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The instability of naked singularities in the gravitational collapse of a scalar field

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1. Introduction

One of the fundamental unanswered questions in the general theory of relativity is whether "naked" singularities, that is singular events which are visible from infinity, may form with positive probability in the process of gravitational collapse. The conjecture that the answer to this question is in the negative has been called "cosmic censorship." The present paper, which is a continuation of the work in [1], [2], [3] and [4] addresses this question in the context of the spherical gravitational collapse of a scalar field.

The problem of a spherically symmetric self-gravitating scalar field is formulated in terms of a 2-dimensional quotient space-time manifold $\mathcal{Q}$ with boundary (see [1]). The boundary of $\mathcal{Q}$ corresponds to the set of fixed points of the group action, the center of symmetry, which is a timelike geodesic Γ. The manifold $\mathcal{Q}$ is endowed with a Lorentzian metric g_{ab}, an area radius function r and a wave function ϕ satisfying the following nonlinear system of partial differential equations:

$$\begin{aligned} r\nabla_a\nabla_b r &= \frac{1}{2}g_{ab}(1-\partial^c r\partial_c r) - r^2 T_{ab} \\ T_{ab} &= \partial_a\phi\partial_b\phi - \frac{1}{2}g_{ab}\partial^c\phi\partial_c\phi \end{aligned} \tag{1.1a}$$

$$\nabla^a(r^2\partial_a\phi) = 0. \tag{1.1b}$$

These imply the following equation for the Gauss curvature of $\mathcal{Q}$:

$$K = r^{-2}(1-\partial^a r\partial_a r) + \partial^a\phi\partial_a\phi. \tag{1.1c}$$

The mass function m is defined by:

$$1 - \frac{2m}{r} = g^{ab}\partial_a r\partial_b r. \tag{1.2}$$

In [2] it was shown that given an initial future light cone with vertex at the center of symmetry and with a region bounded by two spheres such that the ratio of the mass contained in the region to the largest radius is large in comparison to the ratio of the radii minus 1, then a trapped region, namely a region where the future light cones have negative expansion, forms in the future terminating at a strictly spacelike singular boundary. The trapped region contains a sphere whose mass is bounded from below by a positive number depending only on the two initial radii. The results of [2] shall be used in an essential way in the present paper.

Solutions with initial data of bounded variation were considered in [1] and a sharp sufficient condition on the initial data was found for the avoidance of singularities, namely that the total variation be sufficiently small, as well as another condition implying the formation of singularities, complementing the results in [2]. Also, a sharp extension criterion was established, namely that if the ratio of mass to radius of spheres tends to zero as we approach a point at the center of symmetry from its causal past, then the solution extends as a regular solution to include a full neighborhood of the point. The structure of bounded variation solutions was studied and it was shown that at each point in the center of symmetry the solutions are locally scale invariant. Also, the behaviour of the solutions at the singular boundary was analyzed. The present paper relies on the results of [1] as well.

In [3] it was shown that when the final Bondi mass, that is, the infimum of the mass at future null infinity, is different from zero, a black hole forms of mass equal to the final Bondi mass surrounded by vacuum. The rate of growth of the redshift of light seen by faraway observers was determined and the asymptotic wave behaviour at timelike infinity and along the event horizon, the boundary of the past of future null infinity, was analyzed.

In [4] we constructed examples of solutions corresponding to regular asymptotically flat initial data which develop singularities which are not preceeded by a trapped region but have future light cones expanding to infinity. Thus naked singularities do, in fact, occur in the spherical gravitational collapse of a scalar field.

The present paper nevertheless supports the cosmic censorship conjecture. For, we shall show in the following that in the space of initial conditions the subset of initial conditions leading to the formation of naked singularities has, in a certain sense, positive codimension, consequently the occurence of naked singularities is an unstable phenomenon in the context of the spherical self-gravitating scalar field model.

We shall be concerned here with the part of the space-time manifold which lies in the past of the apparent horizon, the past boundary of the trapped region. This part includes the causal past of Γ (see [1]). Consequently, in the region of interest the gradient of the function r is a spacelike outward- directed vectorfield and we may use r as a coordinate. We also use a null coordinate u which is constant along the future-directed null curve from each point on $\Gamma = \partial\mathcal{Q}$ and increases toward the future. In terms of the *Bondi coordinates* u

and r the metric assumes the form:

$$g_{ab}dx^a dx^b = -e^{2\nu}du^2 - 2e^{\nu+\lambda}dudr. \tag{1.3}$$

We shall express the system of equations (1.1a), (1.1b) in Bondi coordinates. It is advantageous to use the pair of null future-directed vectorfields n and l,

$$n = 2e^{-\nu}\frac{\partial}{\partial u} - e^{-\lambda}\frac{\partial}{\partial r} \quad , \qquad l = e^{-\lambda}\frac{\partial}{\partial r}$$
$$(g(n,l) \quad = \quad -2),$$

the integral curves of which are the outgoing and incoming null curves, respectively. We have:

$$lr = e^{-\lambda}, \quad nr = -e^{-\lambda}.$$

We note that the metric function λ has an invariant geometric meaning, for

$$\frac{4}{r^2}e^{-2\lambda}$$

is the square of the length of the mean curvature vector of the spheres which are the orbits of the rotation group in the 4-dimensional space-time manifold. It is also advantageous to express the derivatives of the wave function in terms of

$$\theta = r\left(\frac{l\phi}{lr}\right), \quad \zeta = r\left(\frac{n\phi}{nr}\right). \tag{1.4}$$

We have

$$\theta = r\frac{\partial\phi}{\partial r}, \quad \zeta = -2re^{\lambda-\nu}\frac{\partial\phi}{\partial u} + r\frac{\partial\phi}{\partial r}. \tag{1.5}$$

The mass function m, defined, in general, by (1.2), is, in terms of Bondi coordinates, given by:

$$1 - \frac{2m}{r} = e^{-2\lambda}. \tag{1.6}$$

The components of the energy tensor T_{ab} are:

$$T(n,n) = (n\phi)^2, \quad T(l,l) = (l\phi)^2$$
$$T(n,l) = -\mathrm{tr}T = 0.$$

The trace of equation (1.1a) is

$$r\left(\frac{\partial\nu}{\partial r} - \frac{\partial\lambda}{\partial r}\right) = e^{2\lambda} - 1 \tag{1.7}$$

while the trace-free part of equation (1.1a) reduces to the following pair of equations for m:

$$nm = -(1/2)r^2(lr)T(n,n) = -(1/2)r^2e^{-\lambda}(n\phi)^2$$

$$lm = -(1/2)r^2(nr)T(l,l) = (1/2)r^2e^{-\lambda}(l\phi)^2;$$

that is,

$$2e^{\lambda-\nu}\frac{\partial m}{\partial u} - \frac{\partial m}{\partial r} = -\frac{1}{2}e^{-2\lambda}\zeta^2 \tag{1.8a}$$

$$\frac{\partial m}{\partial r} = \frac{1}{2}e^{-2\lambda}\theta^2. \tag{1.8b}$$

By virtue of (1.7) this last equation can be written as:

$$r\left(\frac{\partial\nu}{\partial r} + \frac{\partial\lambda}{\partial r}\right) = \theta^2. \tag{1.9}$$

Finally, the wave equation (1.1b) takes in Bondi coordinates the form:

$$-2\left(\frac{\partial^2\phi}{\partial u\partial r} + \frac{1}{r}\frac{\partial\phi}{\partial u}\right) + e^{\nu-\lambda}\left[\frac{\partial^2\phi}{\partial r^2} + \left(\frac{2}{r} + \frac{\partial\nu}{\partial r} - \frac{\partial\lambda}{\partial r}\right)\frac{\partial\phi}{\partial r}\right] = 0.$$

In view of (1.7), the wave equation is equivalent to the following pair of equations for θ, ζ:

$$r\left(2e^{\lambda-\nu}\frac{\partial\theta}{\partial u} - \frac{\partial\theta}{\partial r}\right) = (e^{2\lambda}-1)\theta + \zeta \tag{1.10a}$$

$$r\frac{\partial\zeta}{\partial r} = -(e^{2\lambda}-1)\zeta - \theta. \tag{1.10b}$$

Thus ϕ is eliminated and we have a first-order system in the unknowns $\lambda, \nu, \theta, \zeta$.

Given initial data of bounded variation (see [1]) on a future light cone C_0^+ with vertex at the center of symmetry), we consider the maximal sphere S_0 in C_0^+ such that for each sphere S in C_0^+ within S_0 there is a point P on the central geodesic Γ whose past light cone C_P^- intersects C_0^+ at S, and we have a development of bounded variation in the region bounded by C_P^- and C_0^+. Let C_0^- be the incoming null hypersurface through S_0 in the future of C_0^+. Then, as we have shown in [1], the solution extends to C_0^-, however C_0^- cannot terminate at a regular vertex on Γ. We can choose the coordinate u so that

$$u = -2r$$

along the incoming null curve corresponding to C_0^- in the quotient space-time $\mathcal{Q}$. The incoming null curves satisfy the equation:

$$\frac{dr}{du} = -\frac{1}{2}e^{\nu-\lambda}. \tag{1.11}$$

Therefore we have

$$e^{\nu-\lambda}\Big|_{u=-2r} = 1. \tag{1.12}$$

The origin $(0,0)$ in the coordinates u, r set up in this way cannot correspond to a regular point. Let the sphere S_0 correspond to $r = a$ $(u = -2a)$. We define dimensionless coordinates t, s by:

$$u = -2ae^{-t}, \quad r = ae^{s-t}. \tag{1.13}$$

Then t is constant along the outgoing null curves while

$$\frac{ds}{dt} = \beta, \qquad \beta = 1 - e^{\nu-\lambda-s}, \tag{1.14}$$

along the incoming null curves. Since $s = 0$ corresponds to C_0^-,

$$\beta\Big|_{s=0} = 0. \tag{1.15}$$

The future of C_0^+ corresponds to $t > 0$, and the interior of C_0^- corresponds to $s < 0$, the exterior to $s > 0$. We have:

$$r\frac{\partial}{\partial r} = \frac{\partial}{\partial s}, \tag{1.16a}$$

$$r\left(2e^{\lambda-\nu}\frac{\partial}{\partial u} - \frac{\partial}{\partial r}\right) = \frac{1}{(1-\beta)}\left(\frac{\partial}{\partial t} + \beta\frac{\partial}{\partial s}\right), \tag{1.16b}$$

while

$$\frac{\partial}{\partial t} = -u\frac{\partial}{\partial u} - r\frac{\partial}{\partial r}. \tag{1.16c}$$

