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# Some conditions under which derivations are zero on Banach $*$-algebras

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**Abstract.** Let $\mathcal{A}$ be a Banach $*$-algebra. By $\mathcal{S}_{\mathcal{A}}$ we denote the set of all self-adjoint elements of $\mathcal{A}$ and by $\mathcal{O}_{\mathcal{A}}$ we denote the set of those elements in $\mathcal{A}$ which can be represented as finite real-linear combinations of mutually orthogonal projections. The main purpose of this paper is to prove the following result:
Suppose that $\overline{\mathcal{O}_{\mathcal{A}}} = \mathcal{S}_{\mathcal{A}}$ and $\{d_n\}$ is a sequence of uniformly bounded linear mappings satisfying $d_n(p) = \sum_{k=0}^{n} d_{n-k}(p)d_k(p)$, where $p$ is an arbitrary projection in $\mathcal{A}$. Then $d_n(\mathcal{A}) \subseteq \bigcap_{\varphi\in\Phi_{\mathcal{A}}} \ker\varphi$ for each $n \geq 1$. In particular, if $\mathcal{A}$ is semi-prime and further, $\dim(\bigcap_{\varphi\in\Phi_{\mathcal{A}}} \ker\varphi) \leq 1$, then $d_n = 0$ for each $n \geq 1$.

## 1 Introduction and preliminaries

In this paper, $\mathcal{A}$ represents a Banach $*$-algebra over the complex field $\mathbb{C}$. If $\mathcal{A}$ is unital, then $\mathbf{1}$ will stand for its unit element. Moreover, $\mathcal{A}$ is called semi-prime if $a\mathcal{A}a = \{0\}$ implies that $a = 0$. A non-zero linear functional $\varphi$ is called a *character* if $\varphi(ab) = \varphi(a)\varphi(b)$ for every $a, b \in \mathcal{A}$. By $\Phi_{\mathcal{A}}$ we denote the set of all characters on $\mathcal{A}$. It is well known that, $\ker\varphi$ the kernel of $\varphi$ is a maximal ideal of $\mathcal{A}$, where $\varphi$ is an arbitrary element of $\Phi_{\mathcal{A}}$. We denote the set of all self-adjoint projections in $\mathcal{A}$ by $\mathcal{P}_{\mathcal{A}}$ (i.e., $\mathcal{P}_{\mathcal{A}} = \{p \in \mathcal{A} \mid p^2 = p,\ p^* = p\}$), and by $\mathcal{S}_{\mathcal{A}}$

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we denote the set of all self-adjoint elements of $\mathcal{A}$ (i.e., $\mathcal{S}_{\mathcal{A}} = \{a \in \mathcal{A} \mid a^* = a\}$). Next, the set of these elements in $\mathcal{A}$ which can be represented as finite real-linear combinations of mutually orthogonal self-adjoint projections, is denoted by $\mathcal{O}_{\mathcal{A}}$. Hence, we have $\mathcal{P}_{\mathcal{A}} \subseteq \mathcal{O}_{\mathcal{A}} \subseteq \mathcal{S}_{\mathcal{A}}$. Note that if $\mathcal{A}$ is a von Neumann algebra, then $\mathcal{O}_{\mathcal{A}}$ is norm dense in $\mathcal{S}_{\mathcal{A}}$. More generally, the same is true for $AW^*$-algebras. Recall that a $C^*$-algebra is a Banach $*$-algebra in which, for every $a$, $\|a^*a\| = \|a\|^2$. A $W^*$-algebra is a weakly closed self-adjoint algebra of operators on a Hilbert space, and an $AW^*$-algebra is a $C^*$-algebra satisfying:
i) In the partially ordered set of projections, any set of orthogonal projections has a least upper bound (LUB),
ii) Any maximal commutative self-adjoint subalgebra is generated by its self-adjoint projections. That is, it is equal to the smallest closed subalgebra containing its self-adjoint projections.
When $\mathcal{A}$ is an $AW^*$-algebras it can be proved that each maximal commutative $*$-subalgebra of $\mathcal{A}$ is monotone complete and $\mathcal{A}$ is unital.
The above-mentioned definitions and results can all be found in [1], [5] and [10] and reader is referred to this sources for more general information on $W^*$-algebras and $AW^*$-algebras. In this paper, similar to Brešar [1], the author's attention is concentrated on Banach $*$-algebras in which $\mathcal{O}_{\mathcal{A}}$ is norm dense in $\mathcal{S}_{\mathcal{A}}$, i.e. $\overline{\mathcal{O}_{\mathcal{A}}} = \mathcal{S}_{\mathcal{A}}$.

