operatorname{tr}\left(P|B|^2\right)-\operatorname{tr}\left(P\left|A-\frac{\gamma+\Gamma}{2}B\right|^2\right) \qquad (24)\\
&= -\operatorname{tr}\left(P|A|^2\right)-\operatorname{Re}(\Gamma\overline{\gamma})\operatorname{tr}\left(P|B|^2\right)+\frac{\overline{\gamma+\Gamma}}{2}\operatorname{tr}(PB^*A)+\frac{\gamma+\Gamma}{2}\operatorname{tr}(PA^*B)\\
&= -\operatorname{tr}\left(P|A|^2\right)-\operatorname{Re}(\Gamma\overline{\gamma})\operatorname{tr}\left(P|B|^2\right)+\frac{\overline{\gamma+\Gamma}}{2}\operatorname{tr}(PB^*A)+\frac{\gamma+\Gamma}{2}\overline{\operatorname{tr}(PB^*A)}\\
&= -\operatorname{tr}\left(P|A|^2\right)-\operatorname{Re}(\Gamma\overline{\gamma})\operatorname{tr}\left(P|B|^2\right)+\frac{\overline{\gamma+\Gamma}}{2}\operatorname{tr}(PB^*A)+\overline{\frac{\overline{\gamma+\Gamma}}{2}\operatorname{tr}(PB^*A)}\\
&= -\operatorname{tr}\left(P|A|^2\right)-\operatorname{Re}(\Gamma\overline{\gamma})\operatorname{tr}\left(P|B|^2\right)+2\operatorname{Re}\left[\frac{\overline{\gamma+\Gamma}}{2}\operatorname{tr}(PB^*A)\right]\\
&= -\operatorname{tr}\left(P|A|^2\right)-\operatorname{Re}(\Gamma\overline{\gamma})\operatorname{tr}\left(P|B|^2\right)+\operatorname{Re}[\overline{\gamma}\operatorname{tr}(PB^*A)]+\operatorname{Re}\left[\overline{\Gamma}\operatorname{tr}(PB^*A)\right]\\
&= -\operatorname{tr}\left(P|A|^2\right)-\operatorname{Re}(\Gamma\overline{\gamma})\operatorname{tr}\left(P|B|^2\right)+\operatorname{Re}[\overline{\gamma}\operatorname{tr}(PB^*A)]+\operatorname{Re}\left[\overline{\overline{\overline{\Gamma}}\operatorname{tr}(PB^*A)}\right]\\
&= -\operatorname{tr}\left(P|A|^2\right)-\operatorname{Re}(\Gamma\overline{\gamma})\operatorname{tr}\left(P|B|^2\right)+\operatorname{Re}[\overline{\gamma}\operatorname{tr}(PB^*A)]+\operatorname{Re}\left[\Gamma\overline{\operatorname{tr}(PB^*A)}\right].
\end{aligned}
$$

Since

$$
\begin{aligned}
&\operatorname{Re}(\operatorname{tr}[P(A^*-\overline{\gamma}B^*)(\Gamma B-A)])\\
&= \operatorname{Re}[\operatorname{tr}(\Gamma PA^*B+\overline{\gamma}PB^*A-\overline{\gamma}\Gamma PB^*B-PA^*A)]\\
&= \operatorname{Re}\left[\Gamma\operatorname{tr}\left(PA^*B\right)+\overline{\gamma}\operatorname{tr}(PB^*A)\right]-\overline{\gamma}\Gamma\operatorname{tr}\left(P|B|^2\right)-\operatorname{tr}\left(P|A|^2\right)\Big]\\
&= \operatorname{Re}\left[\Gamma\overline{\operatorname{tr}(PB^*A)}+\overline{\gamma}\operatorname{tr}(PB^*A)\right]-\operatorname{tr}\left(P|B|^2\right)\operatorname{Re}(\overline{\gamma}\Gamma)-\operatorname{tr}\left(P|A|^2\right),
\end{aligned}
$$

then we get

$$
\begin{aligned}
&\frac{1}{4}|\Gamma-\gamma|^2\operatorname{tr}\left(P|B|^2\right)-\operatorname{tr}\left(P\left|A-\frac{\gamma+\Gamma}{2}B\right|^2\right)\\
&= \operatorname{Re}(\operatorname{tr}[P(A^*-\overline{\gamma}B^*)(\Gamma B-A)]),
\end{aligned} \qquad (25)
$$