With

$$\kappa = e^{2\lambda}, \tag{1.17}$$

equations (1.6), (1.8a), 1.8b), (1.10a), (1.10b) take in the dimensionless coordinates of the form:

$$\frac{\partial \kappa}{\partial t} + \beta\frac{\partial \kappa}{\partial s} = (1-\beta)\kappa(\kappa - 1 - \zeta^2) \tag{1.18a}$$

$$\frac{\partial \kappa}{\partial s} = \kappa(1 - \kappa + \theta^2) \tag{1.18b}$$

$$\frac{\partial \beta}{\partial s} = (1-\beta)(2-\kappa) \tag{1.18c}$$

$$\frac{\partial \theta}{\partial t} + \beta\frac{\partial \theta}{\partial s} = (1-\beta)[(\kappa-1)\theta + \zeta] \tag{1.18d}$$

$$\frac{\partial \zeta}{\partial s} = -(\kappa-1)\zeta - \theta. \tag{1.18e}$$

Let us denote by a subscript 0 the restriction to $s = 0$. Then by virtue of (1.15), that is

$$\beta_0 = 0, \tag{1.19}$$

the restrictions of equations (1.18a) and (1.18d) to $s = 0$ are:

$$\frac{d\kappa_0}{dt} = \kappa_0(\kappa_0 - 1 - \zeta_0^2) \tag{1.20a}$$

$$\frac{d\theta_0}{dt} = (\kappa_0 - 1)\theta_0 + \zeta_0. \tag{1.20b}$$

Equations (1.18b) and (1.18c) are equivalent to equations (1.9) and (1.7) for ν and λ, which in the dimensionless coordinates read:

$$\frac{\partial\nu}{\partial s} + \frac{\partial\lambda}{\partial s} = \theta^2 \tag{1.21a}$$

$$\frac{\partial\nu}{\partial s} - \frac{\partial\lambda}{\partial s} = e^{2\lambda} - 1. \tag{1.21b}$$

By (1.14), (1.17) and (1.19),

$$\nu_0 = \lambda_0 = \frac{1}{2}\log\kappa_0. \tag{1.22}$$

Hence integrating (1.21a) from $s = 0$ yields:

$$(\nu + \lambda)(t, s) = \log\kappa_0(t) + \int_0^s \theta^2(t, s')ds'. \tag{1.23a}$$

Also, writing (1.21b) in the form

$$\frac{\partial e^{\nu-\lambda}}{\partial s} = e^{\nu+\lambda} - e^{\nu-\lambda},$$

or

$$\frac{\partial e^{\nu-\lambda-s}}{\partial s} = e^{\nu+\lambda+s}$$

and integrating from $s = 0$ yields:

$$e^{(\nu-\lambda)(t,s)+s} = 1 + \int_0^s e^{(\nu+\lambda)(t,s')+s'}ds'. \tag{1.23b}$$

Using (1.21b) we can write (1.18e) in the form:

$$\frac{\partial(e^{\nu-\lambda}\zeta)}{\partial s} = -e^{\nu-\lambda}\theta.$$

Hence, integrating from $s = 0$ we obtain

$$(e^{\nu-\lambda}\zeta)(t, s) = \zeta_0(t) + \xi(t, s) \tag{1.24a}$$

where

$$\xi(t, s) = -\int_0^s (e^{\nu-\lambda}\theta)(t, s')ds'. \tag{1.24b}$$

Let us define the mass ratio

$$\mu = \frac{2m}{r}. \tag{1.25}$$

We then have,

$$\kappa = \frac{1}{1-\mu} \tag{1.26}$$

(see (1.6)). The fact that μ is nonnegative (see [1]) implies:

$$\kappa \geq 1. \tag{1.27a}$$

In particular,

$$\kappa_0 \geq 1. \tag{1.27b}$$

2. The first instability theorem

In the following we confine attention to the exterior of C_0^- and the future of $C_0^+ \, : \, s,t > 0$. Let us define

$$\gamma(t) = \int_0^t (\kappa_0(t') - 1)dt'. \tag{1}$$

Lemma 1.

$$\kappa_0(t) \leq 2\kappa_0(0)e^{\gamma(t)}.$$

Proof. According to (1.8a), m is nonincreasing along incoming null curves. Consequently, by (1.13) and (1.25) the function $\mu_0 e^{-t}$ is nondecreasing; hence, by (1.26),

$$\kappa_0(t) \leq \frac{1}{1-\left(1-\frac{1}{\kappa_0(0)}\right)e^t} \tag{2.2a}$$

provided that

$$e^t < \frac{1}{1-\frac{1}{\kappa_0(0)}}.$$

On the other hand, since $t' \leq t$ implies

$$\mu_0(t')e^{-t'} \geq \mu_0(t)e^{-t},$$

we also have that

$$\kappa_0(t') - 1 = \frac{\mu_0(t')}{1-\mu_0(t')} \geq \frac{\mu_0(t)e^{t'-t}}{1-\mu_0(t)e^{t'-t}}.$$

Hence,

$$\begin{aligned} \gamma(t) &\geq \int_0^t \frac{\mu_0(t)e^{t'-t}}{1-\mu_0(t)e^{t'-t}}dt' \\ &= \log\left[\frac{1-\mu_0(t)e^{-t}}{1-\mu_0(t)}\right] > \log\left[\frac{1-e^{-t}}{1-\mu_0(t)}\right] \end{aligned}$$

since $\mu_0 < 1$; that is,

$$\kappa_0(t) < \frac{e^{\gamma(t)}}{1-e^{-t}}. \tag{2.2b}$$

By setting

$$t_1 = \log\left[\frac{1-\frac{1}{2\kappa_0(0)}}{1-\frac{1}{\kappa_0(0)}}\right],$$

we see that (2.2a) holds for $t \in [0, t_1]$, which yields

$$\kappa_0(t) \leq \frac{1}{1-\left(1-\frac{1}{\kappa_0(0)}\right)e^{t_1}} = 2\kappa_0(0)$$

for $t \in [0, t_1]$, while by (2.2b)

$$\kappa_0(t) < \frac{e^{\gamma(t)}}{1-e^{-t_1}} = (2\kappa_0(0)-1)e^{\gamma(t)}$$

for $t \in (t_1, \infty)$.

Since γ is a nondecreasing function, either γ is bounded, in which case it tends to a finite limit $\gamma(\infty)$ as $t \to \infty$, or γ is unbounded, in which case $\gamma(t) \to \infty$ as $t \to \infty$.

Lemma 2. *If γ is bounded then $\mu_0(t) \to 0$ as $t \to \infty$.*

Proof. Let $t > t_0$. Following the proof of Lemma 1 we obtain

$$\begin{aligned} \gamma(t) - \gamma(t_0) &\geq \int_{t_0}^{t} \frac{\mu_0(t)e^{t'-t}}{1-\mu_0(t)e^{t'-t}}dt' \\ &= \log\left[\frac{1-\mu_0(t)e^{t_0-t}}{1-\mu_0(t)}\right]; \end{aligned}$$

that is,

$$\frac{\mu_0(t)(1-e^{t_0-t})}{1-\mu_0(t)} \leq e^{\gamma(t)-\gamma(t_0)} - 1$$

which implies

$$\mu_0(t) \leq \frac{e^{\gamma(t)-\gamma(t_0)}-1}{1-e^{t_0-t}}. \tag{2.3}$$

The result follows by setting $t_0 = t - 1$ in (2.3), in view of the fact that γ bounded implies $\gamma(t) - \gamma(t-1) \to 0$ as $t \to \infty$.

As we have noted, the incoming null hypersurface C_0^- cannot terminate at a regular vertex on Γ. The past of C_0^- is a *terminal indecomposable past set* in the terminology of [5]. In view of Lemma 2 and the extension criterion mentioned in the introduction, we shall assume in the following that the function

γ is unbounded. Defining another null coordinate v which is constant along the incoming null curves and satisfies

$$v = 2(r - a)$$

along the outgoing null curve corresponding to C_0^+, we see that $\mathcal{Q}$ becomes a domain in the u, v plane in which $u < 0$, C_0^- corresponds to the line $v = 0$ in $\mathcal{Q}$, and the terminal indecomposable past set to the origin O which lies on the boundary of $\mathcal{Q}$ in the u, v plane. The point O is the past end point of the *central component* $\mathcal{B}_0$ of the *singular boundary* $\mathcal{B}$ of $\mathcal{Q}$ (see [1]). The function r extends continuously to O where it vanishes.

The *apparent horizon* $\mathcal{A}$ is the set of points of $\mathcal{Q}$ at which $\partial r/\partial v = 0$. Each point of $\mathcal{A}$ corresponds to a sphere which has maximal area in the future light cone with vertex on Γ in which it is contained. According to the results of [1], $\mathcal{A}$, if nonempty, is a spacelike curve which may contain outgoing null segments but does not contain incoming null segments. In fact $\mathcal{A}$ is given by

$$\mathcal{A} = \{(u, v_0(u)) \,:\, u \in (\underline{u}^*, 0)\} \bigcup \left(\bigcup_n \{u_n\} \times I_n \right), \tag{2.4a}$$

where v_0 is a strictly decreasing function in $(\underline{u}^*, 0)$, $\underline{u}^* > -2a$, and the intervals

$$I_n = \left(\lim_{u \to u_n^+} v_0(u), \lim_{u \to u_n^-} v_0(u) \right) \tag{2.4b}$$

correspond to the (denumerable) points of discontinuity of v_0. Also,

$$v_0(u) \to \infty \ \text{ as } \ u \to \underline{u}^*. \tag{2.4c}$$

The future light cone, with vertex on Γ, which corresponds to the outgoing null curve $u = \underline{u}^*$ is the *event horizon* $\mathcal{H}$ (see [2]).