A linear mapping $d : \mathcal{A} \to \mathcal{A}$ is called a derivation if it satisfies the Leibnitz's rule $d(ab) = d(a)b + ad(b)$ for all $a, b \in \mathcal{A}$. An additive mapping $d : \mathcal{A} \to \mathcal{A}$ is called a Jordan derivation if $d(a^2) = d(a)a + ad(a)$ holds for all $a \in \mathcal{A}$. If we define a sequence $\{d_n\}$ of linear mappings on $\mathcal{A}$ by $d_0 = I$ and $d_n = \frac{d^n}{n!}$, where I is the identity mapping on $\mathcal{A}$, then the Leibnitz's rule ensures us that $d_n$'s satisfy the condition

$$d_n(ab) = \sum_{k=0}^{n} d_{n-k}(a)d_k(b) \tag{1}$$

for each $a, b \in \mathcal{A}$ and each non-negative integer n. This motivates us to consider the sequences $\{d_n\}$ of linear mappings on an algebra $\mathcal{A}$ satisfying (1). Such a sequence is called a higher derivation. A sequence $\{d_n\}$ of linear mappings on an algebra $\mathcal{A}$ satisfying $d_n(p) = \sum_{k=0}^{n} d_{n-k}(p)d_k(p)$, where $p$ is an arbitrary element of $\mathcal{P}_{\mathcal{A}}$, is called a pre-higher derivation. A pre-higher derivation $\{d_n\}$ is called uniformly bounded if there exists an $M > 0$ such that $\|d_n\| \leq M$ for each $n$. In current note, the focus of attention is on uniformly bounded pre-higher derivations. The question under which conditions all derivations are zero on a given $*$-algebra have attracted much attention of authors (for

instance, see [3], [4], [6], [8], [9], and [12]). In this paper, we also concentrate on this topic. Let us provide a background of our study. In 1955, Singer and Wermer [11] achieved a fundamental result which started investigation into the range of derivations on Banach algebras. The result states that if $\mathcal{A}$ is a commutative Banach algebra and $d : \mathcal{A} \to \mathcal{A}$ is a bounded derivation, then $d(\mathcal{A}) \subseteq rad(\mathcal{A})$, where $rad(\mathcal{A})$ denotes the Jacobson radical of $\mathcal{A}$. It is evident that if $\mathcal{A}$ is semi-simple, i.e. $rad(\mathcal{A}) = \{0\}$, then $d$ is zero. In this paper, we prove that there is not any non-zero bounded derivation from $\mathcal{A}$ into $\mathcal{A}$ without considering the commutativity and semi-simplicity assumptions for $\mathcal{A}$. Indeed, we prove the following result:
Suppose that $\mathcal{A}$ is a semi-prime Banach $*$-algebra so that $\mathcal{O}_{\mathcal{A}}$ is norm dense in $\mathcal{S}_{\mathcal{A}}$, and $d : \mathcal{A} \to \mathcal{A}$ is a bounded derivation. If $\dim(\bigcap_{\varphi\in\Phi_{\mathcal{A}}} \ker\varphi) \leq 1$, then $d$ is identically zero. In this case, it is possible that $rad(\mathcal{A}) \neq \{0\}$, and it means that $\mathcal{A}$ is not semi-simple.