which proves the desired equivalence. □

**Corollary 2** *Let, either* $P \in \mathcal{B}_+(H)$, $A$, $B \in \mathcal{B}_2(H)$ *or* $P \in \mathcal{B}_1^+(H)$, $A$, $B \in \mathcal{B}(H)$ *and* $\gamma$, $\Gamma \in \mathbb{C}$. *If the transform* $\mathcal{C}_{\gamma,\Gamma}(A,B)$ *is accretive, then* $(A,B)$ *satisfies the* $P$-$(\gamma,\Gamma)$*-trace property.*

We have the following result:

**Theorem 4** *Let, either* $P \in \mathcal{B}_+(H)$, $A$, $B \in \mathcal{B}_2(H)$ *or* $P \in \mathcal{B}_1^+(H)$, $A$, $B \in \mathcal{B}(H)$ *and* $\gamma$, $\Gamma \in \mathbb{C}$ *with* $\operatorname{Re}(\Gamma\overline{\gamma}) = \operatorname{Re}(\Gamma)\operatorname{Re}(\gamma) + \operatorname{Im}(\Gamma)\operatorname{Im}(\gamma) > 0$.

(i) *If* $(A,B)$ *satisfies the* $P$-$(\gamma,\Gamma)$*-trace property, then we have*

$$\begin{aligned}
&\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|B|^2\right)\\
&\le \frac{1}{4}\cdot\frac{\left[\operatorname{Re}(\gamma+\Gamma)\operatorname{Re}\operatorname{tr}(PB^*A) + \operatorname{Im}(\gamma+\Gamma)\operatorname{Im}\operatorname{tr}(PB^*A)\right]^2}{\operatorname{Re}(\Gamma)\operatorname{Re}(\gamma) + \operatorname{Im}(\Gamma)\operatorname{Im}(\gamma)} \qquad (26)\\
&\le \frac{1}{4}\cdot\frac{|\gamma+\Gamma|^2}{\operatorname{Re}(\Gamma\overline{\gamma})}\left|\operatorname{tr}(PB^*A)\right|^2.
\end{aligned}$$

(ii) *If the transform* $\mathcal{C}_{\gamma,\Gamma}(A,B)$ *is accretive, then the inequality* (26) *also holds.*

**Proof.** (i) If $(A,B)$ satisfies the $P$-$(\gamma,\Gamma)$-trace property, then, on utilizing the calculations above, we have

$$\begin{aligned}
0 &\le \frac{1}{4}|\Gamma-\gamma|^2\operatorname{tr}\left(P|B|^2\right) - \operatorname{tr}\left(P\left|A-\frac{\gamma+\Gamma}{2}B\right|^2\right)\\
&= -\operatorname{tr}\left(P|A|^2\right) - \operatorname{Re}(\Gamma\overline{\gamma})\operatorname{tr}\left(P|B|^2\right) + \operatorname{Re}\left[\overline{\gamma}\operatorname{tr}(PB^*A)\right] + \operatorname{Re}\left[\Gamma\overline{\operatorname{tr}(PB^*A)}\right]\\
&= -\operatorname{tr}\left(P|A|^2\right) - \operatorname{Re}(\Gamma\overline{\gamma})\operatorname{tr}\left(P|B|^2\right) + \operatorname{Re}\left[\overline{\gamma}\operatorname{tr}(PB^*A)\right] + \operatorname{Re}\left[\overline{\Gamma\overline{\operatorname{tr}(PB^*A)}}\right]\\
&= -\operatorname{tr}\left(P|A|^2\right) - \operatorname{Re}(\Gamma\overline{\gamma})\operatorname{tr}\left(P|B|^2\right) + \operatorname{Re}\left[\overline{\gamma}\operatorname{tr}(PB^*A)\right] + \operatorname{Re}\left[\overline{\Gamma}\operatorname{tr}(PB^*A)\right]\\
&= -\operatorname{tr}\left(P|A|^2\right) - \operatorname{Re}(\Gamma\overline{\gamma})\operatorname{tr}\left(P|B|^2\right) + \operatorname{Re}\left[\left(\overline{\gamma}+\overline{\Gamma}\right)\operatorname{tr}(PB^*A)\right],
\end{aligned}$$

