Acta Univ. Sapientiae, Mathematica, **9**, 1 (2017) 94–121

DOI: 10.1515/ausm-2017-0007

Split equality monotone variational inclusions and fixed point problem of set-valued operator

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Abstract. In this paper, we are concerned with the split equality problem of finding an element in the zero point set of the sum of two monotone operators and in the common fixed point set of a finite family of quasi- nonexpansive set-valued mappings. Strong convergence theorems are established under suitable condition in an infinite dimensional Hilbert spaces. Some applications of the main results are also provided.

1 Introduction

Let C and Q be nonempty closed convex subsets of real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. The split feasibility problem (SFP) was recently introduced by Censor and Elfving [1] and is formulated as

$$\text{to finding} \quad x^* \in C \quad \text{such that} \quad \mathcal{A}x^* \in Q, \tag{1}$$

where $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator. Such models were successfully developed for instance in radiation therapy treatment planning, sensor

2010 Mathematics Subject Classification: 47J25, 47N10, 65J15, 90C25
Key words and phrases: split equality problem, monotone variational inclusions, quasi-nonexpansive, set-valued mappings

networks, resolution enhancement and so on [2, 3, 4]. Initiated by SFP, several split type problems have been investigated and studied, for example, the split common fixed point problem (SCFP) [5], the split variational inequality problem (SVIP) [6], and the split null point problem (SCNP) [7]. Many authors have studied the SFP in infinite-dimensional Hilbert spaces, see, for example, [8-13] and some of the references therein.

Many nonlinear problems arising in applied areas such as image recovery, signal processing, and machine learning are mathematically modeled as a nonlinear operator equation and this operator is decomposed as the sum of two nonlinear operators, see [14-17]. The central problem is to iteratively find a zero point of the sum of two monotone operators, that is, $0 \in (A+B)(x)$. Many real world problems can be formulated as a problem of the above form. For instance, a stationary solution to the initial value problem of the evolution equation

$$\begin{cases} 0 \in Fu + \frac{\partial u}{\partial t}, \\ u_0 = u(0), \end{cases} \tag{2}$$

can be recast as the inclusion problem when the governing maximal monotone F is of the form $F = A+B$; for more details, see [14] and the references therein.

Let $F : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ and $G : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ be set-valued mappings with nonempty values, and let $f : \mathcal{H}_1 \to \mathcal{H}_1$ and $g : \mathcal{H}_2 \to \mathcal{H}_2$ be mappings. Then, inspired by the work in [6], Moudafi [18] introduced the following split monotone variational inclusion problem (SMVIP):

$$\begin{cases} \text{find} \quad x^* \in \mathcal{H}_1 \quad \text{such that} \quad 0 \in f(x^*) + F(x^*), \\ \text{and such that} \quad y^* = \mathcal{A}x^* \in \mathcal{H}_1 \quad \text{solves} \quad g(y^*) + G(y^*). \end{cases} \tag{3}$$

Moudafi [18], present an algorithm for solving the SMVIP and obtain a weak convergence theorem for the algorithm.

Very recently, Moudafi [19] introduced the following split equality problem. Let $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ be real Hilbert spaces. Let $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_3$, $\mathcal{B} : \mathcal{H}_2 \to \mathcal{H}_3$ be two bounded linear operators, let C and Q be nonempty closed convex subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$. The split equality problem (SEP) is to find

$$x \in C, \quad y \in Q \qquad \text{such that} \quad \mathcal{A}x = \mathcal{B}y, \tag{4}$$

Obviously, if $\mathcal{B} = I$ and $\mathcal{H}_2 = \mathcal{H}_3$ then (SEP) reduces to (SFP). This kind of split equality problem allows asymmetric and partial relations between the variables x and y. The interest is to cover many situations, such as decomposition methods for PDEs, applications in game theory, and intensity-modulated radiation therapy, (see [20, 21]).

Each nonempty closed convex subset of a Hilbert space can be regarded as a set of fixed points of a projection. In [22], Moudafi introduced the following split equality fixed point problem:

Let $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_3$, $\mathcal{B} : \mathcal{H}_2 \to \mathcal{H}_3$ be two bounded linear operators, let $S : \mathcal{H}_1 \to \mathcal{H}_1$ and $T : \mathcal{H}_2 \to \mathcal{H}_2$ be two nonlinear operators such that $\mathrm{Fix}(S) \neq \emptyset$ and $\mathrm{Fix}(T) \neq \emptyset$. The split equality fixed point problem (SEFP) is to find

$$x \in \mathrm{Fix}(S), \quad y \in \mathrm{Fix}(T) \qquad \text{such that} \quad \mathcal{A}x = \mathcal{B}y. \tag{5}$$

Moudafi [22], proposed some algorithms for solving the split equality fixed point problem. In these algorithms we need to compute norm of the operators, which is difficult. To solve the split equality fixed point problem for quasi-nonexpansive mappings, Zhao [23] proposed the following iteration algorithm which does not require any knowledge of the operator norms:

Theorem 1 *Let $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$, be real Hilbert spaces, $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_3$ and $\mathcal{B} : \mathcal{H}_2 \to \mathcal{H}_3$ be bounded linear operators. Let $S : \mathcal{H}_1 \to \mathcal{H}_1$ and $T : \mathcal{H}_2 \to \mathcal{H}_2$ be quasi-nonexpansive mappings such that $S - I$ and $T - I$ are demiclosed at 0. Suppose $\Omega = \{x \in \mathrm{Fix}(S), y \in \mathrm{Fix}(T) : \mathcal{A}x = \mathcal{B}y\} \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by $x_0 \in \mathcal{H}_1, \quad y_0 \in \mathcal{H}_2$ and by*

$$\begin{cases} u_n = x_n - \gamma_n \mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n) \\ x_{n+1} = \beta_n u_n + (1 - \beta_n)\, S(u_n), \\ w_n = y_n + \gamma_n \mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n) \\ y_{n+1} = \beta_n w_n + (1 - \beta_n)\, T(w_n), \qquad \forall n \geq 0. \end{cases} \tag{6}$$

Assume that the step-size γ_n is chosen in such a way that

$$\gamma_n \in (\epsilon, \frac{2\|\mathcal{A}x_n - \mathcal{B}y_n\|^2}{\|\mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2 + \|\mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2} - \epsilon), n \in \Pi$$

otherwise $\gamma_n = \gamma$ (γ being any nonnegative value), where the index set $\Pi = \{n : \mathcal{A}x_n - \mathcal{B}y_n \neq 0\}$. Let $\{\alpha_n\} \subset (\delta, 1-\delta)$ and $\{\beta_n\} \subset (\eta, 1-\eta)$ for small enough $\delta, \eta > 0$. Then, the sequences $\{(x_n, y_n)\}$ converges weakly to $(x^\star, y^\star) \in \Omega$.

On the other hand, in the last years, many authors studied the problems of finding a common element of the set of zero point of the sum of two monotone operators and the set of fixed points of nonlinear operators, see [24, 25]. The motivation for studying such a problem is in its possible application to mathematical models whose constraints can be expressed as fixed-point problems and/or variational inclusion problem: see, for instance, [26, 27].

Now, we consider the following split equality monotone variational inclusions and fixed point problem:

Let $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$, be real Hilbert spaces, $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_3$ and $\mathcal{B} : \mathcal{H}_2 \to \mathcal{H}_3$ be bounded linear operators. Let $F : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ and $G : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ be set-valued mappings with nonempty values, and let $f : \mathcal{H}_1 \to \mathcal{H}_1$ and $g : \mathcal{H}_2 \to \mathcal{H}_2$ be mappings. Let for $i \in \{1, 2, ..., m\}$, $T_i : \mathcal{H}_1 \to CB(\mathcal{H}_1)$ and $S_i : \mathcal{H}_2 \to CB(\mathcal{H}_2)$ be two finite family of set valued mappings. We find a point

$$
\begin{aligned}
&x \in \bigcap_{i=1}^{m} Fix(T_i) \bigcap (f + F)^{-1}(0), \quad \text{and} \\
&y \in \bigcap_{i=1}^{m} Fix(S_i) \bigcap (g + G)^{-1}(0) \quad \text{such that} \quad \mathcal{A}x = \mathcal{B}y.
\end{aligned}
$$

Motivated by the above works, the purpose of this paper is to introduce a new algorithm for the split equality problem for finding an element in the zero point set of the sum of two operators which are inverse-strongly monotone and a maximal monotone and in the common fixed point set of a finite family of quasi-nonexpansive set-valued mappings. Under suitable conditions, we prove that the sequences generated by the proposed new algorithm converges strongly to a solution of the split equality problem in Hilbert spaces. Our results improve and generalize the result of Takahashi et al. [11], Moudafi [18, 22], Censor et al. [6], Zhao [23], and many others.

2 Preliminaries

A subset $E \subset \mathcal{H}$ is called proximal if for each $x \in \mathcal{H}$, there exists an element $y \in E$ such that

$$\| x - y \| = dist(x, E) = \inf\{\| x - z \| : z \in E\}.$$

We denote by $CB(E), CC(E), K(E)$ and $P(E)$ the collection of all nonempty closed bounded subsets, nonempty closed convex subsets, nonempty compact subsets, and nonempty proximal bounded subsets of E respectively. The Hausdorff metric $\mathfrak{h}$ on $CB(\mathcal{H})$ is defined by

$$\mathfrak{h}(A, B) := \max\{\sup_{x \in A} dist(x, B), \sup_{y \in B} dist(y, A)\},$$

for all $A, B \in CB(\mathcal{H})$.

Let $T : \mathcal{H} \to 2^{\mathcal{H}}$ be a set-valued mapping. An element $x \in \mathcal{H}$ is said to be a fixed point of T, if $x \in Tx$. We use $\mathrm{Fix}(T)$ to denote the set of all fixed points of T. An element $x \in \mathcal{H}$ is said to be an endpoint of a set-valued mapping T if x is a fixed point of T and $T(x) = \{x\}$. We say that T satisfies the endpoint condition if each fixed point of T is an endpoint of T. We also say that a family of set-valued mapping $T_i, (i = 1, 2, ..., m)$ satisfies the common endpoint condition if $T_i(x) = \{x\}$ for all $x \in \bigcap_{i=1}^{m} \mathrm{Fix}(T_i)$.