Moreover, $\mathcal{A}$ can equivalently be defined as the set of points of $\mathcal{Q}$ at which $\mu = 1$. The past of $\mathcal{A}$ in $\mathcal{Q}$, the domain of the u, r coordinates, is the region where $\mu < 1$ and the future light cones have positive expansion: $\partial r/\partial v > 0$, while the future of $\mathcal{A}$ in $\mathcal{Q}$ is the *trapped region* $\mathcal{T}$, where $\mu > 1$ and the future light cones have negative expansion: $\partial r/\partial v < 0$. The future boundary of $\mathcal{T}$ is the *noncentral component* $\mathcal{B} \setminus \mathcal{B}_0$ of the singular boundary $\mathcal{B}$. The function r extends continuously to $\mathcal{B} \setminus \mathcal{B}_0$ where it vanishes. According to the results of [1], $\mathcal{B} \setminus \mathcal{B}_0$ is a strictly spacelike C^1 curve, given by

$$\mathcal{B} \setminus \mathcal{B}_0 = \{(u, v^*(u)) \,:\, u \in (\underline{u}^*, 0)\}. \tag{2.5}$$

Here v^* is a strictly decreasing C^1 function in $(\underline{u}^*, 0)$, $v^* > v_0$. Letting

$$\underline{v}^* = \lim_{u \to 0} v^*(u), \tag{2.6}$$

the central component $\mathcal{B}_0$ of the singular boundary $\mathcal{B}$ is given by

$$\mathcal{B}_0 = \{(0, v) \,:\, v \in [0, \underline{v}^*]\} \tag{2.7a}$$

if $\mathcal{A}$ is nonempty, and

$$\mathcal{B}_0 = \{(0,v) \,:\, v \in [0,\infty)\} \tag{2.7b}$$

if $\mathcal{A}$ is empty. Also, if $\mathcal{A}$ is nonempty we have:

$$\underline{v}_0 = \lim_{u\to 0} v_0(u) \in [0, \underline{v}^*]. \tag{2.8}$$

In [4] we constructed examples where $\mathcal{A}$ is empty and the solutions have a regular extension to $\mathcal{B}_0 \setminus O$, with $r \to \infty$ as $v \to \infty$ on $\mathcal{B}_0$. Then O is a *naked singularity*. We also constructed examples where $\mathcal{B}_0 \setminus O$ is nonempty with r extending continuously to $\mathcal{B}_0$ where it vanishes. Then $\mathcal{B}_0$ corresponds to a *singular future null cone* which has collapsed to a line (see [4]).

Integrating equation (1.20b) yields:

$$\theta_0(t) = e^{\gamma(t)}(\theta_0(0) - I(t)) \tag{2.9a}$$

where

$$I(t) = -\int_0^t e^{-\gamma(t')}\zeta_0(t')dt'. \tag{2.9b}$$

The aim of the present section is the proof of the following theorem.

Theorem 2.1. *Let γ be unbounded. Suppose that either I does not tend to a finite limit as $t \to \infty$ or, otherwise,*

$$\theta_0(0) \neq \lim_{t\to\infty} I(t).$$

Then $\mathcal{A}$ is nonempty,

$$\underline{v}_0 = \underline{v}^* = 0,$$

so that $\mathcal{B}_0 = O$ and both $\mathcal{A}$ and $\mathcal{B} \setminus \mathcal{B}_0$ issue from O.

We recall the following theorem, which was proved in [2].

Theorem*. *Let C_0^+ be a future light cone with vertex on Γ and consider the annular region in C_0^+ bounded by two spheres $S_{1,0}$ and $S_{2,0}$, with $S_{2,0}$ in the exterior of $S_{1,0}$. Let δ_0 and η_0 be the dimensionless size and dimensionless mass content of the region, defined by*:

$$\delta_0 = \frac{r_{2,0}}{r_{1,0}} - 1, \quad \eta_0 = \frac{2(m_{2,0} - m_{1,0})}{r_{2,0}}.$$

Let C_1^- and C_2^- be the incoming null hypersurfaces through $S_{1,0}$ and $S_{2,0}$ and consider the spheres S_1 and S_2 at which C_1^- and C_2^- intersect future light cones C^+ with vertices on Γ in the future of C_0^+. Then there are positive constants $c_0 \leq 1/e$ and $c_1 \geq 1$ such that if $\delta_0 \leq c_0$ and

$$\eta_0 > c_1 \delta_0 \log\left(\frac{1}{\delta_0}\right),$$

then S_2 intersects an apparent horizon before S_1 reduces to a point on Γ, that is, there is a future light cone C_*^+ such that S_{2*} is a maximal sphere in C_*^+ while $r_{1*} > 0$.

In the present context we may apply Theorem* with C_0^- in the role of C_1^-, any future light cone C^+ intersecting C_0^- in the future of C_0^+ in the role of C_0^+, and any incoming null hypersurface C^-, respecting the spherical symmetry in the exterior of C_0^- in the role of C_2^-. Denoting

$$\eta = \frac{2(m - m_0)}{r} = \mu - \mu_0 e^{-s}, \tag{2.10}$$

we conclude that there are positive constants c_0 and c_1 such that if at some (t_0, s_0) with $t_0 \geq 0$ and $s_0 \in (0, c_0]$ we have

$$\eta(t_0, s_0) > c_1 s_0 \log\left(\frac{1}{s_0}\right),$$

then there is a $t_* \in (t_0, \infty)$ such that the incoming null curve through (t_0, s_0) intersects an apparent horizon at $t = t_*$, so $\kappa \to \infty$ along this curve as $t \to t_*$.

Now if Theorem 2.1 is not true then there is an $\varepsilon > 0$ such that for each $s_0 \in [0, \varepsilon]$ the incoming null curve $s = \chi(t; s_0)$ through $s = s_0$ at $t = 0$ does not intersect an apparent horizon at finite t. Let us denote by $\mathcal{R}_\varepsilon$ the region in the half-plane $t \geq 0$,

$$\mathcal{R}_\varepsilon = \{(t, s) \,:\, t \in [0, \infty) \,\&\, s \in [0, \chi(t; \varepsilon)]\} \tag{2.11a}$$

bounded by the incoming null curves $s = 0$ and $s = \chi(t; \varepsilon)$. Given any positive constant c let us set

$$\mathcal{R}_\varepsilon^c = \{(t, s) \in \mathcal{R}_\varepsilon \,:\, s \leq c\}. \tag{2.11b}$$

Then according to the above,

$$\eta \leq c_1 s \log\left(\frac{1}{s}\right) \;:\; \text{in } \mathcal{R}_\varepsilon^{c_0}. \tag{2.12}$$

Let us now consider equation (1.18d) and define:

$$\psi = e^{-\gamma}(\theta e^s - \theta_0). \tag{2.13a}$$

Using (1.24a) and (1.14) we derive from (1.18d) and (1.20b) the following equation of evolution of ψ along incoming null curves:

$$\frac{\partial \psi}{\partial t} + \beta \frac{\partial \psi}{\partial s} = \omega \psi + \rho, \tag{2.13b}$$

where

$$\omega = (1 - \beta)(\kappa - 2) - (\kappa_0 - 2) \tag{2.13c}$$

and

$$\rho = e^{-\gamma}(\omega \theta_0 + \xi). \tag{2.13d}$$

We shall presently derive estimates for ω and ξ, making use of Lemma 1 as well as the bound (2.12). We begin with an upper bound for κ. We have:

$$\begin{aligned}\kappa(t,s) &= \frac{1}{1-\mu(t,s)} = \frac{1}{1-\mu_0(t)e^{-s}-\eta(t,s)} \\ &\leq \frac{1}{\frac{1}{\kappa_0(t)}-\eta(t,s)} \leq \frac{1}{\frac{e^{-\gamma(t)}}{2\kappa_0(0)}-c_1 s\log\left(\frac{1}{s}\right)}\end{aligned}$$

in $\mathcal{R}_\varepsilon^{c_0}$. Thus $(t,s)\in\mathcal{R}_\varepsilon^{c_0}$ and

$$s\log\left(\frac{1}{s}\right) \leq \frac{e^{-\gamma(t)}}{4c_1\kappa_0(0)} \tag{2.14}$$

implies

$$\kappa(t,s) \leq \kappa_0(0)e^{\gamma(t)}. \tag{2.15}$$

Next, we obtain bounds for β. From (1.21b) and the boundary condition $\nu_0-\lambda_0=0$ (see (1.22)),

$$(\nu-\lambda)(t,s) = \int_0^s (\kappa(t,s')-1)ds'. \tag{2.16}$$

Substituting the estimate (2.15) we obtain, for $(t,s)\in\mathcal{R}_\varepsilon^{c_0}$,

$$(\nu-\lambda)(t,s) \leq 4\kappa_0(0)e^{\gamma(t)}s \tag{2.17a}$$

provided that (2.14) holds. Hence,

$$e^{(\nu-\lambda)(t,s)-s} \leq e^{4\kappa_0(0)e^{\gamma(t)}s}, \tag{2.17b}$$

and since for $x\in[0,c]$, $c>0$, we have

$$e^x \leq e^c, \quad e^x-1 \leq \frac{(e^c-1)}{c}x,$$

while (2.14) implies

$$4\kappa_0(0)e^{\gamma(t)}s \leq \frac{1}{c_1}, \tag{2.18}$$

(recall that $s\leq c_0\leq 1/e$), also the following hold:

$$e^{(\nu-\lambda)(t,s)-s} \leq e^{1/c_1} \tag{2.19a}$$

$$e^{(\nu-\lambda)(t,s)-s}-1 \leq 4c_1(e^{1/c_1}-1)\kappa_0(0)e^{\gamma(t)}s. \tag{2.19b}$$

On the other hand for $s\geq 0$, we have

$$(\nu-\lambda)(t,s) \geq 0; \tag{2.20a}$$

hence,

$$1-e^{(\nu-\lambda)(t,s)-s} \leq 1-e^{-s} \leq s. \tag{2.20b}$$

From (2.19a), (2.19b), (2.20b) we conclude, recalling (1.14), that if $(t,s) \in \mathcal{R}_\varepsilon^{c_0}$ and (2.14) holds, then

$$0 < 1 - \beta(t,s) \le e^{1/c_1} \tag{2.21a}$$

$$|\beta(t,s)| \le 4c_1(e^{1/c_1} - 1)\kappa_0(0)e^{\gamma(t)}s. \tag{2.21b}$$

To obtain an estimate for ω (see (2.13c)), we write:

$$\omega = (1-\beta)(\kappa - \kappa_0) + \beta(2 - \kappa_0). \tag{2.22}$$

We have,

$$\kappa - \kappa_0 = \kappa\kappa_0(\mu - \mu_0) = \kappa\kappa_0(\eta - \mu_0(1 - e^{-s})).$$

Using Lemma 1 and estimates (2.12), (2.15), yields:

$$|(\kappa - \kappa_0)(t,s)| \le 8\kappa_0(0)c_1 e^{2\gamma(t)} s \log\left(\frac{1}{s}\right) \tag{2.23}$$

for $(t,s) \in \mathcal{R}_\varepsilon^{c_0}$ where (2.16) holds. Estimates (2.21a), (2.21b), (2.23), allow us to conclude, in view of the expression (2.22) that:

$$\begin{aligned} |\omega(t,s)| &\le c_2(\kappa_0(0))^2 e^{2\gamma(t)} s \log\left(\frac{1}{s}\right) \\ c_2 &= 16c_1 e^{1/c_1} \end{aligned} \tag{2.24}$$

for $(t,s) \in \mathcal{R}_\varepsilon^{c_0}$ where (2.14) holds.