which implies that

$$\begin{aligned}
&\operatorname{tr}\left(P|A|^2\right) + \operatorname{Re}(\Gamma\overline{\gamma})\operatorname{tr}\left(P|B|^2\right) \le \operatorname{Re}\left[\left(\overline{\gamma}+\overline{\Gamma}\right)\operatorname{tr}(PB^*A)\right]\\
&= \operatorname{Re}(\gamma+\Gamma)\operatorname{Re}\operatorname{tr}(PB^*A) + \operatorname{Im}(\gamma+\Gamma)\operatorname{Im}\operatorname{tr}(PB^*A).
\end{aligned} \qquad (27)$$

Making use of the elementary inequality

$$2\sqrt{pq} \le p+q,\ \ p,q \ge 0,$$

we also have

$$2\sqrt{\mathrm{Re}\,(\Gamma\overline{\gamma})\,\mathrm{tr}\big(P\,|A|^2\,\big)\mathrm{tr}\big(P\,|B|^2\,\big)} \le \mathrm{tr}\big(P\,|A|^2\,big\big) + \mathrm{Re}\,(\Gamma\overline{\gamma})\,\mathrm{tr}\big(P\,|B|^2\,\big). \tag{28}$$

Utilising (27) and (28) we get

$$\begin{aligned} &\sqrt{\mathrm{tr}\big(P\,|A|^2\,\big)\mathrm{tr}\big(P\,|B|^2\,\big)} \\ &\le \frac{\mathrm{Re}\,(\gamma+\Gamma)\,\mathrm{Re}\;\mathrm{tr}\,(PB^*A) + \mathrm{Im}\,(\gamma+\Gamma)\,\mathrm{Im}\;\mathrm{tr}\,(PB^*A)}{2\sqrt{\mathrm{Re}\,(\Gamma\overline{\gamma})}} \end{aligned} \tag{29}$$

that is equivalent with the first inequality in (26).

The second inequality in (26) is obvious by Schwarz inequality

$$(ab+cd)^2 \le \left(a^2+c^2\right)\left(b^2+d^2\right),\ \ a,b,c,d \in \mathbb{R}.$$

The (ii) is obvious from (i). □

**Remark 2** *We observe that the inequality between the first and last term in* (26) *is equivalent to*

$$0 \le \mathrm{tr}\big(P\,|A|^2\,\big)\mathrm{tr}\big(P\,|B|^2\,\big) - |\mathrm{tr}\,(PB^*A)|^2 \le \frac{1}{4}\cdot\frac{|\gamma-\Gamma|^2}{\mathrm{Re}\,(\Gamma\overline{\gamma})}\,|\mathrm{tr}\,(PB^*A)|^2\,. \tag{30}$$

**Corollary 3** *Let, either* $P \in \mathcal{B}_+(H)$, $A \in \mathcal{B}_2(H)$ *or* $P \in \mathcal{B}_1^+(H)$, $A \in \mathcal{B}(H)$ *and* $\gamma, \Gamma \in \mathbb{C}$ *with* $\mathrm{Re}\,(\Gamma\overline{\gamma}) = \mathrm{Re}\,(\Gamma)\,\mathrm{Re}\,(\gamma) + \mathrm{Im}\,(\Gamma)\,\mathrm{Im}\,(\gamma) > 0$.

(i) *If* $A$ *satisfies the* $P$-$(\gamma,\Gamma)$*-trace property, namely*

$$\mathrm{Re}\,(\mathrm{tr}\,[P\,(A^* - \overline{\gamma}1_H)\,(\Gamma 1_H - A)]) \ge 0 \tag{31}$$

*or, equivalently*

$$\mathrm{tr}\left(P\left|A - \frac{\gamma+\Gamma}{2}1_H\right|^2\right) \le \frac{1}{4}\,|\Gamma-\gamma|^2\,\mathrm{tr}\,(P)\,, \tag{32}$$