Definition 1 *A set-valued mapping* $T : \mathcal{H} \to CB(\mathcal{H})$ *is called*

(i) *nonexpansive if*

$$\mathfrak{h}(Tx, Ty) \le \|x - y\|, \quad x, y \in \mathcal{H}.$$

(ii) *quasi-nonexpansive if* $\mathrm{Fix}(T) \neq \emptyset$ *and* $\mathfrak{h}(Tx, Tp) \le \|x - p\|$ *for all* $x \in \mathcal{H}$ *and all* $p \in \mathrm{Fix}(T)$.

(iii) *generalized nonexpansive* [28] *if*

$$\mathfrak{h}(Tx, Ty) \le \mu\, \mathrm{dist}(x, Tx) + \|x - y\|, \quad x, y \in \mathcal{H},$$

for some $\mu > 0$.

It is obvious that every generalized nonexpansive set- valued mapping with nonempty fixed point set $\mathrm{Fix}(T)$ is quasi-nonexpansive.

We use the following notion in the sequel:

- $\rightharpoonup$ for weak convergence and $\to$ for strong convergence.

Definition 2 *Let* E *be a nonempty subset of a real Hilbert space* $\mathcal{H}$ *and let* $T : E \to CB(E)$ *be a set-valued mapping. The mapping* $I - T$ *is said to be demiclosed at zero if for any sequence* $\{x_n\}$ *in* E*, the conditions* $x_n \rightharpoonup x^*$ *and* $\lim_{n\to\infty} \mathrm{dist}(x_n, Tx_n) = 0$*, imply* $x^* \in \mathrm{Fix}(T)$.

The proof of the following result is similar to the proof of Theorem 3.4 in [29], and so is not included.

Lemma 1 *Let* E *be a nonempty closed convex subset of a real Hilbert space* $\mathcal{H}$*. Let* $T : E \to K(E)$ *be a generalized nonexpansive set- valued mapping. Then* $I - T$ *is demiclosed in zero.*

Lemma 2 [30] *Let* E *be a closed convex subset of a real Hilbert space* $\mathcal{H}$*. Let* $T : E \to CB(E)$ *be a quasi-nonexpansive set-valued mapping satisfies the endpoint condition. Then* $\mathrm{Fix}(T)$ *is closed and convex.*

Given a nonempty closed convex set $C \subset \mathcal{H}$, the mapping that assigns every point $x \in \mathcal{H}$, to its unique nearest point in C is called the metric projection onto C and is denoted by P_C; i.e., $P_C \in C$ and $\|x - P_C x\| = \inf_{y \in C} \|x - y\|$. The metric projection P_C is characterized by the fact that $P_C(x) \in C$ and

$$\langle y - P_C(x), x - P_C(x) \rangle \leq 0, \qquad \forall x \in \mathcal{H}, y \in C.$$

The metric projection, P_C, satisfies the nonexpansivity condition with $\mathrm{Fix}(P_C) = C$.

Let $f : \mathcal{H} \to \mathcal{H}$ be a nonlinear operator. It is well known that the Variational Inequality Problem is to find $u \in E$ such that

$$\langle fu, v - u \rangle \geq 0, \qquad \forall v \in E. \tag{7}$$

We denote by $VI(E, f)$ the solution set of (7). The operator $f : \mathcal{H} \to \mathcal{H}$ is called Inverse strongly monotone with constant $\beta > 0$, $(\beta - ism)$ if

$$\langle f(x) - f(y), x - y \rangle \geq \beta \|f(x) - f(y)\|^2, \quad \forall x, y \in E.$$

It is known that if f is β- inverse strongly monotone, and $\lambda \in (0, 2\beta)$ then $P_E(I - \lambda f)$ is nonexpansive, where P_E is the metric projection onto E.

Let F be a mapping of $\mathcal{H}$ into $2^{\mathcal{H}}$. The effective domain of F is denoted by $dom(F)$, that is, $dom(F) = \{x \in \mathcal{H} : Fx \neq \emptyset\}$. A multi-valued mapping F is said to be a monotone operator on $\mathcal{H}$ if $\langle u - v, x - y \rangle \geq 0$, for all $x, y \in dom(F)$, $u \in Fx$ and $v \in Fy$. Classical examples of monotone operators are subdifferential operators of functions that are convex, lower semicontinuous, and proper; linear operators with a positive symmetric part. See, e.g. [31, 32]. A monotone operator F on $\mathcal{H}$ is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on $\mathcal{H}$. For a maximal monotone operator F on $\mathcal{H}$ and $r > 0$, the resolvent of F for r is $J_r^F = (I + rF)^{-1} : \mathcal{H} \to dom(F)$. This operator enjoys many important properties that make it a central tool in monotone operator theory and its applications. In particular, it is single-valued, firmly nonexpansive in the sense that

$$\|J_r^F x - J_r^F y\|^2 \leq \langle x - y, J_r^F x - J_r^F y \rangle, \qquad \forall x, y \in \mathcal{H}.$$

Finally, the set $\mathrm{Fix}(J_r^F) = \{x \in \mathcal{H} : J_r^F x = x\}$ of fixed points of J_r^F coincides with $F^{-1}(0)$.

Lemma 3 [33] *For each* $x_1, \cdots, x_m \in \mathcal{H}$ *and* $\alpha_1, \cdots, \alpha_m \in [0, 1]$ *with* $\sum_{i=1}^m \alpha_i = 1$ *the equality*

$$\|\alpha_1 x_1 + ... + \alpha_m x_m\|^2 = \sum_{i=1}^m \alpha_i \|x_i\|^2 - \sum_{1 \leq i < j \leq m} \alpha_i \alpha_j \|x_i - x_j\|^2,$$

holds.

Lemma 4 [34] *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\vartheta_n)a_n + \vartheta_n\delta_n, \qquad n \geq 0,$$

where $\{\vartheta_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty} \vartheta_n = \infty$,

(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\vartheta_n\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 5 [35] *Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $(\Gamma_{n_j})_{j\geq 0}$ of (Γ_n) such that $\Gamma_{n_j} < \Gamma_{n_j+1}$ for all $j \geq 0$. Also consider the sequence of integers $(\tau(n))_{n\geq n_0}$ defined by*

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\}.$$

Then $(\tau(n))_{n\geq n_0}$ is a nondecreasing sequence verifying $\lim_{n\to\infty} \tau(n) = \infty$, and, for all $n \geq n_0$, the following two estimates hold:

$$\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}, \qquad \Gamma_n \leq \Gamma_{\tau(n)+1}.$$

3 Algorithm and convergence theorem

The main result of this paper is the following.

Theorem 2 *Let $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$, be real Hilbert spaces, $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_3$ and $\mathcal{B} : \mathcal{H}_2 \to \mathcal{H}_3$ be bounded linear operators. Let $f : \mathcal{H}_1 \to \mathcal{H}_1$ and $g : \mathcal{H}_2 \to \mathcal{H}_2$ be respectively α and β- inverse strongly monotone operators and F, G two maximal monotone operators on $\mathcal{H}_1, \mathcal{H}_2$. Let for $i \in \{1, 2, ..., m\}$, $T_i : \mathcal{H}_1 \to CB(\mathcal{H}_1)$ and $S_i : \mathcal{H}_2 \to CB(\mathcal{H}_2)$ be two finite families of quasi-nonexpansive set valued mappings such that $S_i - I$ and $T_i - I$ are demiclosed at 0, and S_i and T_i satisfies the common endpoint condition. Suppose $\Omega = \{x \in \bigcap_{i=1}^m Fix(T_i) \bigcap (f+F)^{-1}(0), \quad y \in \bigcap_{i=1}^m Fix(S_i) \bigcap (g+G)^{-1}(0) : \mathcal{A}x = \mathcal{B}y\} \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by $x_0, \vartheta \in \mathcal{H}_1, \quad y_0, \zeta \in \mathcal{H}_2$ and by*

$$\begin{cases} z_n = x_n - \gamma_n \mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n) \\ u_n = J^F_{\lambda_n}(I - \lambda_n f)z_n, \\ x_{n+1} = \alpha_n \vartheta + \beta_n u_n + \sum_{i=1}^m \delta_{n,i} v_{n,i} \\ w_n = y_n + \gamma_n \mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n) \\ t_n = J^G_{\mu_n}(I - \mu_n g)w_n, \\ y_{n+1} = \alpha_n \zeta + \beta_n t_n + \sum_{i=1}^m \delta_{n,i} s_{n,i} \qquad \forall n \geq 0, \end{cases} \tag{8}$$

where $v_{n,i} \in T_i u_n$, $s_{n,i} \in S_i t_n$ *and the step-size* γ_n *is chosen in such a way that*

$$\gamma_n \in (\epsilon, \frac{2\|\mathcal{A}x_n - \mathcal{B}y_n\|^2}{\|\mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2 + \|\mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2} - \epsilon), n \in \Pi$$

otherwise $\gamma_n = \gamma$ *(*γ *being any nonnegative value), where the index set* $\Pi = \{n : \mathcal{A}x_n - \mathcal{B}y_n \neq 0\}$. *Let the sequences* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_{n,i}\}$, $\{\lambda_n\}$ *and* $\{\mu_n\}$ *satisfy the following conditions:*

(i) $\alpha_n + \beta_n + \sum_{i=1}^{m} \delta_{n,i} = 1$, *and* $\liminf_n \beta_n \delta_{n,i} > 0$ *for each* $i \in \{1, 2, ..., m\}$,

(ii) $\{\lambda_n\} \subset [a, b] \subset (0, 2\alpha)$ *and* $\{\mu_n\} \subset [c, d] \subset (0, 2\beta)$,

(iii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then, the sequences $\{(x_n, y_n)\}$ *converges strongly to* $(x^\star, y^\star) \in \Omega$.