To obtain an estimate for ξ we consider the expression (1.24b). By the Schwarz inequality,

$$\xi^2(t,s) \le \int_0^s e^{s' - 2\lambda(t,s')}\theta^2(t,s')ds' \cdot \int_0^s e^{-s' + 2\nu(t,s')}ds'. \tag{2.25}$$

Now according to (1.18b),

$$\frac{\partial(e^s\mu)}{\partial s} = e^{s-2\lambda}\theta^2; \tag{2.26a}$$

thus, in view of the definition (2.10),

$$e^s\eta(t,s) = \int_0^s e^{s'-2\lambda(t,s')}\theta^2(t,s')ds'. \tag{2.26b}$$

This is the first integral on the right in (2.25). Let us define

$$\delta(t,s) = e^{s - 2\nu(t,s)} \int_0^s e^{-s' + 2\nu(t,s')}ds'. \tag{2.27a}$$

By (1.21a), (1.21b),

$$\frac{\partial\nu}{\partial s} = \frac{1}{2}(e^{2\lambda} - 1 + \theta^2) \ge 0; \tag{2.27b}$$

hence for $s \geq 0$

$$\delta(t,s) \leq e^{s} \int_0^s e^{-s'} ds' = e^s - 1 \tag{2.27c}$$

and the second integral on the right in (2.25),

$$\int_0^s e^{-s'+2\nu(t,s')} ds \leq e^{2\nu(t,s)}(1 - e^{-s}). \tag{2.27d}$$

From (2.25), (2.26b), (2.27d), we conclude that

$$\xi^2 \leq \eta e^{2\nu}(e^s - 1). \tag{2.28}$$

Writing

$$e^{2\nu} = \kappa e^{2(\nu-\lambda)},$$

and using (2.15) and the fact that by (2.17a), (2.18),

$$e^{\nu-\lambda} \leq e^{1/c_1}, \tag{2.29a}$$

we obtain

$$e^{2\nu} \leq 4 e^{2/c_1} \kappa_0(0) e^{\gamma(t)} \tag{2.29b}$$

in $\mathcal{R}_\varepsilon^{c_0}$ where (2.14) holds. Hence, using in addition estimate (2.12) we conclude that

$$\xi^2 \leq 4 c_1 e^{2/c_1} \kappa_0(0) e^{\gamma(t)} (e^s - 1) s \log\left(\frac{1}{s}\right). \tag{2.30a}$$

This also implies that

$$|\xi(t,s)| \leq c_2 (\kappa_0(0))^2 e^{2\gamma(t)} s \log\left(\frac{1}{s}\right) \tag{2.30b}$$

(see (2.24)) for $(t,s) \in \mathcal{R}_\varepsilon^{c_0}$ where (2.14) holds.

We now begin the proof of Theorem 2.1. Since nothing *a priori* is known about the asymptotic behaviour of the integral I defined by (2.9b), we must consider all possibilities. Let

$$l_+ = \limsup_{t\to\infty} I(t) \qquad l_- = \liminf_{t\to\infty} I(t). \tag{2.31}$$

Then any $l \in [l_-, l_+]$ is a limit value of I. The following seven cases exhaust all possibilities:

Case 1: $-\infty < l_- = l_+ < \infty$. In this case

$$I(t) \to l \quad \text{as} \quad t \to \infty \quad (l = l_- = l_+)$$

and the hypothesis of Theorem 2.1 states that

$$\theta_0(0) \neq l.$$

Case 2: $l_- = l_+ = \infty$.

Case 3: $l_- = l_+ = -\infty$.
Case 4: $-\infty < l_- < l_+ < \infty$.
Case 5: $-\infty < l_- < l_+ = \infty$.
Case 6: $-\infty = l_- < l_+ < \infty$.
Case 7: $-\infty = l_- < l_+ = \infty$.

In Cases 1 and 4 the integral I is bounded, hence so is $\theta_0 e^{-\gamma}$. Let us set, in these cases,

$$b = \sup_{t\in[0,\infty)} \left|\theta_0(t)e^{-\gamma(t)}\right|. \tag{2.32}$$

In Case 1, setting

$$h = |\theta_0(0) - l| \tag{2.33a}$$

we can find a $T > 0$ such that

$$|I_0(t) - l| \leq \frac{2h}{3} \quad : \quad \text{for all } t \geq T;$$

hence,

$$|\theta_0(t)| \geq \frac{h}{3}e^{\gamma(t)} \quad : \quad \text{for all } t \geq T. \tag{2.33b}$$

In Case 4, setting

$$h = l_+ - l_- \tag{2.34a}$$

we have

$$\max\{|\theta_0(0) - l_-|, |\theta_0(0) - l_+|\} \geq \frac{h}{2}.$$

It follows that there is an increasing sequence $(t_n : n = 1, 2, ...)$, $t_n \to \infty$ as $n \to \infty$, such that

$$|\theta_0(0) - I(t_n)| \geq \frac{h}{3};$$

hence,

$$|\theta_0(t_n)| \geq \frac{h}{3}e^{\gamma(t_n)} \quad : n = 1, 2, \ldots. \tag{2.34b}$$

We shall treat Cases 1 and 4 first. From (2.13d) and estimates (2.24) and (2.30b) we have, in view of (2.32),

$$|\rho(t, s)| \leq c_2(\kappa_0(0))^2(b + 1)e^{2\gamma(t)}s \log\left(\frac{1}{s}\right) \tag{2.35}$$

in $\mathcal{R}_\varepsilon^{c_0}$ where (2.14) holds. Denoting by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \beta\frac{\partial}{\partial s},$$

the derivative along incoming null curves with respect to the parameter t, we then obtain from (2.13b),

$$\frac{d|\psi|}{dt} \le c_2(\kappa_0(0))^2 e^{2\gamma(t)} s \log\left(\frac{1}{s}\right)(|\psi| + b + 1) \tag{2.36}$$

along the incoming null curve

$$s = \chi(t; s_0) \qquad (\chi(0; s_0) = s_0) \tag{2.37}$$

through $s = s_0 \in [0, \varepsilon]$ at $t = 0$, provided that $s \le c_0$ and (2.14) holds. Let us define:

$$\varphi(t; s_0) = c_2(\kappa_0(0))^2 \int_0^t e^{2\gamma(t')} \left[s \log\left(\frac{1}{s}\right)\right]_{s=\chi(t';s_0)} dt'. \tag{2.38}$$

Integrating (2.36) along the incoming null curve (2.37) then yields:

$$|\psi(t, \chi(t; s_0))| \le e^{\varphi(t;s_0)}|\psi(0, s_0)| + (b + 1)(e^{\varphi(t;s_0)} - 1) \tag{2.39}$$

provided that

$$\chi(t'; s_0) \le c_0 \tag{2.40a}$$

and (condition 2.14)

$$\left[s \log\left(\frac{1}{s}\right)\right]_{s=\chi(t';s_0)} \le \frac{e^{-\gamma(t')}}{4c_1\kappa_0(0)} \tag{2.40b}$$

hold for all $t' \in [0, t]$.

To proceed we must estimate $\varphi(t; s_0)$. This requires the bound

$$\kappa(t, s) \le \frac{3}{2}\kappa_0(t) \quad : \quad \text{for all} \quad s \in [0, \chi(t; s_0)]. \tag{2.41a}$$

Now, by (2.12),

$$\begin{aligned} \mu(t, s) &= \mu_0(t)e^{-s} + \eta(t, s) \\ &\le \mu_0(t) + c_1 s \log\left(\frac{1}{s}\right) \end{aligned}$$

for $(t, s) \in \mathcal{R}_\varepsilon^{c_0}$, while (2.41a) is equivalent to:

$$\mu(t, s) \le \frac{1}{3} + \frac{2}{3}\mu_0(t) \quad : \quad \text{for all} \quad s \in [0, \chi(t; s_0)].$$

Consequently, (2.41a) holds when

$$c_1 s \log\left(\frac{1}{s}\right) \le \frac{1}{3}(1 - \mu_0(t)) = \frac{1}{3\kappa_0(t)}.$$

Thus, by Lemma 1, the bound (2.41a) follows if

$$\left[s \log\left(\frac{1}{s}\right)\right]_{s=\chi(t';s_0)} \le \frac{e^{-\gamma(t')}}{6c_1\kappa_0(0)}. \tag{2.41b}$$

We note that condition (2.41b) is stronger than condition (2.40b), so only condition (2.41b) together with condition (2.40a) need be considered from this point.