*then we have*

$$\begin{aligned} \frac{\mathrm{tr}\left(P\,|A|^2\right)}{\mathrm{tr}\,(P)} &\le \frac{1}{4}\cdot\frac{\left[\mathrm{Re}\,(\gamma+\Gamma)\,\frac{\mathrm{Re tr}(PA)}{\mathrm{tr}(P)} + \mathrm{Im}\,(\gamma+\Gamma)\,\frac{\mathrm{Im tr}(PA)}{\mathrm{tr}(P)}\right]^2}{\mathrm{Re}\,(\Gamma)\,\mathrm{Re}\,(\gamma) + \mathrm{Im}\,(\Gamma)\,\mathrm{Im}\,(\gamma)} \\ &\le \frac{1}{4}\cdot\frac{|\gamma+\Gamma|^2}{\mathrm{Re}\,(\Gamma\overline{\gamma})}\left|\frac{\mathrm{tr}\,(PA)}{\mathrm{tr}\,(P)}\right|^2. \end{aligned} \tag{33}$$

(ii) *If the transform* $\mathcal{C}_{\gamma,\Gamma}(A)$ *is accretive, then the inequality* (26) *also holds.*

(iii) *We have*

$$0 \leq \frac{\operatorname{tr}\left(P|A|^2\right)}{\operatorname{tr}(P)} - \left|\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right|^2 \leq \frac{1}{4} \cdot \frac{|\gamma - \Gamma|^2}{\operatorname{Re}(\Gamma\overline{\gamma})} \left|\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right|^2. \tag{34}$$

**Remark 3** *The case of selfadjoint operators is as follows.*

*Let* $A$, $B$ *be selfadjoint operators and either* $P \in \mathcal{B}_+(H)$, $A, B \in \mathcal{B}_2(H)$ *or* $P \in \mathcal{B}_1^+(H)$, $A, B \in \mathcal{B}(H)$ *and* $m, M \in \mathbb{R}$ *with* $mM > 0$.

(i) *If* $(A, B)$ *satisfies the* $P$-$(m, M)$*-trace property, then we have*

$$\operatorname{tr}\left(PA^2\right)\operatorname{tr}\left(PB^2\right) \leq \frac{(m+M)^2}{4mM}[\operatorname{tr}(PBA)]^2 \tag{35}$$

*or, equivalently*

$$0 \leq \operatorname{tr}\left(PA^2\right)\operatorname{tr}\left(PB^2\right) - [\operatorname{tr}(PBA)]^2 \leq \frac{(m-M)^2}{4mM}[\operatorname{tr}(PBA)]^2. \tag{36}$$

(ii) *If the transform* $\mathcal{C}_{m,M}(A, B)$ *is accretive, then the inequality* (35) *also holds.*

(iii) *If* $(A - mB)(MB - A) \geq 0$, *then* (35) *is valid.*

*We observe that the inequality* (35) *is the operator trace inequality version of Cassels' inequality from Introduction.*

## 4 Trace inequalities of Grüss type

Let $P$ be a selfadjoint operator with $P \geq 0$. The functional $\langle\cdot,\cdot\rangle_{2,P}$ defined by

$$\langle A, B\rangle_{2,P} := \operatorname{tr}(PB^*A) = \operatorname{tr}(APB^*) = \operatorname{tr}(B^*AP)$$

is a *nonnegative Hermitian form* on $\mathcal{B}_2(H)$, i.e. $\langle\cdot,\cdot\rangle_{2,P}$ satisfies the properties:

*(h)* $\langle A, A\rangle_{2,P} \geq 0$ for any $A \in \mathcal{B}_2(H)$;

*(hh)* $\langle\cdot,\cdot\rangle_{2,P}$ is linear in the first variable;

*(hhh)* $\langle B, A\rangle_{2,P} = \overline{\langle A, B\rangle_{2,P}}$ for any $A, B \in \mathcal{B}_2(H)$.

Using the properties of the trace we also have the following representations

$$\|A\|_{2,P}^2 := \operatorname{tr}\left(P|A|^2\right) = \operatorname{tr}\left(APA^*\right) = \operatorname{tr}\left(|A|^2 P\right)$$

and

$$\langle A, B\rangle_{2,P} = \operatorname{tr}(APB^*) = \operatorname{tr}(B^*AP)$$

for any $A, B \in \mathcal{B}_2(H)$.

The same definitions can be considered if $P \in \mathcal{B}_1^+(H)$ and $A, B \in \mathcal{B}(H)$.