Let $\{d_n\}$ be a uniformly bounded pre-higher derivation (i.e., $\|d_n\| \leq M$ for some positive number $M$) and $p$ be an arbitrary element of $\mathcal{P}_{\mathcal{A}}$. Then, the function $F$ given by $F(t) = \sum_{n=0}^{\infty} d_n(p)t^n$ is well defined for $|t| < 1$. Indeed,

$$\begin{aligned}\|\sum_{n=0}^{\infty} d_n(p)t^n\| &\leq \sum_{n=0}^{\infty} \|d_n(p)t^n\| = \sum_{n=0}^{\infty} \|d_n(p)\||t^n| \\ &\leq \sum_{n=0}^{\infty} \|d_n\|\|p\||t^n| \leq \sum_{n=0}^{\infty} M\|p\||t^n| < \infty.\end{aligned}$$

Moreover, the m-th derivative of $F$ exists and is given by the formula $F^{(m)}(t) := \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} d_n(p)t^{n-m}$. There is a good match between $F(t)$ and the uniformly bounded pre-higher derivation $\{d_n\}$. Using $F(t)$ the following main result is proved:
Let $\mathcal{A}$ be a Banach $*$-algebra such that $\overline{\mathcal{O}_{\mathcal{A}}} = \mathcal{S}_{\mathcal{A}}$. Suppose that $\{d_n\}$ is a uniformly bounded pre-higher derivation. Then, $d_n(\mathcal{A}) \subseteq \bigcap_{\varphi\in\Phi_{\mathcal{A}}} \ker\varphi$ for each $n \geq 1$. In particular, if $\mathcal{A}$ is semi-prime and further, $\dim(\bigcap_{\varphi\in\Phi_{\mathcal{A}}} \ker\varphi) \leq 1$, then $d_n = 0$ for each $n \geq 1$.

## 2 Results and proofs

Before proving the main results, we present the following lemma:

**Lemma 1** [[1], Lemma 1] *Let $\mathcal{A}$ be a normed complex $*$-algebra. If a linear mapping $\delta$ of $\mathcal{A}$ into a normed $\mathcal{A}$-bimodule $\mathcal{M}$ satisfies* $\delta(p) = \delta(p)p + p\delta(p)$

*for all* $p \in \mathcal{P}_{\mathcal{A}}$*, then* $\delta(w^2) = \delta(w)w + w\delta(w)$ *holds for all* $w \in \mathcal{O}_{\mathcal{A}}$*. Moreover, if* $\mathcal{O}_{\mathcal{A}}$ *is dense in* $\mathcal{S}_{\mathcal{A}}$ *and* $\delta$ *is continuous, then* $\delta$ *is a Jordan derivation.*

Note that each member of $\Phi_{\mathcal{A}}$ is continuous (see [2]). Since the case $\Phi_{\mathcal{A}} = \emptyset$ makes everything trivial, so we will assume that $\Phi_{\mathcal{A}}$ is a non-empty set.

**Theorem 1** *Let* $\mathcal{A}$ *be a Banach* $*$*-algebra such that* $\overline{\mathcal{O}_{\mathcal{A}}} = \mathcal{S}_{\mathcal{A}}$*. Suppose that* $\{d_n\}$ *be a uniformly bounded pre-higher derivation. Then* $d_n(\mathcal{A}) \subseteq \bigcap_{\varphi \in \Phi_{\mathcal{A}}} \ker \varphi$ *for each* $n \geq 1$*. In particular, if* $\mathcal{A}$ *is semi-prime and* $\dim(\bigcap_{\varphi \in \Phi_{\mathcal{A}}} \ker \varphi) \leq 1$*, then* $d_n = 0$ *for each* $n \geq 1$*.*

**Proof.** Let $p$ be an arbitrary element of $\mathcal{P}_{\mathcal{A}}$. We know that the function $F(t) = \sum_{n=0}^{\infty} d_n(p)t^n$ is well-defined for $|t| < 1$. Note that

$$F(t)F(t) = \left(\sum_{n=0}^{\infty} d_n(p)t^n\right)\left(\sum_{n=0}^{\infty} d_n(p)t^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} d_{n-k}(p)d_k(p)\right)t^n$$
$$= \sum_{n=0}^{\infty} d_n(p)t^n = F(t).$$