Substituting the bound (2.41a) in (2.16) yields:

$$(\nu - \lambda)(t, \chi(t; s_0)) \leq \left(\frac{3}{2}\kappa_0(t) - 1\right)\chi(t; s_0); \tag{2.42}$$

hence,

$$\frac{d\chi(t; s_0)}{dt} = \beta(t, \chi(t; s_0)) \geq 1 - e^{\left(\frac{3}{2}\kappa_0(t) - 2\right)\chi(t; s_0)}. \tag{2.43}$$

Now,

$$f(x) = \begin{cases} (e^x - 1 - x)/x^2 & : x \neq 0 \\ 1/2 & : x = 0 \end{cases}$$

is a continuous strictly increasing function on the real line. Thus $x \leq 1$ implies

$$f(x) \leq f(1) = e - 2;$$

that is,

$$e^x - 1 \leq x + (e - 2)x^2.$$

Since condition (2.41b) implies that

$$\left(\frac{3}{2}\kappa_0(t) - 2\right)\chi(t; s_0) \leq 1,$$

it follows that

$$1 - e^{\left(\frac{3}{2}\kappa_0 - 2\right)\chi} \geq -\left(\frac{3}{2}\kappa_0 - 2\right)\chi - (e - 2)\left(\frac{3}{2}\kappa_0 - 2\right)^2\chi^2,$$

and from (2.43),

$$\frac{d\chi}{dt} \geq -\left(\frac{3}{2}\kappa_0 - 2\right)\chi - (e - 2)\left(\frac{3}{2}\kappa_0 - 2\right)^2\chi^2, \tag{2.44a}$$

or

$$\frac{d}{dt}\left(e^{-\frac{3}{2}\gamma + \frac{1}{2}t}\frac{1}{\chi}\right) \leq (e - 2)e^{-\frac{3}{2}\gamma + \frac{1}{2}t}\left(\frac{3}{2}\kappa_0 - 2\right)^2. \tag{2.44b}$$

Let $0 \leq t_0 \leq t_1$. Integrating (2.44b) on $[t_0, t_1]$ yields:

$$e^{-\frac{3}{2}\gamma(t_1) + \frac{1}{2}t_1}\frac{1}{\chi(t_1)} - e^{-\frac{3}{2}\gamma(t_0) + \frac{1}{2}t_0}\frac{1}{\chi(t_0)} \leq g(t_0, t_1), \tag{2.44c}$$

where

$$g(t_0, t_1) = (e - 2)\int_{t_0}^{t_1} e^{-\frac{3}{2}\gamma + \frac{1}{2}t}\left(\frac{3}{2}\kappa_0 - 2\right)^2 dt. \tag{2.44d}$$

We have

$$\left(\frac{3}{2}\kappa_0 - 2\right)^2 \leq \frac{9}{4}(\kappa_0 - 1)^2 + \frac{1}{4} \tag{2.45a}$$

and

$$\int_{t_0}^{t_1} e^{-\frac{3}{2}\gamma+\frac{1}{2}t} dt \leq 2e^{-\frac{3}{2}\gamma(t_0)}\left(e^{\frac{1}{2}t_1} - e^{\frac{1}{2}t_0}\right) \tag{2.45b}$$

while, by Lemma 1,

$$\begin{aligned}\int_{t_0}^{t_1}(\kappa_0-1)^2 e^{-\frac{3}{2}\gamma+\frac{1}{2}t}dt &= \int_{t_0}^{t_1}(\kappa_0-1)\frac{d\gamma}{dt}e^{-\frac{3}{2}\gamma+\frac{1}{2}t}dt \\ &\leq 2\kappa_0(0)\int_{t_0}^{t_1}\frac{d\gamma}{dt}e^{-\frac{1}{2}\gamma+\frac{1}{2}t}dt \\ &= 4\kappa_0(0)\left\{e^{-\frac{1}{2}\gamma(t_0)+\frac{1}{2}t_0} - e^{-\frac{1}{2}\gamma(t_1)+\frac{1}{2}t_1} + \frac{1}{2}\int_{t_0}^{t_1} e^{-\frac{1}{2}\gamma+\frac{1}{2}t}dt\right\},\end{aligned}$$

which, since

$$\frac{1}{2}\int_{t_0}^{t_1} e^{-\frac{1}{2}\gamma+\frac{1}{2}t}dt \leq e^{-\frac{1}{2}\gamma(t_0)}\left(e^{\frac{1}{2}t_1} - e^{\frac{1}{2}t_0}\right)$$

implies

$$\int_{t_0}^{t_1}(\kappa_0-1)^2 e^{-\frac{3}{2}\gamma+\frac{1}{2}t}dt \leq 4\kappa_0(0)e^{\frac{1}{2}t_1}\left(e^{-\frac{1}{2}\gamma(t_0)} - e^{-\frac{1}{2}\gamma(t_1)}\right). \tag{2.45c}$$

By (2.45b) and (2.45c) we conclude, in view of (2.45a), that

$$\begin{aligned} g(t_0,t_1) \quad \leq \quad & (e-2)\left\{\frac{1}{2}e^{-\frac{3}{2}\gamma(t_0)}\left(e^{\frac{1}{2}t_1} - e^{\frac{1}{2}t_0}\right)\right. \\ & \left. +9\kappa_0(0)e^{\frac{1}{2}t_1}\left(e^{-\frac{1}{2}\gamma(t_0)} - e^{-\frac{1}{2}\gamma(t_1)}\right)\right\},\end{aligned} \tag{2.46a}$$

which implies

$$g(t_0,t_1) \leq c_3\kappa_0(0)e^{\frac{1}{2}t_1} \qquad c_3 = \frac{19}{2}(e-2). \tag{2.46b}$$

Consequently,

$$g(t_0,t_1) \leq \frac{1}{2}e^{-\frac{3}{2}\gamma(t_1)+\frac{1}{2}t_1}\frac{1}{\chi(t_1;s_0)} \tag{2.47a}$$

provided that

$$\chi(t_1;s_0) \leq \frac{e^{-\frac{3}{2}\gamma(t_1)}}{2c_3\kappa_0(0)}. \tag{2.47b}$$

We conclude from (2.44c) that under this condition

$$e^{-\frac{3}{2}\gamma(t_0)+\frac{1}{2}t_0}\frac{1}{\chi(t_0;s_0)} \geq \frac{1}{2}e^{-\frac{3}{2}\gamma(t_1)+\frac{1}{2}t_1}\frac{1}{\chi(t_1;s_0)};$$

that is,

$$\chi(t_0;s_0) \leq 2e^{\frac{3}{2}(\gamma(t_1)-\gamma(t_0))-\frac{1}{2}(t_1-t_0)}\chi(t_1;s_0) \tag{2.48}$$

holds.

We turn to the estimation of $\varphi(t;s_0)$ defined by (2.38). Setting $t_0 = t'$, $t_1 = t$ in (2.48) and noting that for $x > 0$ we have

$$x^{1/2}\log\left(\frac{1}{x}\right) \leq \frac{2}{e},$$

and we obtain

$$\begin{aligned} \chi(t';s_0)\log\left(\frac{1}{\chi(t';s_0)}\right) &\leq \frac{2}{e}(\chi(t';s_0))^{1/2} \\ &\leq \frac{2^{3/2}}{e}e^{\frac{3}{4}(\gamma(t)-\gamma(t'))-\frac{1}{4}(t-t')}(\chi(t;s_0))^{1/2}. \end{aligned} \tag{2.49a}$$

Hence,

(2.49b)

$$\begin{aligned} \int_0^t e^{2\gamma(t')}\chi(t';s_0)\log\left(\frac{1}{\chi(t';s_0)}\right)dt' &\leq \frac{2^{3/2}}{e}e^{2\gamma(t)}\int_0^t e^{-\frac{1}{4}(t-t')}dt'\cdot(\chi(t;s_0))^{1/2} \\ &\leq \frac{2^{7/2}}{e}e^{2\gamma(t)}(\chi(t;s_0))^{1/2}, \end{aligned}$$

and

$$\begin{aligned} \varphi(t;s_0) &\leq c_4(\kappa_0(0))^2 e^{2\gamma(t)}(\chi(t;s_0))^{1/2} \\ & c_4 = \frac{2^{7/2}}{e}c_2. \end{aligned} \tag{2.50}$$

This estimate holds provided that condition (2.40a), condition (2.41b) with t replaced by t', that is,

$$\chi(t';s_0)\log\left(\frac{1}{\chi(t';s_0)}\right) \leq \frac{e^{-\gamma(t)}}{6c_1\kappa_0(0)}, \tag{2.51a}$$

is satisfied for all $t' \in [0,t]$ and condition (2.47b) with t_1 replaced by t, that is,

$$\chi(t;s_0) \leq \frac{e^{-\frac{3}{2}\gamma(t)}}{2c_3\kappa_0(0)}, \tag{2.51b}$$

is satisfied as well.

By virtue of estimate (2.50), if

$$\chi(t;s_0) \leq \left(\min\left\{\log 2, \frac{h}{48(b+1)}\right\}\right)^2 \frac{e^{-4\gamma(t)}}{c_4^2(\kappa_0(0))^4} \tag{2.52}$$

then we have

$$e^{\varphi(t;s_0)} \leq 2, \tag{2.53a}$$

and

$$e^{\varphi(t;s_0)} - 1 \leq 2\varphi(t;s_0) \leq \frac{h}{24(b+1)}. \tag{2.53b}$$

Therefore, if also

$$\sup_{s\in[0,s_0]} |\psi(0;s)| \leq \frac{h}{48} \tag{2.54a}$$

then from (2.39) we conclude that

$$|\psi(t,\chi(t;s_0))| \leq \frac{h}{12}. \tag{2.54b}$$

By (2.13a) and (2.34b) this implies:

$$|\theta(t_n,s)e^s| \geq \frac{h}{4}e^{\gamma(t_n)} \qquad : \ n = 1,2,... \tag{2.55a}$$

for all $s \in [0, s_n]$, where

$$s_n = \chi(t_n; s_0). \tag{2.55b}$$

For, if conditions (2.40a), (2.51a), (2.51b), (2.52), (2.54a) hold for s_0, then they hold *a fortiori* if s_0 is replaced by $s_0' \in [0, s_n]$; thus (2.54b) also holds with s_0 replaced by s_0'. Since for each $s \in [0, s_n]$ there is a $s_0' \in [0, s_0]$ such that $\chi(t_n; s_0') = s$, the result follows. We remark that in Case 1, the sequence $(t_n \ : \ n = 1, 2, ...)$ can be chosen to be an arbitrary increasing sequence contained in $[T,\infty)$ such that $t_n \to \infty$ as $n \to \infty$. Now, (2.55a) implies (see (2.26b)):

$$\begin{aligned} \eta(t_n, s_n) &= e^{-s_n}\int_0^{s_n} e^s \frac{\theta^2(t_n,s)}{\kappa(t_n,s)}ds \\ &\geq \frac{h^2 e^{\gamma(t_n)}}{64\kappa_0(0)}(1-e^{-s_n}) \end{aligned} \tag{2.56a}$$

where we have used the bound (2.15). Since $s_n \leq c_0$ and the function

$$\frac{1-e^{-x}}{x}$$

is decreasing for $x > 0$, we have

$$\frac{1-e^{-s_n}}{s_n} \geq \frac{1-e^{-c_0}}{c_0}.$$

Thus (2.56a) implies:

$$\begin{aligned} \eta(t_n, s_n) &\geq \frac{c_5 h^2}{\kappa_0(0)}e^{\gamma(t_n)}s_n \\ c_5 &= \frac{1}{64}\frac{1-e^{-c_0}}{c_0}. \end{aligned} \tag{2.56b}$$

The lower bound (2.56b) contadicts the upper bound (2.12) if we choose

$$s_n = e^{-c_6 h^2 e^{\gamma(t_n)}/\kappa_0(0)} \tag{2.57}$$

where c_6 is some constant such that

$$c_6 > \frac{c_5}{c_1}.$$

We shall show that the choice (2.57) is, for sufficiently large n, consistent with conditions (2.40a), (2.51a), (2.51b), (2.52), (2.54a).