Proof. Firstly, we prove that $\{x_n\}$ and $\{y_n\}$ are bounded. Take $(x^\star, y^\star) \in \Omega$. It is obvious that $J^F_{\lambda_n}(x^\star - \lambda_n f x^\star) = x^\star$. Since the operator $J^F_{\lambda_n}$ is nonexpansive and f is $\alpha-$ inverse strongly monotone we have

$$\begin{aligned}
\|u_n - x^\star\|^2 &= \|J^F_{\lambda_n}(z_n - \lambda_n f z_n) - J^F_{\lambda_n}(x^\star - \lambda_n f x^\star)\|^2 \\
&\leq \|(z_n - \lambda_n f z_n) - (x^\star - \lambda_n f x^\star)\|^2 \\
&= \|(z_n - x^\star) - \lambda_n(f z_n - f x^\star)\|^2 \\
&= \|z_n - x^\star\|^2 - 2\lambda_n\langle z_n - x^\star, f z_n - f x^\star\rangle + \lambda_n^2\|f z_n - f x^\star\|^2 \\
&\leq \|z_n - x^\star\|^2 - 2\lambda_n\alpha\|f z_n - f x^\star\|^2 + \lambda_n^2\|f z_n - f x^\star\|^2 \\
&= \|z_n - x^\star\|^2 + \lambda_n(\lambda_n - 2\alpha)\|f z_n - f x^\star\|^2.
\end{aligned} \tag{9}$$

Similarly, we obtain that

$$\|t_n - y^\star\|^2 \leq \|w_n - y^\star\|^2 + \mu_n(\mu_n - 2\beta)\|g w_n - g y^\star\|^2. \tag{10}$$

By Lemma 3 and inequality (9), we have

$$\begin{aligned}
\|x_{n+1}-x^\star\|^2 &= \|\alpha_n\vartheta+\beta_n u_n+\sum_{i=1}^{m}\delta_{n,i}v_{n,i}-x^\star\|^2\\
&\leq \alpha_n\|\vartheta-x^\star\|^2+\beta_n\|u_n-x^\star\|^2\\
&\quad+\sum_{i=1}^{m}\delta_{n,i}\|v_{n,i}-x^\star\|^2-\sum_{i=1}^{m}\beta_n\delta_{n,i}\|v_{n,i}-u_n\|^2\\
&= \alpha_n\|\vartheta-x^\star\|^2+\beta_n\|u_n-x^\star\|^2\\
&\quad+\sum_{i=1}^{m}\delta_{n,i}\mathrm{dist}(v_{n,i},T_i x^\star)^2-\sum_{i=1}^{m}\beta_n\delta_{n,i}\|v_{n,i}-u_n\|^2\\
&\leq \alpha_n\|\vartheta-x^\star\|^2+\beta_n\|u_n-x^\star\|^2\\
&\quad+\sum_{i=1}^{m}\delta_{n,i}\mathfrak{h}(T_i u_n,T_i x^\star)^2-\sum_{i=1}^{m}\beta_n\delta_{n,i}\|v_{n,i}-u_n\|^2\\
&\leq \alpha_n\|\vartheta-x^\star\|^2+\beta_n\|u_n-x^\star\|^2\\
&\quad+\sum_{i=1}^{m}\delta_{n,i}\|u_n-x^\star\|^2-\sum_{i=1}^{m}\beta_n\delta_{n,i}\|v_{n,i}-u_n\|^2\\
&\leq \alpha_n\|\vartheta-x^\star\|^2+(1-\alpha_n)\|z_n-x^\star\|^2\\
&\quad-\sum_{i=1}^{m}\beta_n\delta_{n,i}\|v_{n,i}-u_n\|^2\\
&(1-\alpha_n)\lambda_n(\lambda_n-2\alpha)\|fz_n-fx^\star\|^2.
\end{aligned} \tag{11}$$

Similarly, from inequality (10) we have

$$\begin{aligned}
\|y_{n+1}-y^\star\|^2 &= \|\alpha_n\zeta+\beta_n t_n+\sum_{i=1}^{m}\delta_{n,i}s_{n,i}-y^\star\|^2\\
&\leq \alpha_n\|\zeta-y^\star\|^2+\beta_n\|t_n-y^\star\|^2\\
&\quad+\sum_{i=1}^{m}\delta_{n,i}\|s_{n,i}-y^\star\|^2-\sum_{i=1}^{m}\beta_n\delta_{n,i}\|s_{n,i}-t_n\|^2\\
&\leq \alpha_n\|\zeta-y^\star\|^2+(1-\alpha_n)\|w_n-y^\star\|^2\\
&\quad-\sum_{i=1}^{m}\beta_n\delta_{n,i}\|s_{n,i}-t_n\|^2\\
&\quad+(1-\alpha_n)\mu_n(\mu_n-2\beta)\|gw_n-gy^\star\|^2.
\end{aligned} \tag{12}$$

From algorithm (8) we have that

$$\begin{aligned}
\|z_n - x^\star\|^2 &= \|x_n - \gamma_n \mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n) - x^\star\|^2 \\
&= \|x_n - x^\star\|^2 + \gamma_n^2 \|\mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2 \\
&\quad - 2\gamma_n \langle x_n - x^\star, \mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n)\rangle \\
&= \|x_n - x^\star\|^2 + \gamma_n^2 \|\mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2 \\
&\quad - 2\gamma_n \langle \mathcal{A}x_n - \mathcal{A}x^\star, (\mathcal{A}x_n - \mathcal{B}y_n)\rangle \\
&= \|x_n - x^\star\|^2 + \gamma_n^2 \|\mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2 - \gamma_n \|\mathcal{A}x_n - \mathcal{A}x^\star\|^2 \\
&\quad - \gamma_n \|\mathcal{A}x_n - \mathcal{B}y_n\|^2 + \gamma_n \|\mathcal{B}y_n - \mathcal{A}x^\star\|^2.
\end{aligned} \tag{13}$$

By similar way we obtain that

$$\begin{aligned}
\|w_n - y^\star\|^2 &= \|y_n + \gamma_n \mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n) - y^\star\|^2 \\
&= \|y_n - y^\star\|^2 + \gamma_n^2 \|\mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2 - \gamma_n \|\mathcal{B}y_n - \mathcal{B}y^\star\|^2 \\
&\quad - \gamma_n \|\mathcal{A}x_n - \mathcal{B}y_n\|^2 + \gamma_n \|\mathcal{A}x_n - \mathcal{B}y^\star\|^2.
\end{aligned} \tag{14}$$

By adding the two last inequalities and by taking into account the fact that $\mathcal{A}x^\star = \mathcal{B}y^\star$ we obtain

$$\begin{aligned}
\|z_n - x^\star\|^2 + \|w_n - y^\star\|^2 &= \|x_n - x^\star\|^2 + \|y_n - y^\star\|^2 \\
&\quad - \gamma_n [2\|\mathcal{A}x_n - \mathcal{B}y_n\|^2 \\
&\quad - \gamma_n (\|\mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2 \\
&\quad + \|\mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2)] \\
&\leq \|x_n - x^\star\|^2 + \|y_n - y^\star\|^2.
\end{aligned} \tag{15}$$

This implies that

$$\begin{aligned}
&\|x_{n+1} - x^\star\|^2 + \|y_{n+1} - y^\star\|^2 \leq (1 - \alpha_n)(\|z_n - x^\star\|^2 \\
&\qquad + \|w_n - y^\star\|^2) + \alpha_n (\|\vartheta - x^\star\|^2 + \|\zeta - y^\star\|^2) \\
&\quad \leq (1 - \alpha_n)(\|x_n - x^\star\|^2 + \|y_n - y^\star\|^2) + \alpha_n (\|\vartheta - x^\star\|^2 + \|\zeta - y^\star\|^2) \\
&\quad \leq \max\{\|x_n - x^\star\|^2 + \|y_n - y^\star\|^2, \|\vartheta - x^\star\|^2 + \|\zeta - y^\star\|^2\} \\
&\quad \vdots \\
&\quad \leq \max\{\|x_0 - x^\star\|^2 + \|y_0 - y^\star\|^2, \|\vartheta - x^\star\|^2 + \|\zeta - y^\star\|^2\}.
\end{aligned} \tag{16}$$

Thus $\|x_{n+1} - x^\star\|^2 + \|y_{n+1} - y^\star\|^2$ is bounded. Therefore $\{x_n\}$ and $\{y_n\}$ are bounded. Consequently $\{z_n\}, \{w_n\}, \{u_n\}$ and $\{v_n\}$ are all bounded. From (11),

(12) and (15) we have that

$$\begin{aligned}
&\|x_{n+1}-x^\star\|^2+\|y_{n+1}-y^\star\|^2 \\
&\quad \le (1-\alpha_n)(\|z_n-x^\star\|^2+\|w_n-y^\star\|^2)+\alpha_n(\|\vartheta-x^\star\|^2+\|\zeta-y^\star\|^2) \\
&\qquad -\sum_{i=1}^{m}\beta_n\delta_{n,i}\|v_{n,i}-u_n\|^2-\sum_{i=1}^{m}\beta_n\delta_{n,i}\|s_{n,i}-t_n\|^2 \\
&\qquad -(1-\alpha_n)\lambda_n(2\alpha-\lambda_n)\|fz_n-fx^\star\|^2 \\
&\qquad -(1-\alpha_n)\mu_n(2\beta-\mu_n)\|gw_n-gy^\star\|^2 \\
&\quad \le (1-\alpha_n)(\|x_n-x^\star\|^2+\|y_n-y^\star\|^2)+\alpha_n(\|\vartheta-x^\star\|^2+\|\zeta-y^\star\|^2) \qquad (17)\\
&\qquad -(1-\alpha_n)\gamma_n[2\|\mathcal{A}x_n-\mathcal{B}y_n\|^2-\gamma_n(\|\mathcal{B}^*(\mathcal{A}x_n-\mathcal{B}y_n)\|^2 \\
&\qquad +\|\mathcal{A}^*(\mathcal{A}x_n-\mathcal{B}y_n)\|^2)] \\
&\qquad -\sum_{i=1}^{m}\beta_n\delta_{n,i}\|v_{n,i}-u_n\|^2-\sum_{i=1}^{m}\beta_n\delta_{n,i}\|s_{n,i}-t_n\|^2 \\
&\qquad -(1-\alpha_n)\lambda_n(2\alpha-\lambda_n)\|fz_n-fx^\star\|^2 \\
&\qquad -(1-\alpha_n)\mu_n(2\beta-\mu_n)\|gw_n-gy^\star\|^2.
\end{aligned}$$

From above inequality we have that

$$\begin{aligned}
(1-\alpha_n)\lambda_n(2\alpha-\lambda_n)\|fz_n-fx^\star\|^2 &\le (1-\alpha_n)(\|x_n-x^\star\|^2+\|y_n-y^\star\|^2) \\
&\quad -\|x_{n+1}-x^\star\|^2-\|y_{n+1}-y^\star\|^2 \qquad (18)\\
&\quad +\alpha_n(\|\vartheta-x^\star\|^2+\|\zeta-y^\star\|^2).
\end{aligned}$$