Hence, $\varphi(F(t)) = 0$ or $\varphi(F(t)) = 1$, where $\varphi$ is an arbitrary fixed element of $\Phi_{\mathcal{A}}$. Let $G(t) := \varphi(F(t))$. We have $G(t) = \varphi(\sum_{n=0}^{\infty} d_n(p)t^n) = \sum_{n=0}^{\infty} \varphi(d_n(p))t^n$. It is observed that $G(t)$ is a power series in $\mathbb{C}$. Thus, the m-th derivative of G exists and is given by the formula $G^{(m)}(t) := \sum_{n=m}^{\infty} \frac{n!}{(n-m)!}\varphi(d_n(p))t^{n-m}$. Since the function G is constant, we have $G^{(m)}(t) = 0$ for every $m \in \mathbb{N}\backslash\{0\}$ and every $|t| < 1$. We have $\varphi(d_1(p)) + 2\varphi(d_2(p))t + 3\varphi(d_3(p))t^2 + 4\varphi(d_4(p))t^3 + \ldots = G^{(1)}(t) = 0$. Putting $t = 0$ in the former equation, we obtain that $\varphi(d_1(p)) = 0$. Using an argument similar to what was described concerning $\varphi(d_1(p))$, we conclude that $\varphi(d_2(p)) = 0$. By continuing this procedure, we prove that $\varphi(d_n(p)) = 0$ for all $n \geq 1$. Our next task is to show that $\varphi(d_n(a)) = 0$ for every $a \in \mathcal{A}$. Let $x$ be an arbitrary element of $\mathcal{O}_{\mathcal{A}}$. Hence, $x = \sum_{i=1}^{m} r_i p_i$, where $p_1, p_2, \ldots, p_m$ are mutually orthogonal self-adjoint projections and $r_1, r_2, \ldots, r_m$ are real numbers. We have $\varphi(d_n(x)) = \varphi(d_n(\sum_{i=1}^{m} r_i p_i)) = \sum_{i=1}^{m} r_i \varphi(d_n(p_i)) = 0$. Since $\overline{\mathcal{O}_{\mathcal{A}}} = \mathcal{S}_{\mathcal{A}}$, $\varphi(d_n(a)) = 0$ for every $a \in \mathcal{S}_{\mathcal{A}}$. It is well-known that each $a$ in $\mathcal{A}$ can be represented as $a = a_1 + ia_2$, $a_1, a_2 \in \mathcal{S}_{\mathcal{A}}$; therefore, $\varphi(d_n(a)) = \varphi(d_n(a_1 + ia_2)) = \varphi(d_n(a_1)) + i\varphi(d_n(a_2)) = 0$ for all $n \geq 1$, $a \in \mathcal{A}$ and $\varphi \in \Phi_{\mathcal{A}}$. It means that $d_n(\mathcal{A}) \subseteq \bigcap_{\varphi \in \Phi_{\mathcal{A}}} \ker \varphi$. Now, suppose that $\dim(\bigcap_{\varphi \in \Phi_{\mathcal{A}}} \ker \varphi) \leq 1$. It is obvious that if $\dim(\bigcap_{\varphi \in \Phi_{\mathcal{A}}} \ker \varphi) = 0$, then $d_n(\mathcal{A}) = \{0\}$ for all $n \geq 1$. Assume that $\dim(\bigcap_{\varphi \in \Phi_{\mathcal{A}}} \ker \varphi) = 1$. First we reduce our discussion to the case $d_1 = 0$. Since $\dim(\bigcap_{\varphi\in\Phi_{\mathcal{A}}} \ker\varphi) = 1$, there exists a non-zero element $x_0$ of $\mathcal{A}$ such that $\bigcap_{\varphi\in\Phi_{\mathcal{A}}} \ker\varphi = \{\alpha x_0 \mid \alpha \in \mathbb{C}\}$. Let $a_0$ be an element of $\mathcal{A}$ so that $d_1(a_0) \neq 0$. We have $d_1(a_0) = \psi(a_0)x_0$, where $\psi$ is a function from $\mathcal{A}$ into the complex numbers. Having put $b = \frac{1}{\psi(a_0)}a_0$, we obtain $d_1(b) = d_1(\frac{1}{\psi(a_0)}a_0) = \frac{1}{\psi(a_0)}\psi(a_0)x_0 = x_0$ and it implies that $\psi(b) = 1$. First we will show $ax_0 + x_0a$ is a scalar multiple of $x_0$ for any $a$ in $\mathcal{A}$. Let $a$ be an element of $\mathcal{A}$. Then, $d_1(a^2) = \psi(a^2)x_0$ (*). Lemma 1 is just what we need to tell us that $d_1$ is a Jordan derivation, i.e. $d_1(a^2) = d_1(a)a + ad_1(a)$ for all $a \in \mathcal{A}$. Using the fact that $d_1$ is a Jordan derivation and the identity $ab+ba = (a+b)^2 - a^2 - b^2$, we get $d_1(ab + ba) = d_1(a)b + ad_1(b) + d_1(b)a + bd_1(a)$ for all $a, b \in \mathcal{A}$. Since $d_1$ is a Jordan derivation and $\dim(\bigcap_{\varphi\in\Phi_{\mathcal{A}}} \ker\varphi) = 1$, we have $d_1(a^2) = d_1(a)a + ad_1(a) = \psi(a)x_0a + a\psi(a)x_0 = \psi(a)(x_0a + ax_0)$ (**). Comparing (*) and (**) , we find that $\psi(a^2)x_0 = \psi(a)(ax_0 + x_0a)$. If $\psi(a) \neq 0$, then $ax_0 + x_0a = \frac{\psi(a^2)}{\psi(a)}x_0$. But if $\psi(a) = 0$, then we have