We have the following Grüss type inequality:

**Theorem 5** *Let, either* $P \in \mathcal{B}_+(H)$, $A$, $B$, $C \in \mathcal{B}_2(H)$ *or* $P \in \mathcal{B}_1^+(H)$, $A$, $B$, $C \in \mathcal{B}(H)$ *with* $P|A|^2$, $P|B|^2$, $P|C|^2 \neq 0$ *and* $\lambda$, $\Gamma$, $\delta$, $\Delta \in \mathbb{C}$ *with* $\operatorname{Re}(\Gamma\overline{\gamma})$, $\operatorname{Re}(\Delta\overline{\delta}) > 0$. *If* $(A, C)$ *has the trace* $P$-$(\lambda, \Gamma)$*-property and* $(B, C)$ *has the trace* $P$-$(\delta, \Delta)$*-property, then*

$$\left|\frac{\operatorname{tr}(PB^*A)\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(PC^*A)\operatorname{tr}(PB^*C)} - 1\right| \leq \frac{1}{4} \cdot \frac{|\gamma - \Gamma|\,|\delta - \Delta|}{\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})\operatorname{Re}\left(\Delta\overline{\delta}\right)}}. \tag{37}$$

**Proof.** We prove in the case that $P \in \mathcal{B}_+(H)$ and $A$, $B$, $C \in \mathcal{B}_2(H)$.

Making use of the Schwarz inequality for the nonnegative hermitian form $\langle\cdot,\cdot\rangle_{2,P}$ we have

$$|\langle A, B\rangle_{2,P}|^2 \leq \langle A, A\rangle_{2,P}\,\langle B, B\rangle_{2,P}$$

for any $A$, $B \in \mathcal{B}_2(H)$.

Let $C \in \mathcal{B}_2(H)$, $C \neq 0$. Define the mapping $[\cdot,\cdot]_{2,P,C} : \mathcal{B}_2(H) \times \mathcal{B}_2(H) \to \mathbb{C}$ by

$$[A, B]_{2,P,C} := \langle A, B\rangle_{2,P}\,\|C\|_{2,P}^2 - \langle A, C\rangle_{2,P}\,\langle C, B\rangle_{2,P}.$$

Observe that $[\cdot,\cdot]_{2,P,C}$ is a nonnegative Hermitian form on $\mathcal{B}_2(H)$ and by Schwarz inequality we also have

$$\begin{aligned}
&\left|\langle A, B\rangle_{2,P}\,\|C\|_{2,P}^2 - \langle A, C\rangle_{2,P}\,\langle C, B\rangle_{2,P}\right|^2 \\
&\leq \left[\|A\|_{2,P}^2\,\|C\|_{2,P}^2 - |\langle A, C\rangle_{2,P}|^2\right]\left[\|B\|_{2,P}^2\,\|C\|_{2,P}^2 - |\langle B, C\rangle_{2,P}|^2\right]
\end{aligned}$$

for any $A$, $B \in \mathcal{B}_2(H)$, namely

$$\begin{aligned}
&\left|\operatorname{tr}(PB^*A)\operatorname{tr}\left(P|C|^2\right) - \operatorname{tr}(PC^*A)\operatorname{tr}(PB^*C)\right|^2 \\
&\leq \left[\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|C|^2\right) - |\operatorname{tr}(PC^*A)|^2\right] \\
&\times \left[\operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|C|^2\right) - |\operatorname{tr}(PB^*C)|^2\right],
\end{aligned} \tag{38}$$

where for the last term we used the equality $|\langle B, C\rangle_{2,P}|^2 = |\langle C, B\rangle_{2,P}|^2$.