By our assumption that

$$\gamma_n\in\left(\epsilon,\frac{2\|\mathcal{A}x_n-\mathcal{B}y_n\|^2}{\|\mathcal{B}^*(\mathcal{A}x_n-\mathcal{B}y_n)\|^2+\|\mathcal{A}^*(\mathcal{A}x_n-\mathcal{B}y_n)\|^2}-\epsilon\right),$$

we have that

$$(\gamma_n+\epsilon)\|\mathcal{B}^*(\mathcal{A}x_n-\mathcal{B}y_n)\|^2+\|\mathcal{A}^*(\mathcal{A}x_n-\mathcal{B}y_n)\|^2\le 2\|\mathcal{A}x_n-\mathcal{B}y_n\|^2.$$

From above inequality and inequality (17) we have that

$$\begin{aligned}
&(1-\alpha_n)\gamma_n\epsilon(\|\mathcal{B}^*(\mathcal{A}x_n-\mathcal{B}y_n)\|^2+\|\mathcal{A}^*(\mathcal{A}x_n-\mathcal{B}y_n)\|^2) \\
&\quad \le (1-\alpha_n)\gamma_n[2\|\mathcal{A}x_n-\mathcal{B}y_n\|^2-\gamma_n(\|\mathcal{B}^*(\mathcal{A}x_n-\mathcal{B}y_n)\|^2 \\
&\qquad +\|\mathcal{A}^*(\mathcal{A}x_n-\mathcal{B}y_n)\|^2)] \qquad (19)\\
&\quad \le (1-\alpha_n)(\|x_n-x^\star\|^2+\|y_n-y^\star\|^2)-\|x_{n+1}-x^\star\|^2-\|y_{n+1} \\
&\qquad -y^\star\|^2+\alpha_n(\|\vartheta-x^\star\|^2+\|\zeta-y^\star\|^2).
\end{aligned}$$

Put $\Gamma_n = \|x_n - x^\star\|^2 + \|y_n - y^\star\|^2$ for all $n \in \mathbb{N}$. We finally analyze the inequalities (18) and (19) by considering the following two cases.

Case A. Suppose that $\Gamma_{n+1} \le \Gamma_n$ for all $n \ge n_0$ (for n_0 large enough). In this case, since Γ_n is bounded, the limit $\lim_{n\to\infty} \Gamma_n$ exists. Since $\lim_{n\to\infty} \alpha_n = 0$, from (19) and by our assumption that on $\{\gamma_n\}$ we have

$$\lim_{n\to\infty} (\|\mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2 + \|\mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2) = 0.$$

So we obtain that $\lim_{n\to\infty} \|\mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n)\| = 0$ and $\lim_{n\to\infty} \|\mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n)\| = 0$. This implies that $\lim_{n\to\infty} \|\mathcal{A}x_n - \mathcal{B}y_n\| = 0$. Also from (18) we deduce

$$\lim_{n\to\infty} (1-\alpha_n)\lambda_n(2\alpha - \lambda_n)\|fz_n - fx^\star\|^2 = 0$$

By our assumption that $\{\lambda_n\} \subset [a, b] \subset (0, 2\alpha)$, we obtain that

$$\lim_{n\to\infty} \|fz_n - fx^\star\| = 0. \tag{20}$$

By similar argument, from inequality (17) we get that

$$\lim_{n\to\infty} \|gw_n - gy^\star\| = \lim_{n\to\infty} \|v_{n,i} - u_n\| = \lim_{n\to\infty} \|s_{n,i} - t_n\| = 0, \tag{21}$$

Since $\mathtt{dist}(u_n, T_i u_n) \le \|v_{n,i} - u_n\|$ we have

$$\lim_{n\to\infty} \mathtt{dist}(u_n, T_i u_n) = 0, \qquad i \in \{1, 2, \dots, m\}. \tag{22}$$

Similarly, from (21) we arrive at

$$\lim_{n\to\infty} \mathtt{dist}(t_n, S_i t_n) = 0, \qquad i \in \{1, 2, \dots, m\}. \tag{23}$$

By using the firm nonexpansivity of $J^F_{\lambda_n}$ and noticing that $J^F_{\lambda_n}(x^\star - \lambda_n fx^\star) = x^\star$ we obtain

$$\begin{aligned}
\|u_n - x^\star\|^2 &= \|J^F_{\lambda_n}(z_n - \lambda_n fz_n) - J^F_{\lambda_n}(x^\star - \lambda_n fx^\star)\|^2 \\
&\le \langle (z_n - \lambda_n fz_n) - (x^\star - \lambda_n fx^\star), J^F_{\lambda_n}(z_n - \lambda_n fz_n) - J^F_{\lambda_n}(x^\star - \lambda_n fx^\star)\rangle \\
&= \frac{1}{2}(\|(z_n - \lambda_n fz_n) - (x^\star - \lambda_n fx^\star)\|^2 + \|J^F_{\lambda_n}(z_n - \lambda_n fz_n) - x^\star)\|^2 \\
&\quad - \|(z_n - \lambda_n fz_n) - (x^\star - \lambda_n fx^\star) - (J^F_{\lambda_n}(z_n - \lambda_n fz_n) - x^\star)\|^2)
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{2}(\|z_n - x^\star\|^2 + \|J^F_{\lambda_n}(z_n - \lambda_n f z_n) - x^\star\|^2 \\
&\quad - \|z_n - J^F_{\lambda_n}(z_n - \lambda_n f z_n) - \lambda_n(f z_n - f x^\star)\|^2) \\
&= \frac{1}{2}(\|z_n - x^\star\|^2 + \|J^F_{\lambda_n}(z_n - \lambda_n f z_n) - x^\star)\|^2 \\
&\quad - \|z_n - J^F_{\lambda_n}(z_n - \lambda_n f z_n)\|^2) \\
&\quad + 2\lambda_n\langle z_n - J^F_{\lambda_n}(z_n - \lambda_n f z_n), f z_n - f x^\star\rangle - \lambda_n^2\|f z_n - f x^\star\|^2).
\end{aligned} \tag{24}$$

Which implies that

$$\begin{aligned}
&\|u_n - x^\star\|^2 \|J^F_{\lambda_n}(z_n - \lambda_n f z_n) - x^\star\|^2 \\
&\qquad \leq \|z_n - x^\star\|^2 - \|z_n - J^F_{\lambda_n}(z_n - \lambda_n f z_n)\|^2 \\
&\qquad\quad + 2\lambda_n\langle z_n - J^F_{\lambda_n}(z_n - \lambda_n f z_n), f z_n - f x^\star\rangle - \lambda_n^2\|f z_n - f x^\star\|^2.
\end{aligned} \tag{25}$$

Utilizing Lemma 3 and inequality (25) we get

$$\begin{aligned}
\|x_{n+1} - x^\star\|^2 &= \|\alpha_n \vartheta + \beta_n u_n + \sum_{i=1}^{m} \delta_{n,i} v_{n,i} - x^\star\|^2 \\
&\leq \alpha_n\|\vartheta - x^\star\|^2 + \beta_n\|u_n - x^\star\|^2 + \sum_{i=1}^{m} \delta_{n,i}\|v_{n,i} - x^\star\|^2 \\
&\leq \alpha_n\|\vartheta - x^\star\|^2 + \beta_n\|u_n - x^\star\|^2 + \sum_{i=1}^{m} \delta_{n,i}\|u_n - x^\star\|^2 \\
&\leq \alpha_n\|\vartheta - x^\star\|^2 + (1 - \alpha_n)\|u_n - x^\star\|^2 \\
&\leq \alpha_n\|\vartheta - x^\star\|^2 + (1 - \alpha_n)\|z_n - x^\star\|^2 \\
&\quad - (1 - \alpha_n)\|z_n - J^F_{\lambda_n}(z_n - \lambda_n f z_n)\|^2 \\
&\quad + 2(1 - \alpha_n)\lambda_n\langle z_n - J^F_{\lambda_n}(z_n - \lambda_n f z_n), f z_n - f x^\star\rangle \\
&\quad - (1 - \alpha_n)\lambda_n^2\|f z_n - f x^\star\|^2 \\
&\leq \alpha_n\|\vartheta - x^\star\|^2 + (1 - \alpha_n)\|z_n - x^\star\|^2 \\
&\quad - (1 - \alpha_n)\|z_n - J^F_{\lambda_n}(z_n - \lambda_n f z_n)\|^2 \\
&\quad + 2(1 - \alpha_n)\lambda_n\|z_n - J^F_{\lambda_n}(z_n - \lambda_n f z_n)\|\|f z_n - f x^\star\|.
\end{aligned} \tag{26}$$

By similar argument we obtain

$$\begin{aligned}\|y_{n+1}-y^\star\|^2 &= \|\alpha_n\zeta+\beta_n t_n+\sum_{i=1}^m \delta_{n,i}s_{n,i}-y^\star\|^2\\ &\le \alpha_n\|\zeta-y^\star\|^2+(1-\alpha_n)\|w_n-y^\star\|^2\\ &\quad -(1-\alpha_n)\|w_n-J^G_{\mu_n}(w_n-\mu_n gw_n)\|^2\\ &\quad +2(1-\alpha_n)\mu_n\|w_n-J^G_{\mu_n}(w_n-\mu_n gw_n)\|\|gw_n-gy^\star\|.\end{aligned}\tag{27}$$

By adding the inequality (26) and the inequality (27) we get

$$\begin{aligned}&\|x_{n+1}-x^\star\|^2+\|y_{n+1}-y^\star\|^2\\ &\quad\le(1-\alpha_n)(\|x_n-x^\star\|^2+\|y_n-y^\star\|^2)+\alpha_n(\|\vartheta-x^\star\|^2+\|\zeta-y^\star\|^2)\\ &\qquad-(1-\alpha_n)\|(z_n-J^F_{\lambda_n}(z_n-\lambda_n fz_n)\|^2\\ &\qquad-(1-\alpha_n)\|(w_n-J^G_{\mu_n}(w_n-\mu_n gw_n)\|^2\\ &\qquad+2(1-\alpha_n)\lambda_n\|z_n-J^F_{\lambda_n}(z_n-\lambda_n fz_n)\|\|fz_n-fx^\star\|\\ &\qquad+2(1-\alpha_n)\mu_n\|w_n-J^G_{\mu_n}(w_n-\mu_n gw_n)\|\|gw_n-gy^\star\|.\end{aligned}\tag{28}$$