We note that in the above conditions t stands for t_n. Thus conditions (2.51b) and (2.52) read:

$$s_n \leq \frac{e^{-\frac{3}{2}\gamma(t_n)}}{2c_3\kappa_0(0)} \tag{2.58a}$$

and

$$s_n \leq \left(\min\left\{\log 2, \frac{h}{48(b+1)}\right\}\right)^2 \frac{e^{-4\gamma(t_n)}}{c_4^2(\kappa_0(0))^4}, \tag{2.58b}$$

respectively. Since, according to the hypotheses of Theorem 2.1,

$$\gamma(t_n) \to \infty \quad \text{as}\, n \to \infty,$$

both these conditions are satisfied by the choice (2.57) if n is sufficiently large. Also, conditions (2.40a) and (2.51a) are satisfied at $t' = t_n$ if n is sufficiently large:

$$s_n \quad \leq \quad c_0 \tag{2.59a}$$

$$s_n \log\left(\frac{1}{s_n}\right) \quad \leq \quad \frac{e^{-\gamma(t_n)}}{6c_1\kappa_0(0)}. \tag{2.59b}$$

Let t_* be the least value of $t \in [0, t_n]$ such that (2.40a) and (2.51a) are satisfied for all $t' \in [t, t_n]$. Then if $t_* > 0$, either

$$\chi(t_*; s_0) = c_0 \tag{2.60a}$$

or

$$\chi(t_*; s_0) \log\left(\frac{1}{\chi(t_*; s_0)}\right) = \frac{e^{-\gamma(t_*)}}{6c_1\kappa_0(0)}. \tag{2.60b}$$

However under these circumstances estimate (2.48) holds with t_0, t_1 replaced by t_*, t_n, respectively

$$\chi(t_*; s_0) \leq 2e^{\frac{3}{2}(\gamma(t_n)-\gamma(t_*))-\frac{1}{2}(t_n-t_*)}s_n;$$

so, *a fortiori*,

$$\chi(t_*; s_0) \leq 2e^{\frac{3}{2}\gamma(t_n)}s_n. \tag{2.61}$$

We see that the choice (2.57) contradicts both (2.60a), (2.60b), if n is sufficiently large. We conclude that $t_* = 0$ so conditions (2.40a) and (2.51a) are satisfied for all $t' \in [0, t_n]$ and (2.61) reads

$$s_0 \leq 2e^{\frac{3}{2}\gamma(t_n)} s_n \to 0 \;\text{ as}\, n \to \infty. \tag{2.62}$$

In view of the fact that solutions of bounded variation have the property that $\theta(t, s)$ is at each t, in particular at $t = 0$, continuous from the right with respect to s, condition (2.54a) also follows for sufficiently large n. We conclude that the choice (2.57) is, for sufficiently large n, consistent with all conditions. We therefore reach a contradiction if we suppose Theorem 2.1 to be false in Cases 1 and 4.

To treat the remaining Cases 2, 3, 5, 6, 7, in which I is unbounded, we set:

$$b(t) = \sup_{t' \in [0,t]} \left| \theta_0(t') e^{-\gamma(t')} \right|. \tag{2.63a}$$

Then b is a nondecreasing function tending to infinity as $t \to \infty$, and there is an increasing sequence $(t_n : n = 1, 2, ...)$, $t_n \to \infty$ as $n \to \infty$, such that

$$\left| \theta_0(t_n) e^{-\gamma(t_n)} \right| = b_0(t_n) \to \infty \;\text{ as}\, n \to \infty. \tag{2.63b}$$

In $[0, t_n]$ we have, as in (2.36),

$$\frac{d|\psi|}{dt} \leq c_2 (\kappa_0(0))^2 e^{2\gamma(t)} s \log\left(\frac{1}{s}\right) (|\psi| + b(t_n) + 1);$$

hence integrating we obtain

$$|\psi(t_n, s_n)| \leq e^{\varphi(t_n; s_0)} |\psi(0, s_0)| + (b(t_n) + 1) \left(e^{\varphi(t_n; s_0)} - 1\right) \tag{2.64}$$

where again

$$s_n = \chi(t_n; s_0).$$

As before, the conditions

$$\chi(t; s_0) \leq c_0 \;:\; \text{for all } t \in [0, t_n], \tag{2.65a}$$

$$\chi(t; s_0) \log\left(\frac{1}{\chi(t; s_0)}\right) \leq \frac{e^{-\gamma(t)}}{6c_1 \kappa_0(0)} \;:\; \text{for all } t \in [0, t_n], \tag{2.65b}$$

and

$$s_n \leq \frac{e^{-\frac{3}{2}\gamma(t_n)}}{2c_3 \kappa_0(0)} \tag{2.65c}$$

imply

$$\varphi(t_n; s_0) \leq c_4 (\kappa_0(0))^2 e^{2\gamma(t_n)} s_n^{1/2} \tag{2.66}$$

(see (2.50)). Thus if

$$s_n \leq \left(\min\left\{\log 2, \frac{1}{4(b(t_n)+1)}\right\}\right)^2 \frac{e^{-4\gamma(t_n)}}{c_4^2(\kappa_0(0))^4}, \tag{2.67}$$

then we have

$$e^{\varphi(t_n;s_0)} \leq 2 \tag{2.68a}$$

and

$$e^{\varphi(t_n;s_0)} - 1 \leq 2\varphi(t_n;s_0) \leq \frac{1}{2(b(t_n)+1)}. \tag{2.68b}$$

Therefore if also

$$\sup_{s\in[0,s_0]} |\psi(0,s)| \leq \frac{1}{4} \tag{2.69}$$

we conclude from (2.64) that

$$|\psi(t_n,s_n)| \leq 1. \tag{2.70a}$$

In fact, since conditions (2.65a)–(2.65c), (2.67), (2.69), hold *a fortiori* if s_0 is replaced by $s_0' \in [0,s_0]$, we conclude that

$$\sup_{s\in[0,s_n]} |\psi(t_n,s)| \leq 1. \tag{2.70b}$$

Hence,

$$\begin{aligned} |\theta(t_n,s)e^s| &\geq |\theta_0(t_n)| - e^{\gamma(t_n)}|\psi(t_n,s)| \\ &\geq (b(t_n)-1)e^{\gamma(t_n)} \geq \frac{1}{2}b(t_n)e^{\gamma(t_n)} \end{aligned} \tag{2.71}$$

for all $s \in [0,s_n]$, if n is large enough. It follows that (see (2.56a), (2.56b))

$$\begin{aligned} \eta(t_n,s_n) &\geq c_7\frac{(b(t_n))^2}{\kappa_0(0)}e^{\gamma(t_n)}s_n \\ c_7 &= \frac{1}{16}\frac{1-e^{-1/c_0}}{1/c_0}. \end{aligned} \tag{2.72}$$

The lower bound (2.72) contradicts the upper bound (2.12) if we choose

$$s_n = e^{-c_8(b(t_n))^2 e^{\gamma(t_n)}/\kappa_0(0)}, \tag{2.73}$$

where c_8 is some constant such that

$$c_8 > \frac{c_7}{c_1}.$$

Since

$$b(t_n) \to \infty \quad \text{as} \quad n \to \infty,$$

as well as

$$\gamma(t_n) \to \infty \quad \text{as} \quad n \to \infty,$$

the conditions (2.65c), (2.67) are satisfied by the choice (2.73) if n is sufficiently large. Also, the conditions (2.65a), (2.65b) are satisfied at $t = t_n$. The continuity argument which we applied previously then shows that conditions (2.65a), (2.65b) are satisfied for all $t \in [0, t_n]$ and, moreover, (2.62) holds, which implies that condition (2.69) is verified as well. We conclude that the choice (2.73) is, for sufficiently large n, consistent with all conditions. We therefore again reach a contradiction if we suppose Theorem 2.1 to be false in the remaining Cases 2, 3, 5, 6, 7. This completes the proof of Theorem 2.1.

3. The second instability theorem

In the following we confine attention to the case not covered by Theorem 2.1, the case where I tends to a finite limit as $t \to \infty$ and

$$\theta_0(0) = \lim_{t \to \infty} I(t). \tag{3.1a}$$

Following the proof of Theorem 2.1 we see that the argument of Cases 1 and 4 still applies if there is a positive constant $p < 1$ such that

$$\limsup_{t \to \infty} \left\{ |\theta_0(0) - I(t)| e^{\frac{1}{2} p \gamma(t)} \right\} \neq 0.$$

Therefore we can assume in the following that

$$(\theta_0(0) - I(t)) e^{\frac{1}{2} p \gamma(t)} \to 0 \;\text{ as }\; t \to \infty \tag{3.1b}$$

for all positive constants $p < 1$.