Since $(A, C)$ has the trace $P$-$(\lambda, \Gamma)$-property and $(B, C)$ has the trace $P$-$(\delta, \Delta)$ -property, then by (30) we have

$$0 \leq \operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|C|^2\right) - |\operatorname{tr}(PC^*A)|^2 \leq \frac{1}{4} \cdot \frac{|\gamma - \Gamma|^2}{\operatorname{Re}(\Gamma\overline{\gamma})}\,|\operatorname{tr}(PC^*A)|^2 \tag{39}$$

and

$$0 \leq \operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|C|^2\right) - |\operatorname{tr}(PB^*C)|^2 \leq \frac{1}{4} \cdot \frac{|\delta - \Delta|^2}{\operatorname{Re}\left(\Delta\bar{\delta}\right)} |\operatorname{tr}(PB^*C)|^2 . \tag{40}$$

If we multiply the inequalities (39) and (40) we get

$$\begin{aligned} &\left[\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|C|^2\right) - |\operatorname{tr}(PC^*A)|^2\right] \\ &\times \left[\operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|C|^2\right) - |\operatorname{tr}(PB^*C)|^2\right] \\ &\leq \frac{1}{16} \cdot \frac{|\gamma - \Gamma|^2}{\operatorname{Re}(\Gamma\bar{\gamma})} \frac{|\delta - \Delta|^2}{\operatorname{Re}\left(\Delta\bar{\delta}\right)} |\operatorname{tr}(PC^*A)|^2 |\operatorname{tr}(PB^*C)|^2 . \end{aligned} \tag{41}$$

If we use (38) and (41) we get

$$\begin{aligned} &\left|\operatorname{tr}(PB^*A)\operatorname{tr}\left(P|C|^2\right) - \operatorname{tr}(PC^*A)\operatorname{tr}(PB^*C)\right|^2 \\ &\leq \frac{1}{16} \cdot \frac{|\gamma - \Gamma|^2}{\operatorname{Re}(\Gamma\bar{\gamma})} \frac{|\delta - \Delta|^2}{\operatorname{Re}\left(\Delta\bar{\delta}\right)} |\operatorname{tr}(PC^*A)|^2 |\operatorname{tr}(PB^*C)|^2 . \end{aligned} \tag{42}$$

Since $P$, $A$, $B$, $C \neq 0$ then by (39) and (40) we get $\operatorname{tr}(PC^*A) \neq 0$ and $\operatorname{tr}(PB^*C) \neq 0$. Now, if we take the square root in (42) and divide by $|\operatorname{tr}(PC^*A) \operatorname{tr}(PB^*C)|$ we obtain the desired result (37). □

**Corollary 4** *Let, either* $P \in \mathcal{B}_+(H)$, $A, B \in \mathcal{B}_2$ *or* $P \in \mathcal{B}_1^+(H)$, $A, B \in \mathcal{B}(H)$ *with* $P|A|^2$, $P|B|^2 \neq 0$ *and* $\lambda$, $\Gamma$, $\delta$, $\Delta \in \mathbb{C}$ *with* $\operatorname{Re}(\Gamma\bar{\gamma})$, $\operatorname{Re}\left(\Delta\bar{\delta}\right) > 0$. *If* $A$ *has the trace* $P$-$(\lambda, \Gamma)$*-property and* $B$ *has the trace* $P$-$(\delta, \Delta)$*-property, then*

$$\left|\frac{\operatorname{tr}(PB^*A)\operatorname{tr}(P)}{\operatorname{tr}(PA)\operatorname{tr}(PB^*)} - 1\right| \leq \frac{1}{4} \cdot \frac{|\gamma - \Gamma||\delta - \Delta|}{\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})\operatorname{Re}\left(\Delta\bar{\delta}\right)}}. \tag{43}$$

The case of selfadjoint operators is useful for applications.

**Remark 4** *Assume that* $A$, $B$, $C$ *are selfadjoint operators. If, either* $P \in \mathcal{B}_+(H)$, $A$, $B$, $C \in \mathcal{B}_2(H)$ *or* $P \in \mathcal{B}_1^+(H)$, $A$, $B$, $C \in \mathcal{B}(H)$ *with* $PA^2$, $PB^2$, $PC^2 \neq 0$ *and* $m$, $M$, $n$, $N \in \mathbb{R}$ *with* $mM$, $nN > 0$. *If* $(A, C)$ *has the trace* $P$-$(m, M)$*-property and* $(B, C)$ *has the trace* $P$-$(n, N)$*-property, then*

$$\left|\frac{\operatorname{tr}(PBA)\operatorname{tr}\left(PC^2\right)}{\operatorname{tr}(PCA)\operatorname{tr}(PBC)} - 1\right| \leq \frac{1}{4} \cdot \frac{(M - m)(N - n)}{\sqrt{mnMN}}. \tag{44}$$

*If* $A$ *has the trace* $P$-$(k,K)$*-property and* $B$ *has the trace* $P$-$(l,L)$*-property, then*

$$\left|\frac{\operatorname{tr}(PBA)\operatorname{tr}(P)}{\operatorname{tr}(PA)\operatorname{tr}(PB)}-1\right|\le\frac{1}{4}\cdot\frac{(K-k)(L-l)}{\sqrt{klKL}},\tag{45}$$

*where* $kK,\ lL>0$.