Consequently,

$$\begin{aligned}&(1-\alpha_n)\|z_n-J^F_{\lambda_n}(z_n-\lambda_n fz_n)\|^2\le\Gamma_n-\Gamma_{n+1}+\alpha_n(\|\vartheta-x^\star\|^2+\|\zeta-y^\star\|^2)\\ &\quad+2(1-\alpha_n)\lambda_n\|z_n-J^F_{\lambda_n}(z_n-\lambda_n fz_n)\|\|fz_n-fx^\star\|\\ &\quad+2(1-\alpha_n)\mu_n\|w_n-J^G_{\mu_n}(w_n-\mu_n gw_n)\|\|gw_n-gy^\star\|.\end{aligned}\tag{29}$$

This implies that

$$\lim_{n\to\infty}\|z_n-J^F_{\lambda_n}(z_n-\lambda_n fz_n)\|=0.\tag{30}$$

By similar argument we obtain

$$\lim_{n\to\infty}\|w_n-J^G_{\mu_n}(w_n-\mu_n gw_n)\|=0.\tag{31}$$

Since $\|z_n-x_n\|=\gamma_n\|\mathcal{A}^*(\mathcal{A}x_n-\mathcal{B}y_n)\|$ and $\{\gamma_n\}$ is bounded, we have

$$\lim_{n\to\infty}\|z_n-x_n\|=0.\tag{32}$$

From (30) and (32) we have

$$\|x_n-u_n\|\le\|x_n-z_n\|+\|z_n-u_n\|\to 0,\quad \text{as}\quad n\to\infty.$$

Therefore

$$\|x_{n+1}-x_n\|\le\alpha_n\|\vartheta-x_n\|+\beta_n\|u_n-x_n\|+\sum_{i=1}^m\delta_{n,i}\|v_{n,i}-x_n\|\to 0,\ \text{as}\ n\to\infty.\tag{33}$$

Similarly we have that $\lim_{n\to\infty}\|y_{n+1}-y_n\|=0$.

Now we claim that $(\omega_w(x_n),\omega_w(y_n))\subset\Omega$, where

$$\omega_w(x_n)=\{x\in\mathcal{H}_1 : x_{n_i}\rightharpoonup x \text{ for some subsequence} \{x_{n_i}\} \text{ of } \{x_n\}\}.$$

Since the sequences $\{x_n\}$ and $\{y_n\}$ are bounded we have $\omega_w(x_n)$ and $\omega_w(y_n)$ are nonempty. Now, take $\widehat{x}\in\omega_w(x_n)$ and $\widehat{y}\in\omega_w(y_n)$. Thus, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to $\widehat{x}$. Without loss of generality, we can assume that $x_n\rightharpoonup\widehat{x}$. Now, we are in a position to show that $\widehat{x}\in(f+F)^{-1}(0)$. Since $\lim_{n\to\infty}\|z_n-x_n\|=0$, we have $z_n\rightharpoonup\widehat{x}$. By our assumption that f is α- inverse strongly monotone mapping we have

$$\langle z_n-\widehat{x}, fz_n-f\widehat{x}\rangle\geq\alpha\|fz_n-f\widehat{x}\|^2.$$

Now, from $z_n\rightharpoonup\widehat{x}$ we deduce $fz_n\to f\widehat{x}$. From $u_n=J^F_{\lambda_n}(z_n-\lambda_n fz_n)$, we have $z_n-\lambda_n fz_n\in(I+\lambda_n F)u_n$, hence $\frac{z_n-u_n}{\lambda_n}-fz_n\in Fu_n$. Since F is monotone, we get, for any $(u,v)\in F$ that

$$\langle u_n-u,\frac{z_n-u_n}{\lambda_n}-fz_n-v\rangle\geq 0.$$

Since $\lim_{n\to\infty}\|z_n-u_n\|=0$, we have $u_n\rightharpoonup\widehat{x}$. Now above inequality implies that

$$\langle\widehat{x}-u,-f\widehat{x}-v\rangle\geq 0.$$

This gives that $-f\widehat{x}\in F\widehat{x}$, that is $0\in(f+F)\widehat{x}$. This proves that $\widehat{x}\in(f+F)^{-1}(0)$. By similar argument we can obtain that $\widehat{y}\in(g+G)^{-1}(0)$. Next we show that $\widehat{x}\in\bigcap_{i=1}^m \mathrm{Fix}(T_i)$ and $\widehat{y}\in\bigcap_{i=1}^m \mathrm{Fix}(S_i)$. Since $\lim_{n\to\infty}\mathrm{dist}(T_iu_n,u_n)=0$ and $u_n\rightharpoonup\widehat{x}$, noticing the demiclosedness of T_i-I in 0, we get that $\widehat{x}\in\mathrm{Fix}(T_i)$ (for each $i\in\{1,2,...,m\}$). By similar argument we obtain that $\widehat{y}\in\bigcap_{i=1}^m\mathrm{Fix}(S_i)$. On the other hand, $\mathcal{A}\widehat{x}-\mathcal{B}\widehat{y}\in\omega_w(\mathcal{A}x_n-\mathcal{B}y_n)$ and weakly lower semi continuity of the norm imply that

$$\|\mathcal{A}\widehat{x}-\mathcal{B}\widehat{y}\|\leq\liminf_{n\to\infty}\|\mathcal{A}x_n-\mathcal{B}y_n\|=0.$$

Thus $(\widehat{x},\widehat{y})\in\Omega$. We also have the uniqueness of the weak cluster point of $\{x_n\}$ are $\{y_n\}$, (see [23] for details) which implies that the whole sequences $\{(x_n,y_n)\}$ weakly convergence to a point $(\widehat{x},\widehat{y})\in\Omega$. Put $C=\bigcap_{i=1}^m\mathrm{Fix}(T_i)\bigcap(f+F)^{-1}(0)$ and $Q=\bigcap_{i=1}^m\mathrm{Fix}(S_i)\bigcap(g+G)^{-1}(0)$. Next we prove that the sequences $\{(x_n,y_n)\}$ converges strongly to $(\vartheta^\star,\zeta^\star)$ where $\vartheta^\star=P_C\,\vartheta$ and $\zeta^\star=P_Q\,\zeta$. First we show that

$$\limsup_{n\to\infty}\langle\vartheta-\vartheta^\star,x_n-\vartheta^\star\rangle\leq 0. \tag{34}$$

To show this inequality, we choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\lim_{k\to\infty}\langle \vartheta-\vartheta^\star, x_{n_k}-\vartheta^\star\rangle = \limsup_{n\to\infty}\langle \vartheta-\vartheta^\star, x_n-\vartheta^\star\rangle.$$

Since $\{x_{n_k}\}$ converges weakly to $\widehat{x}$, it follows that

$$\limsup_{n\to\infty}\langle \vartheta-\vartheta^\star, x_n-\vartheta^\star\rangle = \lim_{k\to\infty}\langle \vartheta-\vartheta^\star, x_{n_k}-\vartheta^\star\rangle = \langle \vartheta-\vartheta^\star, \widehat{x}-\vartheta^\star\rangle \leq 0. \quad (35)$$

By similar argument we obtain that

$$\limsup\nolimits_{n\to\infty}\langle \zeta-\zeta^\star, y_n-\zeta^\star\rangle \leq 0. \quad (36)$$

From the inequality, $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \qquad (\forall x, y \in \mathcal{H}_1)$, we find that

$$\begin{aligned}
\|x_{n+1}-\vartheta^\star\|^2 &\leq \|\beta_n u_n + \sum_{i=1}^{m}\delta_{n,i}v_{n,i} - (1-\alpha_n)\vartheta^\star\|^2 + 2\alpha_n\langle \vartheta-\vartheta^\star, x_{n+1}-\vartheta^\star\rangle \\
&= (1-\alpha_n)^2\|\frac{\beta_n}{(1-\alpha_n)}u_n + \frac{\sum_{i=1}^{m}\delta_{n,i}}{(1-\alpha_n)}v_{n,i} - \vartheta^\star\|^2 \\
&\quad + 2\alpha_n\langle \vartheta-\vartheta^\star, x_{n+1}-\vartheta^\star\rangle \\
&\leq \beta_n(1-\alpha_n)\|u_n-\vartheta^\star\|^2 + \sum_{i=1}^{m}\delta_{n,i}(1-\alpha_n)\|v_{n,i}-\vartheta^\star\|^2 \\
&\quad + 2\alpha_n\langle \vartheta-\vartheta^\star, x_{n+1}-\vartheta^\star\rangle \\
&= (1-\alpha_n)(\beta_n + \sum_{i=1}^{m}\delta_{n,i})\|u_n-\vartheta^\star\|^2 + 2\alpha_n\langle \vartheta-\vartheta^\star, x_{n+1}-\vartheta^\star\rangle \\
&\leq (1-\alpha_n)^2\|u_n-\vartheta^\star\|^2 + 2\alpha_n\langle \vartheta-\vartheta^\star, x_{n+1}-\vartheta^\star\rangle.
\end{aligned}$$

Similarly we obtain that

$$\|y_{n+1}-\zeta^\star\|^2 \leq (1-\alpha_n)^2\|t_n-\zeta^\star\|^2 + 2\alpha_n\langle \zeta-\zeta^\star, y_{n+1}-\zeta^\star\rangle. \quad (37)$$

By adding the two last inequalities we have that

$$\begin{aligned}
\|x_{n+1}-\vartheta^\star\|^2 \quad &+ \|y_{n+1}-\zeta^\star\|^2 \\
&\leq (1-\alpha_n)^2(\|x_n-\vartheta^\star\|^2 + \|y_n-\zeta^\star\|^2) \\
&+ 2\alpha_n(\langle \vartheta-\vartheta^\star, x_{n+1}-\vartheta^\star\rangle + \langle \zeta-\zeta^\star, y_{n+1}-\zeta^\star\rangle).
\end{aligned} \quad (38)$$

It immediately follows that

$$\begin{aligned}\Gamma_{n+1} &\leq (1-\alpha_n)^2\Gamma_n + 2\alpha_n\eta_n \\ &= (1-2\alpha_n)\Gamma_n + \alpha_n^2\Gamma_n + 2\alpha_n\eta_n \\ &\leq (1-2\alpha_n)\Gamma_n + 2\alpha_n\{\tfrac{\alpha_n N}{2} + \eta_n) \\ &\leq (1-\rho_n)\Gamma_n + \rho_n\delta_n,\end{aligned} \tag{39}$$

where $\eta_n = \langle \vartheta - \vartheta^\star, x_{n+1} - \vartheta^\star\rangle + \langle \zeta - \zeta^\star, y_{n+1} - \zeta^\star\rangle$, $N = \sup\{\|x_n - x^\star\|^2 + \|y_n - y^\star\|^2 : n \geq 0\}$, $\rho_n = 2\alpha_n$ and $\delta_n = \frac{\alpha_n N}{2} + \eta_n$. It is easy to see that $\rho_n \to 0$, $\sum_{n=1}^{\infty}\rho_n = \infty$ and $\limsup_{n\to\infty}\delta_n \leq 0$. Hence, all conditions of Lemma 4 are satisfied. Therefore, we immediately deduce that $\lim_{n\to\infty}\Gamma_n = 0$. Consequently $\lim_{n\to\infty}\|x_n - \vartheta^\star\| = \lim_{n\to\infty}\|y_n - \zeta^\star\| = 0$, that is $(x_n, y_n) \to (\vartheta^\star, \zeta^\star)$.