Let us define a function τ by

$$\frac{\partial \tau}{\partial t} + \beta \frac{\partial \tau}{\partial s} = \omega, \qquad \tau(0, s) = 0. \tag{3.2a}$$

We then have

$$\tau(t, \chi(t; s_0)) = \int_0^t \omega(t', \chi(t'; s_0)) dt'. \tag{3.2b}$$

Let us also define the functions

$$\tilde{\psi} = e^{-\tau} \psi \tag{3.3a}$$

$$\tilde{\rho} = e^{-\tau} \rho. \tag{3.3b}$$

Then, from (2.13b) we have

$$\frac{\partial \tilde{\psi}}{\partial t} + \beta \frac{\partial \tilde{\psi}}{\partial s} = \tilde{\rho}, \qquad \tilde{\psi}(0, s) = \psi(0, s). \tag{3.3c}$$

Thus, if we also define a function σ by

$$\frac{\partial\sigma}{\partial t}+\beta\frac{\partial\sigma}{\partial s}=\tilde{\rho}, \qquad \sigma(0,s)=0, \tag{3.3d}$$

that is,

$$\sigma(t,\chi(t;s_0))=\int_0^t \tilde{\rho}(t',\chi(t';s_0))dt', \tag{3.3e}$$

then

$$\tilde{\psi}=\hat{\psi}+\sigma \tag{3.3f}$$

where $\hat{\psi}$ satisfies

$$\frac{\partial\hat{\psi}}{\partial t}+\beta\frac{\partial\hat{\psi}}{\partial s}=0, \qquad \hat{\psi}(0,s)=\psi(0,s) \tag{3.3g}$$

so that

$$\hat{\psi}(t,\chi(t;s_0))=\psi(0,s_0). \tag{3.3h}$$

Let us fix

$$p=\frac{1}{2}$$

in (3.1b). Then from (2.9a) there is a constant b such that

$$|\theta_0(t)|\leq be^{\frac{3}{4}\gamma(t)} \quad : \text{ for all } \ t\geq 0. \tag{3.4}$$

If the conclusion of Theorem 2.1 is false, there is an $\varepsilon>0$ such that for each $s_0\in[0,\varepsilon]$ the incoming null curve $s=\chi(t;s_0)$ through $s=s_0$ at $t=0$ does not intersect an apparent horizon at finite t. Estimates (2.24) and (2.30b) then hold in $\mathcal{R}_\varepsilon^{c_0}$ where condition (2.14) holds. It follows that:

$$|\rho(t,s)|\leq c_2(\kappa_0(0))^2(b+1)e^{\frac{7}{4}\gamma(t)}s\log\left(\frac{1}{s}\right) \tag{3.5}$$

in $\mathcal{R}_\varepsilon^{c_0}$ where (2.14) holds. Also, conditions (2.51a) and (2.51b) imply (2.48) with t_1,t_0 replaced by $t,t'\in[0,t]$, respectively, that is

$$s'\leq 2e^{\frac{3}{2}(\gamma(t)-\gamma(t'))-\frac{1}{4}(t-t')}s \tag{3.6}$$

with

$$s'=\chi(t';s_0), \quad s=\chi(t;s_0).$$

The same conditions imply estimate (2.50), which if

$$\chi(t;s_0)\leq(\log 2)^2\frac{e^{-4\gamma(t)}}{c_4^2(\kappa_0(0))^4} \tag{3.7}$$

then it, in turn, implies

$$e^{\varphi(t;s_0)}\leq 2. \tag{3.8}$$

Hence, in view of the fact that by estimate (2.24),

$$|\tau(t,\chi(t;s_0))| \leq \varphi(t;s_0) \tag{3.9}$$

we have

$$\begin{aligned} |\tilde{\rho}(t,s)| &\leq 2|\rho(t,s)| \\ &\leq 2c_2(\kappa_0(0))^2(b+1)e^{\frac{7}{4}\gamma(t)}s\log\left(\frac{1}{s}\right). \end{aligned} \tag{3.10a}$$

Moreover, if (2.51b) and (3.7) hold with t replaced by t', for all $t' \in [0,t]$, then also (3.8) and (3.10a) hold with $t, s = \chi(t;s_0)$, replaced by $t', s' = \chi(t';s_0)$, for all $t' \in [0,t]$. It follows that:

$$\begin{aligned} |\sigma(t,s)| &\leq \int_0^t |\tilde{\rho}(t',s')|dt' \\ &\leq 2c_2(\kappa_0(0))^2(b+1)\int_0^t e^{\frac{7}{4}\gamma(t')}s'\log\left(\frac{1}{s'}\right)dt'. \end{aligned} \tag{3.10b}$$

Now, by virtue of (3.6) we have (see (2.49a))

$$\begin{aligned} s'\log\left(\frac{1}{s'}\right) &\leq \frac{2}{e}s'^{1/2} \\ &\leq \frac{2^{3/2}}{e}e^{\frac{3}{4}(\gamma(t)-\gamma(t'))-\frac{1}{4}(t-t')}s^{1/2}; \end{aligned}$$

hence

$$\begin{aligned} &\int_0^t e^{\frac{7}{4}\gamma(t')}s'\log\left(\frac{1}{s'}\right)dt' \\ &\leq \frac{2^{3/2}}{e}e^{\frac{7}{4}\gamma(t)}\int_0^t e^{-\frac{1}{4}(t-t')}dt'\cdot s^{1/2} \\ &\leq \frac{2^{7/2}}{e}e^{\frac{7}{4}\gamma(t)}s^{1/2}. \end{aligned}$$

Thus we obtain

$$|\sigma(t,s)| \leq 2c_4(\kappa_0(0))^2e^{\frac{7}{4}\gamma(t)}s^{1/2} \tag{3.10c}$$

(see (2.50)), which, being valid for all $s \in [0,s_*]$ implies

$$\begin{aligned} ||\sigma(t)||_{L^2(0,s_*)} &:= \sqrt{\int_0^{s_*}\sigma^2(t,s)ds} \\ &\leq c_4(\kappa_0(0))^2e^{\frac{7}{4}\gamma(t)}s_*. \end{aligned} \tag{3.10d}$$

On the other hand, from (3.3h),

$$\begin{aligned} ||\hat{\psi}(t)||^2_{L^2(0,s_*)} &:= \int_0^{s_*}\hat{\psi}^2(t,s)ds \\ &= \int_0^{s_{0*}}\psi^2(0,s_0)\frac{\partial\chi}{\partial s_0}(t;s_0)ds_0, \end{aligned} \tag{3.11}$$

where

$$s = \chi(t; s_0), \quad s_* = \chi(t; s_{0*}).$$

Now, differentiating the equation

$$\frac{d\chi(t; s_0)}{dt} = \beta(t, \chi(t; s_0)), \quad \chi(0; s_0) = s_0$$

with respect to s_0 yields:

$$\text{(3.12a)} \qquad \frac{d}{dt}\left(\frac{\partial \chi}{\partial s_0}(t; s_0)\right) = \frac{\partial \beta}{\partial s}(t, \chi(t; s_0))\frac{\partial \chi}{\partial s_0}(t; s_0), \quad \frac{\partial \chi}{\partial s_0}(0; s_0) = 1.$$

By equations (1.18c) and (2.13c),

$$\text{(3.12b)} \qquad \frac{\partial \beta}{\partial s} = 2 - \kappa_0 - \omega.$$

Thus, substituting in (3.12a) and integrating,

$$\frac{\partial \chi}{\partial s_0}(t; s_0) = \exp\left[\int_0^t (2 - \kappa_0(t') - \omega(t', \chi(t'; s_0)))dt'\right]$$

or, in view of (2.1) and (3.2b), we see that

$$\text{(3.12c)} \qquad \frac{\partial \chi}{\partial s_0}(t; s_0) = e^{t - \gamma(t) - \tau(t, \chi(t; s_0))}.$$

By (3.8), (3.9) we then have

$$\text{(3.12d)} \qquad \frac{\partial \chi}{\partial s_0}(t; s_0) \geq \frac{1}{2}e^{t - \gamma(t)}.$$

Substituting in (3.11) we conclude that

$$\text{(3.13)} \qquad ||\hat{\psi}(t)||_{L^2(0, s_*)} \geq \frac{1}{2^{1/2}} e^{\frac{1}{2}t - \frac{1}{2}\gamma(t)} s_{0*}^{1/2} h(s_{0*}),$$

where

$$\text{(3.14)} \qquad h(s) = \sqrt{\frac{1}{s}\int_0^s \psi^2(0, s')ds'}.$$

Now we wish to achieve

$$\text{(3.15a)} \qquad ||\hat{\psi}(t)||_{L^2(0, s_*)} \geq 2||\sigma||_{L^2(0, s_*)}$$

so that

$$\text{(3.15b)} \qquad \begin{aligned} ||\tilde{\psi}(t)||_{L^2(0, s_*)} &\geq ||\hat{\psi}(t)||_{L^2(0, s_*)} - ||\sigma||_{L^2(0, s_*)} \\ &\geq \frac{1}{2}||\hat{\psi}(t)||^2_{L^2(0, s_*)} \end{aligned}$$

(see (3.3f)). In view of estimates (3.10d) and (3.13), this requires an upper bound for s_* in terms of s_{0*}, which in turn requires an upper bound for β. Since $\eta \geq 0$, we have

$$\kappa(t, s) = \frac{1}{1 - \mu(t, s)} \geq \frac{1}{1 - \mu_0(t)e^{-s}};$$

hence (see (2.16)),

$$\begin{aligned}(\nu-\lambda)(t,s) &\geq \int_0^s \left(\frac{1}{1-\mu_0(t)e^{-s'}}-1\right)ds' \\ &= \log\left(\frac{1-\mu_0(t)e^{-s}}{1-\mu_0(t)}\right),\end{aligned}$$

and

$$\begin{aligned}\beta(t,s) &= 1-e^{(\nu-\lambda)(t,s)-s} \\ &\leq 1-e^{-2s}-(e^{-s}-e^{-2s})\kappa_0(t).\end{aligned} \tag{3.16a}$$

Since

$$1-e^{-2s} \leq 2s-\frac{3}{2}s^2 \quad : \text{ if } \ s \leq c_0,$$

(3.16a) implies

$$\beta(t,s) \leq (2-\kappa_0(t))s+\frac{3}{2}(\kappa_0(t)-1)s^2. \tag{3.16b}$$

Thus,

$$\frac{d\chi}{dt} \leq (2-\kappa_0)\chi+\frac{3}{2}(\kappa_0-1)\chi^2 \tag{3.17a}$$

or

$$\frac{d}{dt}\left(e^{t-\gamma}\frac{1}{\chi}\right) \geq -\frac{3}{2}(\kappa_0-1)e^{t-\gamma}. \tag{3.17b}$$

Integrating (3.17b) on $[0,t]$ yields

$$\begin{aligned}\frac{e^{t-\gamma(t)}}{s}-\frac{1}{s_0} &\geq -\frac{3}{2}\int_0^t(\kappa_0(t')-1)e^{t'-\gamma(t')}dt' \\ &= -\frac{3}{2}\int_0^t \frac{d\gamma}{dt'}e^{t'-\gamma(t')}dt' \\ &= -\frac{3}{2}\left(1-e^{t-\gamma(t)}+\int_0^t e^{t'-\gamma(t')}dt'\right) \\ &\geq -\frac{3}{2}e^t(1-e^{-\gamma(t)}) \geq -\frac{3}{2}e^t,\end{aligned} \tag{3.17c}$$

where $s=\chi(t;s_0)$. It follows that if

$$s_0 \leq \frac{1}{3}e^{-t}, \tag{3.18a}$$

then

$$s \leq 2e^{t-\gamma(t)}s_0. \tag{3.18b}$$

In particular, if

$$s_{0*} \leq \frac{1}{3}e^{-t}, \tag{3.19a}$$

then

$$s_* \leq 2e^{t-\gamma(t)} s_{0*}. \tag{3.19b}$$

Substituting this in (3.10d) we conclude that under condition (3.19a),

$$h(s_{0*}) \geq 2^{5/2} c_4 (\kappa_0(0))^2 e^{\frac{1}{2}t + \frac{5}{4}\gamma(t)} s_{0*}^{1/2} \tag{3.20}$$

implies (3.15a), hence also (3.15b).