*We observe that, if* $0<k1_H\le A\le K1_H$ *and* $0<l1_H\le B\le L1_H$*, then by* (46) *we have*

$$|\operatorname{tr}(PBA)\operatorname{tr}(P)-\operatorname{tr}(PA)\operatorname{tr}(PB)|\le\frac{1}{4}\cdot\frac{(K-k)(L-l)}{\sqrt{klKL}}\operatorname{tr}(PA)\operatorname{tr}(PB)\tag{46}$$

*or, equivalently*

$$\left|\frac{\operatorname{tr}(PBA)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PB)}{\operatorname{tr}(P)}\right|\le\frac{1}{4}\cdot\frac{(K-k)(L-l)}{\sqrt{klKL}}\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PB)}{\operatorname{tr}(P)}.\tag{47}$$

## 5 Applications for convex functions

In the paper [30] we obtained amongst other the following reverse of the Jensen trace inequality:

$$\begin{aligned}
0&\le\frac{\operatorname{tr}(Pf(A))}{\operatorname{tr}(P)}-f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\\
&\le\frac{\operatorname{tr}(Pf'(A)A)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\cdot\frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)}\\
&\le\begin{cases}\frac{1}{2}[f'(M)-f'(m)]\,\dfrac{\operatorname{tr}\left(P\left|A-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}1_H\right|\right)}{\operatorname{tr}(P)}\\[2ex]
\frac{1}{2}(M-m)\,\dfrac{\operatorname{tr}\left(P\left|f'(A)-\frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)}1_H\right|\right)}{\operatorname{tr}(P)}\end{cases}\\
&\le\begin{cases}\frac{1}{2}[f'(M)-f'(m)]\left[\frac{\operatorname{tr}(PA^2)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^2\right]^{1/2}\\[2ex]
\frac{1}{2}(M-m)\left[\frac{\operatorname{tr}(P[f'(A)]^2)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)}\right)^2\right]^{1/2}\end{cases}\\
&\le\frac{1}{4}\left[f'(M)-f'(m)\right](M-m).
\end{aligned}\tag{48}$$

Let $\mathcal{M}_n(\mathbb{C})$ be the space of all square matrices of order $n$ with complex elements and $A\in\mathcal{M}_n(\mathbb{C})$ be a Hermitian matrix such that $\operatorname{Sp}(A)\subseteq[m,M]$

for some scalars $m$, $M$ with $m < M$. If $f$ is a continuously differentiable convex function on $[m, M]$, then by taking $P = I_n$ in (48) we get

$$
\begin{aligned}
0 &\leq \frac{\operatorname{tr}(f(A))}{n} - f\left(\frac{\operatorname{tr}(A)}{n}\right) \\
&\leq \frac{\operatorname{tr}(f'(A)A)}{n} - \frac{\operatorname{tr}(A)}{n} \cdot \frac{\operatorname{tr}(f'(A))}{n} \\
&\leq \begin{cases} \frac{1}{2}[f'(M) - f'(m)] \frac{\operatorname{tr}\left(\left|A - \frac{\operatorname{tr}(A)}{n} 1_H\right|\right)}{n} \\ \\ \frac{1}{2}(M - m) \frac{\operatorname{tr}\left(\left|f'(A) - \frac{\operatorname{tr}(f'(A))}{n} 1_H\right|\right)}{n} \end{cases} \\
&\leq \begin{cases} \frac{1}{2}[f'(M) - f'(m)] \left[\frac{\operatorname{tr}(A^2)}{n} - \left(\frac{\operatorname{tr}(A)}{n}\right)^2\right]^{1/2} \\ \\ \frac{1}{2}(M - m) \left[\frac{\operatorname{tr}([f'(A)]^2)}{n} - \left(\frac{\operatorname{tr}(f'(A))}{n}\right)^2\right]^{1/2} \end{cases} \\
&\leq \frac{1}{4}\left[f'(M) - f'(m)\right](M - m).
\end{aligned} \tag{49}
$$