Case B. Assume that $\{\Gamma_n\}$ is not a monotone sequence. Then, we can define an integer sequence $\{\tau(n)\}$ for all $n \geq n_0$ (for some n_0 large enough) by

$$\tau(n) = max\{k \leq n : \Gamma_k < \Gamma_{k+1}\}.$$

Clearly, τ is a nondecreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and for all $n \geq n_0$, $\Gamma_{\tau(n)} < \Gamma_{\tau(n)+1}$. From (17), we deduce

$$\Gamma_{\tau(n)+1} - \Gamma_{\tau(n)} \leq \alpha_n(\|\vartheta - \vartheta^\star\|^2 + \|\upsilon - \zeta^\star\|^2) - \alpha_n(\|x_n - \vartheta^\star\|^2 + \|y_n - \zeta^\star\|^2). \tag{40}$$

Since $\lim_{n\to\infty}\alpha_n = 0$ and $\{y_n\}$ and $\{x_n\}$ are bounded, we derive that

$$\lim_{n\to\infty}(\Gamma_{\tau(n)+1} - \Gamma_{\tau(n)}) = 0. \tag{41}$$

Following an argument similar to that in $\mathsf{Case\ A}$ we have

$$\Gamma_{\tau(n)+1} \leq (1-\rho_{\tau(n)})\Gamma_{\tau(n)} + \rho_{\tau(n)}\delta_{\tau(n)},$$

where $\limsup_{n\to\infty}\delta_{\tau(n)} \leq 0$. Since $\Gamma_{\tau(n)} < \Gamma_{\tau(n)+1}$, we have

$$\rho_{\tau(n)}\Gamma_{\tau(n)} \leq \rho_{\tau(n)}\delta_{\tau(n)}.$$

Since $\rho_{\tau(n)} > 0$ we deduce that

$$\Gamma_{\tau(n)} \leq \delta_{\tau(n)}.$$

Hence $\lim_{n\to\infty}\Gamma_{\tau(n)} = 0$. This together with (41), implies that $\lim_{n\to\infty}\Gamma_{\tau(n)+1} = 0$. Applying Lemma 5 to get

$$0 \leq \Gamma_n \leq \max\{\Gamma_{\tau(n)}, \Gamma_n\} \leq \Gamma_{\tau(n)+1}. \tag{42}$$

Therefore $(x_n, y_n) \to (\vartheta^\star, \zeta^\star)$. This completes the proof.

$\square$

As a consequence of our main result we have the following theorem for single valued mappings.

Theorem 3 *Let $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$, be real Hilbert spaces, $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_3$ and $\mathcal{B} : \mathcal{H}_2 \to \mathcal{H}_3$ be bounded linear operators. Let $f : \mathcal{H}_1 \to \mathcal{H}_1$ and $g : \mathcal{H}_2 \to \mathcal{H}_2$ be respectively α and β- inverse strongly monotone operators and F, G two maximal monotone operators on $\mathcal{H}_1, \mathcal{H}_2$. Let for $i \in \{1, 2, ..., m\}$, $T_i : \mathcal{H}_1 \to \mathcal{H}_1$ and $S_i : \mathcal{H}_2 \to \mathcal{H}_2$ be two finite families of quasi-nonexpansive mappings such that $S_i - I$ and $T_i - I$ are demiclosed at 0. Suppose $\Omega = \{x \in \bigcap_{i=1}^m \mathrm{Fix}(T_i) \bigcap (f + F)^{-1}(0), \quad y \in \bigcap_{i=1}^m \mathrm{Fix}(S_i) \bigcap (g + G)^{-1}(0) : \mathcal{A}x = \mathcal{B}y\} \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by $x_0, \vartheta \in \mathcal{H}_1, \quad y_0, \zeta \in \mathcal{H}_2$ and by*

$$\begin{cases} z_n = x_n - \gamma_n \mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n) \\ u_n = J^F_{\lambda_n}(I - \lambda_n f) z_n, \\ x_{n+1} = \alpha_n \vartheta + \beta_n u_n + \sum_{i=1}^m \delta_{n,i} T_i u_n \\ w_n = y_n + \gamma_n \mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n) \\ t_n = J^G_{\mu_n}(I - \mu_n g) w_n, \\ y_{n+1} = \alpha_n \zeta + \beta_n t_n + \sum_{i=1}^m \delta_{n,i} S_i t_n \qquad \forall n \geq 0. \end{cases} \tag{43}$$

Let the sequences $\{\gamma_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$,$\{\delta_{n,i}\}$, $\{\lambda_n\}$ and $\{\mu_n\}$ satisfy the conditions of Theorem 3.1. Then, the sequences $\{(x_n, y_n)\}$ converges strongly to $(x^\star, y^\star) \in \Omega$.

Now, let $T : \mathcal{H} \to P(\mathcal{H})$ be a set- valued mapping and let

$$P_T(x) = \{y \in Tx : \|x - y\| = \mathrm{dist}(x, Tx)\}, \quad x \in \mathcal{H}.$$

It can be easily seen $\mathrm{Fix}(T) = \mathrm{Fix}(P_T)$. From this we have the following theorem.

Theorem 4 *Let $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$, be real Hilbert spaces, $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_3$ and $\mathcal{B} : \mathcal{H}_2 \to \mathcal{H}_3$ be bounded linear operators. Let $f : \mathcal{H}_1 \to \mathcal{H}_1$ and $g : \mathcal{H}_2 \to \mathcal{H}_2$ be respectively α and β- inverse strongly monotone operators and F, G two maximal monotone operators on $\mathcal{H}_1, \mathcal{H}_2$. Let for $i \in \{1, 2, ..., m\}$, $T_i : \mathcal{H}_1 \to CC(\mathcal{H}_1)$ and $S_i : \mathcal{H}_2 \to CC(\mathcal{H}_2)$ be two finite families of set valued mappings such that $P_{S_i} : \mathcal{H}_1 \to \mathcal{H}_1$ and $P_{T_i} : \mathcal{H}_2 \to \mathcal{H}_2$ are generalized nonexpansive. Suppose $\Omega = \{x \in \bigcap_{i=1}^m \mathrm{Fix}(T_i) \bigcap (f + F)^{-1}(0), \quad y \in \bigcap_{i=1}^m \mathrm{Fix}(S_i) \bigcap (g + G)^{-1}(0) : \mathcal{A}x =$*

$\mathcal{B}y\} \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by $x_0, \vartheta \in \mathcal{H}_1,\quad y_0, \zeta \in \mathcal{H}_2$ and by

$$\begin{cases} z_n = x_n - \gamma_n \mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n) \\ u_n = J^F_{\lambda_n}(I - \lambda_n f)z_n, \\ x_{n+1} = \alpha_n \vartheta + \beta_n u_n + \sum_{i=1}^m \delta_{n,i} P_{T_i} u_n \\ w_n = y_n + \gamma_n \mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n) \\ t_n = J^G_{\mu_n}(I - \mu_n g)w_n, \\ y_{n+1} = \alpha_n \zeta + \beta_n t_n + \sum_{i=1}^m \delta_{n,i} P_{S_i} t_n \qquad \forall n \geq 0. \end{cases} \tag{44}$$

Let the sequences $\{\gamma_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$,$\{\delta_{n,i}\}$, $\{\lambda_n\}$ and $\{\mu_n\}$ satisfy the conditions of Theorem 3.1. Then, the sequences $\{(x_n, y_n)\}$ converges strongly to $(x^\star, y^\star) \in \Omega$.

Remark 1 *In [11], Takahashi et al. present some algorithms for generalized split feasibility problem for finding fixed point of nonlinear single valued mappings and the zero point of a maximal monotone operator. They proved some weak convergence theorems for finding a solution of the generalized split feasibility problem. In this paper we present an algorithm for solving split equality problem for finding common fixed point of a finite family of quasi-nonexpansive set-valued mappings and the zero point of the sum of two monotone operators. Our algorithm do not require any knowledge of the operator norms. We also present a strong convergence theorem which is more desirable than weak convergence.*

Remark 2 *In [23], Zhao present a weak convergence theorem for solving split equality fixed point problem of quasi-nonexpansive mapping (see theorem 1.1 of this paper). In this paper we extend the result for solving split equality common fixed problem of a finite family of quasi-nonexpansive set valued mappings. We also present a strong convergence theorem which is more desirable than weak convergence.*

Remark 3 *Moudafi [18] and Censor et al. [6] present some algorithms for solving the split monotone variational inclusion problem. They establish some weak convergence theorems for these algorithms. In this paper we present an algorithm for split equality monotone variational inclusion problem. Our algorithm do not require any knowledge of the operator norms. We also present a strong convergence theorem which is more desirable than weak convergence.*

4 Application

In this section, using Theorem 3.1, we can obtain well-known and new strong convergence theorems in a Hilbert space.