$$\begin{aligned}\psi(ab + ba)x_0 &= d_1(ab + ba)\\ &= d_1(a)b + ad_1(b) + d_1(b)a + bd_1(a)\\ &= \psi(a)x_0b + a\psi(b)x_0 + \psi(b)x_0a + b\psi(a)x_0\\ &= ax_0 + x_0a.\end{aligned}$$

It means that $ax_0 + x_0a$ is a scalar multiple of $x_0$ for any $a$ in $\mathcal{A}$. Next, it will be shown that $x_0^2 = 0$. Suppose that $\psi(x_0) = 0$. We have $\psi(b^2)x_0 = d_1(b^2) = d_1(b)b + bd_1(b) = \psi(b)x_0b + b\psi(b)x_0 = x_0b + bx_0$. Applying $d_1$ on this equality and then using the fact that $d_1(x_0) = \psi(x_0)x_0 = 0$, we obtain that $x_0^2 = 0$. Now, suppose $\psi(x_0) \neq 0$. We therefore have

$$\psi(x_0^2)x_0 = d_1(x_0^2) = d_1(x_0)x_0 + x_0d_1(x_0) = 2\psi(x_0)x_0^2. \tag{2}$$

If $\psi(x_0^2) = 0$, then it follows from previous equality that $x_0^2 = 0$. Assume that $\psi(x_0^2) \neq 0$; so $x_0^2 = \frac{\psi(x_0^2)}{2\psi(x_0)}x_0$. Simplifying the notation, we put $\lambda = \frac{\psi(x_0^2)}{2\psi(x_0)}$. Replacing $x_0^2$ by $\lambda x_0$ in $2\psi(x_0)x_0^2 = d_1(x_0^2)$, we have $2\psi(x_0)\lambda x_0 = \lambda d_1(x_0) = \lambda\psi(x_0)x_0$. Since $\psi(x_0) \neq 0$, $\lambda x_0 = 0$ and it implies that either $\lambda = 0$ or $x_0 = 0$, which is a contradiction. This contradiction shows that $\psi(x_0^2) = 0$ and by using (2) it is obtained that $x_0^2 = 0$. We know that $x_0a + ax_0 = \mu x_0$, where $\mu \in \mathbb{C}$. Multiplying the previous equality by $x_0$ and using the fact that $x_0^2 = 0$, we see that $x_0ax_0 = 0$ for any $a$ in $\mathcal{A}$. Since $\mathcal{A}$ is semi-prime, $x_0 = 0$. From this contradiction we deduce that $d_1 = 0$. Hence, $d_2(p) = d_2(p)p + pd_2(p) + (d_1(p))^2 = d_2(p)p + pd_2(p)$ for every $p \in \mathcal{P}_{\mathcal{A}}$. Reusing