The following reverse inequality also holds:

**Proposition 1** *Let* $A$ *be a selfadjoint operator on the Hilbert space* $H$ *and assume that* $\operatorname{Sp}(A) \subseteq [m, M]$ *for some scalars* $m$, $M$ *with* $0 < m < M$. *If* $f$ *is a continuously differentiable convex function on* $[m, M]$ *with* $f'(m) > 0$ *and* $P \in \mathcal{B}_1(H) \setminus \{0\}$, $P \geq 0$, *then we have*

$$
\begin{aligned}
0 &\leq \frac{\operatorname{tr}(Pf(A))}{\operatorname{tr}(P)} - f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) \\
&\leq \frac{\operatorname{tr}(Pf'(A)A)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \cdot \frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)} \\
&\leq \frac{1}{4} \cdot \frac{(M - m)[f'(M) - f'(m)]}{\sqrt{mMf'(m)f'(M)}} \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)}.
\end{aligned} \tag{50}
$$

The proof follows by the inequality (47) and the details are omitted.

Let $A \in \mathcal{M}_n(\mathbb{C})$ be a Hermitian matrix such that $\operatorname{Sp}(A) \subseteq [m, M]$ for some scalars $m$, $M$ with $m < M$. If $f$ is a continuously differentiable convex function on $[m, M]$ with $f'(m) > 0$ then by taking $P = I_n$ in (50) we get

$$
\begin{aligned}
0 &\leq \frac{\operatorname{tr}(f(A))}{n} - f\left(\frac{\operatorname{tr}(A)}{n}\right) \\
&\leq \frac{\operatorname{tr}(f'(A)A)}{n} - \frac{\operatorname{tr}(A)}{n} \cdot \frac{\operatorname{tr}(f'(A))}{n} \\
&\leq \frac{1}{4} \cdot \frac{(M-m)[f'(M) - f'(m)]}{\sqrt{mMf'(m)f'(M)}} \frac{\operatorname{tr}(A)}{n} \frac{\operatorname{tr}(f'(A))}{n}.
\end{aligned}
\tag{51}
$$

We consider the power function $f : (0, \infty) \to (0, \infty)$, $f(t) = t^r$ with $t \in \mathbb{R} \backslash \{0\}$. For $r \in (-\infty, 0) \cup [1, \infty)$, $f$ is convex while for $r \in (0, 1)$, $f$ is concave.

Let $r \geq 1$ and $A$ be a selfadjoint operator on the Hilbert space $H$ and assume that $\operatorname{Sp}(A) \subseteq [m, M]$ for some scalars $m$, $M$ with $0 < m < M$. If $P \in \mathcal{B}_1^+(H) \setminus \{0\}$, then

$$
\begin{aligned}
0 &\leq \frac{\operatorname{tr}(PA^r)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^r \\
&\leq r\left[\frac{\operatorname{tr}(PA^r)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \cdot \frac{\operatorname{tr}\left(PA^{r-1}\right)}{\operatorname{tr}(P)}\right] \\
&\leq \frac{1}{4} r \frac{(M-m)\left(M^{r-1} - m^{r-1}\right)}{m^{r/2} M^{r/2}} \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \frac{\operatorname{tr}\left(PA^{r-1}\right)}{\operatorname{tr}(P)}.
\end{aligned}
\tag{52}
$$

If we take the first and last term in (52) we get the inequality:

$$
\begin{aligned}
0 &\leq \frac{\operatorname{tr}(P)\operatorname{tr}(PA^r)}{\operatorname{tr}(PA)\operatorname{tr}(PA^{r-1})} - \frac{\operatorname{tr}(P)[\operatorname{tr}(PA)]^{r-1}}{\operatorname{tr}(PA^{p-1})[\operatorname{tr}(P)]^{r-1}} \\
&\leq \frac{1}{4} r \frac{(M-m)\left(M^{r-1} - m^{r-1}\right)}{m^{r/2} M^{r/2}}.
\end{aligned}
\tag{53}
$$

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