Variational inequality

Let $\mathcal{H}$ be a Hilbert space, and let h be a proper lower semicontinuous convex function of $\mathcal{H}$ into $\mathbb{R}$. Then the subdifferential ∂h of h is defined as follows:

$$\partial h(x) = \{z \in \mathcal{H} : h(x) + \langle z, u - x\rangle \leq h(u), \forall u \in \mathcal{H}\}$$

for all $x \in \mathcal{H}$. From Rockafellar [31] , we know that ∂h is amaximal monotone operator. Let C be a nonempty closed convex subset of $\mathcal{H}$, and let i_C be the indicator function of C, i.e.,

$$i_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ +\infty, & \text{if } x \notin C. \end{cases} \tag{45}$$

Then, i_C is a proper lower semicontinuous convex function on $\mathcal{H}$. So, we can define the resolvent operator $J_r^{\partial i_C}$ of i_C for $r > 0$, i.e.,

$$J_r^{\partial i_C}(x) = (I + r\partial i_C)^{-1}(x), \quad x \in \mathcal{H}.$$

We know that $J_r^{\partial i_C}(x) = P_C x$ for all $x \in \mathcal{H}$ and $r > 0$; see [32]. Moreover, for the single valued operator $f : \mathcal{H} \to \mathcal{H}$ we have

$$x \in (\partial i_C + f)^{-1}(0) \Leftrightarrow x \in VI(C, f).$$

Theorem 5 *Let $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$, be real Hilbert spaces, $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_3$ and $\mathcal{B} : \mathcal{H}_2 \to \mathcal{H}_3$ be bounded linear operators, and let C and Q, be two nonempty closed convex subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Let $f : \mathcal{H}_1 \to \mathcal{H}_1$ and $g : \mathcal{H}_2 \to \mathcal{H}_2$ be respectively α and β- inverse strongly monotone operators. Let for $i \in \{1, 2, \ldots, m\}$, $T_i : \mathcal{H}_1 \to K(\mathcal{H}_1)$ and $S_i : \mathcal{H}_2 \to K(\mathcal{H}_2)$ be two finite families of generalized nonexpansive set-valued mappings such that S_i and T_i satisfies the common endpoint condition. Suppose $\Omega = \{x \in \bigcap_{i=1}^m Fix(T_i) \bigcap VI(C, f), \quad y \in \bigcap_{i=1}^m Fix(S_i) \bigcap VI(Q, g) : \mathcal{A}x = \mathcal{B}y\} \neq \emptyset$. Let*

$\{x_n\}$ and $\{y_n\}$ be sequences generated by $x_0, \vartheta \in \mathcal{H}_1, \quad y_0, \zeta \in \mathcal{H}_2$ and by

$$\begin{cases} z_n = x_n - \gamma_n \mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n) \\ u_n = P_C(I - \lambda_n f)z_n, \\ x_{n+1} = \alpha_n \vartheta + \beta_n u_n + \sum_{i=1}^m \delta_{n,i} v_{n,i} \\ w_n = y_n + \gamma_n \mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n) \\ t_n = P_Q(I - \mu_n g)w_n, \\ y_{n+1} = \alpha_n \zeta + \beta_n t_n + \sum_{i=1}^m \delta_{n,i} s_{n,i} \qquad \forall n \geq 0, \end{cases} \tag{46}$$

where $v_{n,i} \in T_i u_n$, $s_{n,i} \in S_i t_n$. Let the sequences $\{\gamma_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_{n,i}\}$, $\{\lambda_n\}$ and $\{\mu_n\}$ satisfy the conditions of Theorem 3.1. Then, the sequences $\{(x_n, y_n)\}$ converges strongly to $(x^\star, y^\star) \in \Omega$.

Equilibrium problem

Let C be a closed convex subset of a real Hilbert space $\mathcal{H}$. Let Φ be a bifunction from $C \times C$ to $\mathbb{R}$. The equilibrium problem for Φ is to find $x^\star \in C$ such that

$$\Phi(x^\star, y) \geq 0, \qquad \forall y \in C. \tag{47}$$

The set of such solutions $x^\star$ is denoted by $\mathrm{EP}(\Phi)$.

It has been a connection between the equilibrium problem and the related problems in applied sciences such as variational inequalities, optimal theory, complementarity problems, Nash equilibrium in game theory and so on (see [36, 37]). In other words, numerous problems in physics, optimization, and economics can be nicely reduced to find a solution of (47) as well. In the recent years iterative algorithms for finding a common element of the set of solutions of equilibrium problem and the set of fixed points of nonlinear mappings have been studied by many authors (see, e.g., [38-42]).

For solving the equilibrium problem, let us assume that the bifunction Φ satisfies the following conditions:

(A1) $\Phi(x, x) = 0$ for all $x \in C$,

(A2) Φ is monotone, i.e., $\Phi(x, y) + \Phi(y, x) \leq 0$, for any $x, y \in C$,

(A3) for each $x, y, z \in C$,

$$\limsup_{t \to 0^+} \Phi(tz + (1-t)x, y) \leq \Phi(x, y),$$

(A4) for each $x \in C$, $y \to \Phi(x, y)$ is convex and lower semi-continuous.

We know the following lemma which appears implicitly in Blum et al. [36] and Combettes et al. [37].

Lemma 6 [36, 37] *Let* C *be a nonempty closed convex subset of* $\mathcal{H}$ *and let* Φ *be a bifunction of* $C \times C$ *into* $\mathbb{R}$ *satisfying* $(A1) - (A4)$. *Let* $r > 0$ *and* $x \in \mathcal{H}$. *Then, there exists* $z \in C$ *such that*

$$\Phi(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \qquad \forall y \in C.$$

Further, if

$$U_r^{\Phi} x = \{z \in C : \Phi(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}.$$

Then, the following hold:

(i) U_r^{Φ} *is single valued and firmly nonexpansive;*

(ii) $\mathrm{Fix}(U_r^{\Phi}) = \mathrm{EP}(\Phi)$;

(iii) $\mathrm{EP}(\Phi)$ *is closed and convex.*

We call such U_r^{Φ} the resolvent of Φ for $r > 0$. Using above lemma, we have the following lemma, see [24] for a more general result.

Lemma 7 [24] *Let* C *be a nonempty closed convex subset of* $\mathcal{H}$ *and let* Φ *be a bifunction of* $C \times C$ *into* $\mathbb{R}$ *satisfy* $(A1) - (A4)$. *Let* B_Φ *be a set-valued mapping of* $\mathcal{H}$ *into itself defined by*

$$B_\Phi(x) = \begin{cases} \{z \in \mathcal{H} : \Phi(x, y) + \langle y - x, z\rangle \geq 0, \quad \forall\, y \in C\}, & \forall\, x \in C \\ \emptyset, \quad \forall\, x \notin C. \end{cases} \tag{48}$$

Then $\mathrm{EP}(\Phi) = B_\Phi^{-1}(0)$ *and* B_Φ *is a maximal monotone operator with* $\mathrm{dom}(B_\Phi) \subset C$. *Furthermore, for any* $x \in \mathcal{H}$ *and* $r > 0$, *the resolvent* U_r^{Φ} *of* Φ *coincides with the resolvent of* B_Φ, *i.e.,*

$$U_r^{\Phi}(x) = (I + rB_\Phi)^{-1}(x).$$

Form Lemma 4.3 and Theorems 3.1 we have the following results.

Theorem 6 *Let $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$, be real Hilbert spaces, $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_3$ and $\mathcal{B} : \mathcal{H}_2 \to \mathcal{H}_3$ be bounded linear operators, and let C and Q, be two nonempty closed convex subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Let $\Phi : C \times C \to \mathbb{R}$ and $\Psi : Q \times Q \to \mathbb{R}$ be functions satisfying conditions $(A1) - (A4)$. Let for $i \in \{1, 2, \ldots, m\}$, $T_i : \mathcal{H}_1 \to K(\mathcal{H}_1)$ and $S_i : \mathcal{H}_2 \to K(\mathcal{H}_2)$ be two finite families of generalized nonexpansive set-valued mappings such that S_i and T_i satisfies the common endpoint condition. Suppose $\Omega = \{x \in \bigcap_{i=1}^m \mathrm{Fix}(T_i) \bigcap EP(\Phi), y \in \bigcap_{i=1}^m \mathrm{Fix}(S_i) \bigcap EP(\Psi) : \mathcal{A}x = \mathcal{B}y\} \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by $x_0, \vartheta \in \mathcal{H}_1, \quad y_0, \zeta \in \mathcal{H}_2$ and by*

$$\begin{cases} z_n = x_n - \gamma_n \mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n) \\ u_n = U_{r_n}^{\Phi} z_n, \\ x_{n+1} = \alpha_n \vartheta + \beta_n u_n + \sum_{i=1}^m \delta_{n,i} v_{n,i} \\ w_n = y_n + \gamma_n \mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n) \\ t_n = U_{\kappa_n}^{\Phi} w_n, \\ y_{n+1} = \alpha_n \zeta + \beta_n t_n + \sum_{i=1}^m \delta_{n,i} s_{n,i} \qquad \forall n \geq 0, \end{cases} \tag{49}$$

where $v_{n,i} \in T_i u_n$, $s_{n,i} \in S_i t_n$. Let the sequences $\{\gamma_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\delta_n\}$ satisfy the conditions of Theorem 3.1. Assume that $\liminf_n r_n > 0$ and $\liminf_n \kappa_n > 0$. Then, the sequences $\{(x_n, y_n)\}$ converges strongly to $(x^\star, y^\star) \in \Omega$.

Proof. For the bifunctions $\Phi : C \times C \to \mathbb{R}$ and $\Psi : Q \times Q \to \mathbb{R}$ we can define B_Φ and B_Ψ in Lemma 7. Putting $F = B_\Phi$ and $G = B_\Psi$ in Theorem 3.1, we obtain from Lemma 7 that $U_{r_n}^{\Phi}(x) = (I + r_n B_\Phi)^{-1}(x)$ and $U_{\kappa_n}^{\Psi}(x) = (I + \kappa_n B_\Psi)^{-1}(x)$. Thus by setting $f = g = 0$, we obtain the desired result by Theorem 3.1. $\square$

Numerical example

Let $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}_3 = \mathbb{R}$. For each $x \in \mathbb{R}$ define set- valued mappings T_i and S_i as follows:

$$T_1 x = [0, \frac{x}{2}], \qquad T_2(x) = \begin{cases} 0, & x < 0 \\ [0, \frac{x}{3}], & 0 \leq x < 3 \\ [1, 2] & x \geq 3, \end{cases}$$

and

$$S_1 x = [0, \frac{x}{5}], \qquad S_2 x = [0, \frac{x}{2}].$$

It is easy to see that T_2 is generalized nonexpansive mapping and T_1, S_1, S_2 are nonexpansive mappings. We put $C = Q = [0, \infty)$ and define the bifunctions