Lemma 1, we get $d_2$ is a Jordan derivation. Now, by a procedure similar to what was described concerning $d_1$, we obtain that $d_2 = 0$. Consequently, by continuing this procedure, we prove that $d_n = 0$ for all $n \geq 1$. □

An immediate but noteworthy corollary to Theorem 1 is:

**Corollary 1** *Let $\mathcal{A}$ be a semi-prime Banach $*$-algebra such that $\overline{\mathcal{O}_{\mathcal{A}}} = \mathcal{S}_{\mathcal{A}}$. If $\dim(\bigcap_{\varphi \in \Phi_{\mathcal{A}}} \ker \varphi) \leq 1$, then every bounded linear mapping $d : \mathcal{A} \to \mathcal{A}$ satisfying $d(p) = d(p)p + pd(p)$ for all $p \in \mathcal{P}_{\mathcal{A}}$, is identically zero.*

**Proof.** First, let us define a sequence $\{d_n\}$ of linear mappings on $\mathcal{A}$ by $d_0 = I$ and $d_n = \frac{d^n}{n!}$, where I is the identity mapping on $\mathcal{A}$. A straightforward verification shows that $d_n(p) = \sum_{k=0}^{n} d_{n-k}(p)d_k(p)$ for all $p \in \mathcal{P}_{\mathcal{A}}$. We have

$$\|d_n\| = \|\frac{d^n}{n!}\| \leq \frac{1}{n!}\|d\|^n < \sum_{n=0}^{\infty} \frac{\|d\|^n}{n!} = e^{\|d\|}$$

for each non-negative integer $n$. It means that $\{d_n\}$ is a uniformly bounded pre-higher derivation. It follows from Theorem 1 that $0 = d_1 = d$. Furthermore, Lemma 1 implies that every bounded Jordan derivation from $\mathcal{A}$ into $\mathcal{A}$ is zero. □

**Remark 1** *Let $\{d_n\}$ be a higher derivation on an algebra $\mathcal{A}$ with $d_0 = I$, where I is the identity mapping on $\mathcal{A}$. Based on Proposition 2.1 of [7] there is a sequence $\{\delta_n\}$ of derivations on $\mathcal{A}$ such that*

$$(n+1)d_{n+1} = \sum_{k=0}^{n} \delta_{k+1} d_{n-k}$$

*for each non-negative integer $n$. Therefore, we have*

$$\begin{aligned}
&d_0 = I, \\
&d_1 = \delta_1, \\
&2d_2 = \delta_1 d_1 + \delta_2 d_0 = \delta_1\delta_1 + \delta_2, \\
&d_2 = \frac{1}{2}\delta_1^2 + \frac{1}{2}\delta_2, \\
&3d_3 = \delta_1 d_2 + \delta_2 d_1 + \delta_3 d_0 = \delta_1(\frac{1}{2}\delta_1^2 + \frac{1}{2}\delta_2) + \delta_2\delta_1 + \delta_3, \\
&d_3 = \frac{1}{6}\delta_1^3 + \frac{1}{6}\delta_1\delta_2 + \frac{1}{3}\delta_2\delta_1 + \frac{1}{3}\delta_3.
\end{aligned}$$

*Now, assume that* $\{d_n\}$ *is a bounded higher derivation (i.e.,* $d_n$ *is a bounded linear map for every non-negative integer* $n$*). Obviously,* $\delta_1 = d_1$ *is bounded. Hence,* $\delta_2 = 2d_2 - \delta_1^2$ *is also bounded. Based on the* $d_3$ *formula, we have* $\delta_3 = 3d_3 - \frac{1}{2}\delta_1^3 - \frac{1}{2}\delta_1\delta_2 - \delta_2\delta_1$. *Using the boundedness of* $d_3$, $\delta_1$ *and* $\delta_2$, *we obtain that* $\delta_3$ *is a bounded derivation. In the next step, we will show that every* $\delta_n$ *is a bounded derivation for every* $n \in \mathbb{N}$. *To reach this aim, we use induction on* $n$. *According to the above-mentioned discussion,* $\delta_1$, $\delta_2$ *and* $\delta_3$ *are bounded derivations. Now, suppose that* $\delta_k$ *is a bounded derivation for* $k \leq n$. *We will show that* $\delta_{n+1}$ *is also a bounded derivation. Based on the proof of Theorem 2.3 in* [7], *we have*