$\Phi : C \times C \to \mathbb{R}$ and $\Psi : Q \times Q \to \mathbb{R}$ as follows:

$$\Phi = y^2 + xy - 2x^2, \qquad \Psi = x(y - x).$$

We observe that the functions Φ and Ψ satisfying the conditions $(A1) - (A4)$. We also have $U_r^{\Phi} = \frac{x}{3r+1}$ and $U_r^{\Psi} = \frac{x}{r+1}$. Also we define $\mathcal{A}x = 2x$ and $\mathcal{B}x = 3x$, hence $\mathcal{A}^{\star}x = 2x$ and $\mathcal{B}^{\star}x = 3x$. Put $\alpha_n = \frac{1}{n+1}$, $\beta_n = \delta_{n,1} = \delta_{n,2} = \frac{n}{3n+3}$, $r_n = \kappa_n = 1$ and $\gamma_n = \frac{1}{6}$. Then these sequences satisfy the conditions of Theorem 4.4. We have the following algorithm:

$$\begin{cases} z_n = x_n - \gamma_n \mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n) = \frac{1}{3}x_n + y_n, \\ u_n = U_{r_n}^{\Phi} z_n = \frac{z_n}{4}, \\ x_{n+1} = \alpha_n \vartheta + \beta_n u_n + \delta_{n,1} v_{n,1} + \delta_{n,2} v_{n,2} \\ w_n = y_n + \gamma_n \mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n) = x_n - \frac{1}{2}y_n, \\ t_n = U_{\kappa_n}^{\Phi} w_n = \frac{w_n}{2}, \\ y_{n+1} = \alpha_n \zeta + \beta_n t_n + \delta_{n,1} s_{n,1} + \delta_{n,2} s_{n,2} \qquad \forall n \geq 0. \end{cases} \tag{50}$$

Taking $(x_0, y_0) = (1, 1), \vartheta = \zeta = 2$, $v_{n,1} = v_{n,2} = \frac{u_n}{5}$ and $s_{n,1} = s_{n,2} = \frac{t_n}{5}$, we have the following algorithm:

$$\begin{cases} z_n = \frac{1}{3}x_n + y_n, \\ u_n = \frac{z_n}{4} = \frac{x_n}{12} + \frac{y_n}{4}, \\ x_{n+1} = \frac{2}{n+1} + \frac{7n}{15n+15}u_n = \frac{2}{n+1} + \frac{(7n)\,x_n}{180n+180} + \frac{(7n)\,y_n}{60n+60}, \\ w_n = x_n - \frac{1}{2}y_n, \\ t_n = \frac{w_n}{2} = \frac{x_n}{2} - \frac{y_n}{4}, \\ y_{n+1} = \frac{2}{n+1} + \frac{7n}{15n+15}t_n = \frac{2}{n+1} + \frac{(7n)\,x_n}{30n+30} - \frac{(7n)\,y_n}{60n+60} \qquad \forall n \geq 0. \end{cases} \tag{51}$$

We observe that, $\{(x_n, y_n)\}$ is convergent to $(0, 0)$. We note that $\Omega = \{(0, 0)\}$.

References

[1] Y. Censor, T. Elfving, A multiprojection algorithms using Bragman projection in a product space, *Numer. Algorithm*, **8** (1994), 221–239.

[2] C. Byrne, A unified treatment of some iterative algorithms in signal processing and image reconstruction, *Inverse Problem*, **20** (2004), 103–120.

[3] Y. Censor, T. Bortfeld, B. Martin, A. Trofimov, A unified approach for inversion problems in intensity-modulated radiation therapy, *Phys. Med. Biol.*, **51** (2006), 2353–2365.

[4] Y. Censor, W. Chen, P. L. Combettes, R. Davidi, G. T. Herman, On the effectiveness of the projection methods for convex feasibility problems with linear inequality constraints, *Comput. Optim. Appl.*, **17** (2011), 1–24.

[5] Y. Censor, A. Segal, The split common fixed point problem for directed operators, *J. Convex Anal.*, **16** (2009), 587–600.

[6] Y. Censor, A. Gibali, S. Reich, Algorithms for the split variational inequality problem, *Numerical Algorithms*, **59** (2012), 301–323.

[7] C. Byrne, Y. Censor, A. Gibali, S. Reich, The split common null point problem, *J. Nonlinear Convex Anal.*, **13** (2012), 759–775.

[8] S. S. Chang, R. P. Agarwal, Strong convergence theorems of general split equality problems for quasi-nonexpansive mappings, *J. Inequal. Appl.*, (2014), **2014**:367.

[9] R. Kraikaew, S. Saejung, On split common fixed point problems, *J. Math. Anal. Appl.*, **415** (2014), 513–524.

[10] M. Eslamian, J. Vahidi, Split Common Fixed Point Problem of Nonexpansive Semigroup, *Mediterr. J. Math.*, **13** (2016), 1177–1195.

[11] W. Takahashi, H. K. Xu, J. C. Yao, Iterative Methods for Generalized Split Feasibility Problems in Hilbert Spaces, *Set-Valued and Variational Analysis.*, **23** (2015), 205–221.

[12] Y. Shehu, F. U. Ogbuisi, O. S. Iyiola, Convergence Analysis of an iterative algorithm for fixed point problems and split feasibility problems in certain Banach spaces, *Optimization*, 2015. doi:10.1080/02331934.2015.1039533

[13] M. Eslamian, General algorithms for split common fixed point problem of demicontractive mappings, *Optimization*, **65** (2016), 443–465.

[14] P. L. Lions, B. Mercier, Splitting algorithms for the sum of two nonlinear operators, *SIAM J. Numer. Anal.*, **16** (1979), 964–979.

[15] G. B. Passty, Ergodic convergence to a zero of the sum of monotone operators in Hilbert space, *J. Math. Anal. Appl.* **72** (1979), 383–390.

[16] S. P. Han, G. Lou, A parallel algorithm for a class of convex programs, *SIAM J. Control Optim.* **26** (1988), 345–355.

[17] J. Eckstein, B. F. Svaiter, A family of projective splitting methods for the sum of two maximal monotone operators, *Math. Program*, **111** (2008), 173–199.

[18] A. Moudafi, Split Monotone Variational Inclusions, *J. Optim. Theory Appl.* **150** (2011), 275–283.

[19] A. Moudafi, A relaxed alternating CQ algorithm for convex feasibility problems, *Nonlinear Analysis*, **79** (2013), 117–121.

[20] H. Attouch, A. Cabot, F. Frankel, J. Peypouquet, Alternating proximal algorithms for constrained variational inequalities. Application to domain decomposition for PDEs, *Nonlinear Anal.* **74** (2011), 7455–7473.

[21] H. Attouch, J. Bolte, P. Redont, A. Soubeyran, Alternating proximal algorithms for weakly coupled minimization problems. Applications to dynamical games and PDEs, *J. Convex Anal.* **15** (2008), 485–506.

[22] A. Moudafi, Alternating CQ-algorithm for convex feasibility and split fixed-point problems, *J. Nonlinear Convex Anal.*, **15** (2014), 809–818.

[23] J. Zhao, Solving split equality fixed-point problem of quasi-nonexpansive mappings without prior knowledge of operators norms, *Optimization*, (2014), doi(10.1080/02331934.2014.883515).

[24] S. Takahashi, W. Takahashi, M. Toyoda, Strong convergence theorems for maximal monotone operators with nonlinear mappings in Hilbert spaces, *J. Optim. Theory Appl.*, **147** (2010), 27–41.

[25] N. Shahzad, H. Zegeye, Approximating a common point of fixed points of a pseudocontractive mapping and zeros of sum of monotone mappings, *Fixed Point Theory and Applications*, **2014**, 2014:85.

[26] P. E. Mainge, A hybrid extragradient-viscosity method for monotone operators and fixed point problems, *SIAM J. Control Optim.* **47** (2008), 1499–1515.

[27] H. Iiduka, I. Yamada, A use of conjugate gradient direction for the convex optimization problem over the fixed point set of a nonexpansive mapping, *SIAM J. Optim.* **19** (2009), 1881–1893.

[28] A. Latif, M. Eslamian, Strong convergence and split common fixed point problem for set-valued operators, *J. Nonlinear Convex Anal.* **17** (2016), 967–986.

[29] A. Abkar, M. Eslamian, Convergence theorems for a finite family of generalized nonexpansive multivalued mappings in CAT(0) spaces, *Nonlinear Analysis*, **75** (2012), 1895–1903.

[30] A. Abkar, M. Eslamian, Geodesic metric spaces and generalized nonexpansive multivalued mappings, *Bull. Iran. Math. Soc.*, **39** (2013), 993–1008.

[31] R. T. Rockafellar, On the maximal monotonicity of subdifferential mappings, *Pac. J. Math.*, **33** (1970), 209–216.

[32] W. Takahashi, *Introduction to Nonlinear and Convex Analysis.* Yokohama Publishers, Yokohama (2009).

[33] H. Zegeye, N. Shahzad, Convergence of Mann's type iteration method for generalized asymptotically nonexpansive mappings, *Comput. Math. Appl.*, **62** (2011), 4007–4014.

[34] H. K. Xu, Iterative algorithms for nonlinear operators, *J. Lond. Math. Soc.*, **66** (2002), 240–256.

[35] P. E. Mainge, Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization, *Set-Valued Analysis*, **16** (2008), 899–912.

[36] E. Blum, W. Oettli, From optimization and variational inequalities to equilibrium problems, *Math. Student*, **63** (1994), 123–145.

[37] P. L. Combettes, S. A. Hirstoaga, Equilibrium programming in Hilbert spaces, *J. Nonlinear Convex Anal.*, **6** (2005), 117–136.

[38] S. Takahashi, W. Takahashi, Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces, *J. Math. Anal. Appl.*, **331** (2007), 506–515.

[39] V. Colao, G. Marino, L. Muglia, Viscosity methods for common solutions for equilib- rium and hierarchical fixed point problems, *Optimization*, **60** (2011), 553–573.

[40] M. Eslamian, Hybrid method for equilibrium problems and fixed point problems of finite families of nonexpansive semigroups, *Rev. R. Acad. Cienc. Exactas Fs. Nat., Ser. A Mat.*, **107** (2013), 299–307.

[41] L. C. Ceng, A. Petrusel, J. C. Yao, Composite viscosity approximation methods for equilibrium problem, variational inequality and common fixed points. *J. Nonlinear Convex Anal.*, **15** (2014), 219–240.

[42] P. T. Vuong, J. J. Strodiot, V. H. Hien Nguyen, On extragradient-viscosity methods for solving equilibrium and fixed point problems in a Hilbert space, *Optimization*, **64** (2015), 429–451.

Received: July 31, 2016