$$\delta_{n+1} = (n+1)d_{n+1} - \sum_{i=2}^{n+1}\left(\sum_{\sum_{j=1}^{i} r_j = n+1} (n+1)a_{r_1,\ldots,r_i}\delta_{r_1}\ldots\delta_{r_i}\right), \qquad (3)$$

*where the inner summation is taken over all positive integers* $r_j$ *with* $\sum_{j=1}^{i} r_j = n+1$. *From* $\sum_{j=1}^{i} r_j = r_1 + r_2 + \ldots + r_i = n+1$ *along with the condition that* $r_j$ *is a positive integer for every* $1 \leq j \leq i$, *we find that* $1 \leq r_j \leq n$ *for every* $1 \leq j \leq i$. *Since we are assuming* $d_n$ *and* $\delta_k$ *are bounded linear mappings for all non-negative integer* $n$ *and* $k \leq n$, *it follows from* (3) *that* $\delta_{n+1}$ *is a bounded derivation.*

We are now ready for Corollary 2.

**Corollary 2** *Let* $\mathcal{A}$ *be a semi-prime Banach* $*$*-algebra such that* $\overline{\mathcal{O}_{\mathcal{A}}} = \mathcal{S}_{\mathcal{A}}$, *and* $\{d_n\}$ *be a bounded higher derivation from* $\mathcal{A}$ *into* $\mathcal{A}$.
*If* $\dim(\bigcap_{\varphi\in\Phi_{\mathcal{A}}} \ker\varphi) \leq 1$, *then* $d_n = 0$ *for all* $n \in \mathbb{N}$.

**Proof.** Let $\{d_n\}$ be the above-mentioned higher derivation. According to Theorem 2.3 of [7] there exists a sequence $\{\delta_n\}$ of derivations on $\mathcal{A}$ such that

$$d_n = \sum_{i=1}^{n}\Big(\sum_{\sum_{j=1}^{i} r_j = n}\Big(\prod_{j=1}^{i}\frac{1}{r_j + \ldots + r_i}\Big)\delta_{r_1}\ldots\delta_{r_i}\Big)$$

, where the inner summation is taken over all positive integers $r_j$ with $\sum_{j=1}^{i} r_j = n$. It follows from Remark 1 that $\delta_n$ is a bounded derivation for every positive integer $n$. At this point, Corollary 1 completes the proof. □

**Corollary 3** *Let* $\mathcal{A}$ *be a semi-prime Banach* $*$*-algebra such that* $\overline{\mathcal{O}_{\mathcal{A}}} = \mathcal{S}_{\mathcal{A}}$. *If* $\dim(\bigcap_{\varphi\in\Phi_{\mathcal{A}}} \ker\varphi) \leq 1$, *then* $\mathcal{A}$ *is commutative.*

**Proof.** Let $x_0$ be a non-zero arbitrary fixed element of $\mathcal{A}$. Define $d_{x_0} : \mathcal{A} \to \mathcal{A}$ by $d_{x_0}(a) = ax_0 - x_0 a$. Obviously, $d_{x_0}$ is a bounded derivation. It follows from Corollary 1 that $d_{x_0}(a) = 0$, i.e. $ax_0 = x_0 a$ for all $a \in \mathcal{A}$. Since $x_0$ is arbitrary, $\mathcal{A}$ is commutative. □

The above results lead us to the following conjecture:

**Conjecture 1** *Let $\mathcal{A}$ be a semi-prime Banach $*$-algebra such that $\overline{\mathcal{O}_{\mathcal{A}}} = \mathcal{S}_{\mathcal{A}}$. If $\dim(\bigcap_{\varphi \in \Phi_{\mathcal{A}}} \ker \varphi) < \infty$, then every bounded derivation from $\mathcal{A}$ into $\mathcal{A}$ is zero.*

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