Now, we have (see (2.56a)):

$$\begin{aligned} \eta(t, s_*) &= e^{-s_*} \int_0^{s_*} e^s \frac{\theta^2(t,s)}{\kappa(t,s)} ds \\ &\geq \frac{e^{-2c_0}}{2\kappa_0(0)} e^{-\gamma(t)} \int_0^{s_*} e^{2s} \theta^2(t,s) ds \end{aligned} \tag{3.21}$$

where we have used the bound (2.15) and the condition

$$s_* \leq c_0.$$

From (2.13a) and (3.3a), we have

$$e^s \theta = \theta_0 + e^{\gamma + \tau} \tilde{\psi}. \tag{3.22a}$$

It follows that

$$||e^s \theta(t)||_{L^2(0,s_*)} \geq ||e^{\gamma+\tau} \tilde{\psi}(t)||_{L^2(0,s_*)} - ||\theta_0(t)||_{L^2(0,s_*)}. \tag{3.22b}$$

By (3.8) and (3.15b),

$$||e^{\gamma+\tau} \tilde{\psi}(t)||_{L^2(0,s_*)} \geq \frac{1}{4} e^{\gamma(t)} ||\hat{\psi}(t)||_{L^2(0,s_*)} \tag{3.23a}$$

while by (3.4) and (3.19b),

$$||\theta_0(t)||_{L^2(0,s_*)} = |\theta_0(t)| s_*^{1/2} \leq 2^{1/2} e^{\frac{1}{2}t + \frac{1}{4}\gamma(t)} b s_{0*}^{1/2}. \tag{3.23b}$$

In view of the lower bound (3.13) we conclude that

$$h(s_{0*}) \geq 16 b e^{-\frac{1}{4}\gamma(t)} \tag{3.24a}$$

implies

$$||\theta_0(t)||_{L^2(0,s_*)} \leq \frac{1}{8} e^{\gamma(t)} ||\hat{\psi}||_{L^2(0,s_*)}, \tag{3.24b}$$

and

$$\begin{aligned} ||e^s \theta(t)||_{L^2(0,s_*)} &\geq \frac{1}{8} e^{\gamma(t)} ||\hat{\psi}(t)||_{L^2(0,s_*)} \\ &\geq \frac{1}{2^{7/2}} e^{\frac{1}{2}t + \frac{1}{2}\gamma(t)} s_{0*}^{1/2} h(s_{0*}). \end{aligned} \tag{3.24c}$$

Substituting this in (3.21) yields:

$$\eta(t, s_*) \geq \frac{e^{-2c_0}}{2^8 \kappa_0(0)} e^t s_{0*} h^2(s_{0*}). \tag{3.25}$$

On the other hand, the bound (2.12) at (t, s_*) reads

$$\eta(t, s_*) \leq c_1 s_* \log\left(\frac{1}{s_*}\right), \tag{3.26a}$$

which by (3.19b) implies

$$\eta(t, s_*) \leq 2c_1 e^{t-\gamma(t)} s_{0*} \left\{\log\left(\frac{1}{s_{0*}}\right) + \gamma(t) - t\right\}. \tag{3.26b}$$

This contradicts (3.25) if

$$\begin{aligned} h^2(s_{0*}) &> c_9 \kappa_0(0) e^{-\gamma(t)} \left\{\log\left(\frac{1}{s_{0*}}\right) + \gamma(t) - t\right\} \\ c_9 &= 2^9 e^{2c_0} c_1. \end{aligned} \tag{3.27}$$

Summarizing, the hypothesis that the conclusion of Theorem 2.1 is false leads to a contradiction if for every $\varepsilon > 0$ there is a $s_{0*} \in (0, \varepsilon]$ and a $t \in [0, \infty)$ such that the requirements (3.20), (3.24a) and (3.27) are satisfied, and, moreover, the conditions (2.40a), (2.51a), (2.51b), (3.7), that is, with

$$s'_* = \chi(t'; s_{0*}),$$

the conditions:

$$s'_* \leq c_0 \ : \text{for all } t' \in [0, t] \tag{3.28a}$$

$$s'_* \log\left(\frac{1}{s'_*}\right) \leq \frac{e^{-\gamma(t')}}{6c_1 \kappa_0(0)} \ : \text{for all } t' \in [0, t] \tag{3.28b}$$

$$s'_* \leq \frac{e^{-\frac{3}{2}\gamma(t')}}{2c_3 \kappa_0(0)} \ : \text{for all } t' \in [0, t] \tag{3.28c}$$

$$s'_* \leq (\log 2)^2 \frac{e^{-4\gamma(t')}}{c_4^2 (\kappa_0(0))^4} \ : \text{for all } t' \in [0, t], \tag{3.28d}$$

respectively, hold, and, finally, condition (3.19a) holds as well.

Given s_{0*}, let us define t by

$$t + 5\gamma(t) = \log\left(\frac{1}{s_{0*}}\right), \tag{3.29a}$$

i.e.

$$s_{0*} = e^{-t-5\gamma(t)}. \tag{3.29b}$$

Then if t is large enough so that

$$\gamma(t) \geq \frac{1}{5} \log 3, \tag{3.30}$$

then condition (3.19a) is verified. This implies (3.19b) with t replaced by any $t' \in [0,t]$, that is:

$$s'_* \leq 2e^{t'-\gamma(t')} s_{0*} \quad : \text{for all } \; t' \in [0,t]. \tag{3.31a}$$

Substituting (3.29b) we obtain

$$s'_* \leq 2e^{-5\gamma(t)} \quad : \text{for all } \; t' \in [0,t]. \tag{3.31b}$$

It follows that (3.28a), (3.28c), (3.28d) are verified if t is large enough so that

$$\gamma(t) \geq \frac{1}{5}\log\left(\frac{2}{c_0}\right) \tag{3.32a}$$

$$\gamma(t) \geq \frac{2}{7}\log(4c_3\kappa_0(0)) \tag{3.32b}$$

$$\gamma(t) \geq 2\log\left(\frac{c_4(\kappa_0(0))^2}{\log 2}\right) + \log 2, \tag{3.32c}$$

respectively. Also, since

$$s'_* \log\left(\frac{1}{s'_*}\right) \leq \frac{2}{e} s'^{1/2}_*,$$

(3.28b) holds *a fortiori* if

$$s'_* \leq \frac{e^2 e^{-2\gamma(t')}}{144c_1^2(\kappa_0(0))^2} \quad : \text{for all } \; t' \in [0,t] \tag{3.33a}$$

which by virtue of (3.31b) is verified if t is large enough so that

$$\gamma(t) \geq \frac{1}{3}\left\{2\log(12c_1\kappa_0(0)) + \log 2 - 2\right\}. \tag{3.33b}$$

We turn to the requirements (3.20), (3.24a) and (3.27). Substituting the definition (3.29b), these requirements read

$$h(e^{-t-5\gamma(t)}) \geq 2^{5/2}c_4(\kappa_0(0))^2 e^{-\frac{5}{4}\gamma(t)} \tag{3.34a}$$

$$h(e^{-t-5\gamma(t)}) \geq 16be^{-\frac{1}{4}\gamma(t)} \tag{3.34b}$$

$$h(e^{-t-5\gamma(t)}) > (6c_9)^{1/2}(\kappa_0(0))^{1/2}(\gamma(t))^{1/2}e^{-\frac{1}{2}\gamma(t)}, \tag{3.34c}$$

respectively. For large t, (3.34b) is the strongest requirement. Let us assume that

$$\limsup_{t\to\infty}\left\{e^{\frac{1}{4}\gamma(t)}h(e^{-t-5\gamma(t)})\right\} = \infty. \tag{3.35}$$

Then given any $T \in (0,\infty)$ we can find a $t \in [T,\infty)$ such that with s_{0*} defined by (3.29b) all conditions and requirements are satisfied. We have therefore proved:

THEOREM 3.1. *Let γ be unbounded and let*

$$I(t) \to \theta_0(0) \;\; \textit{as } t \to \infty.$$

Let $g(s)$ be the function on $(0,1)$ defined by

$$g(s) = e^{-\frac{1}{4}\gamma(t)}, \quad s = e^{-t-5\gamma(t)}.$$

Consider the function

$$h(s) = \sqrt{\frac{1}{s}\int_0^s \left(e^{s'}\theta(0,s') - \theta(0,0)\right)^2 ds'}$$

defined for $s > 0$. Then if

$$\limsup_{s\to 0+} \left\{\frac{h(s)}{g(s)}\right\} = \infty,$$

then the conclusion of Theorem 2.1 holds.

4. The exceptional set

We now investigate the subset $\mathcal{E}$ of the space of initial conditions of bounded variation on C_0^+, which lead to the formation of a singular boundary $\mathcal{B}$, with the function γ, defined along C_0^-, the past light cone of O, the past end point of the central component $\mathcal{B}_0$ of $\mathcal{B}$, being unbounded, while the conclusion of Theorem 2.1 fails.

According to the definitions of [1] *initial conditions of bounded variation* means that

$$\alpha = \frac{\partial}{\partial r}(r\phi) = \theta + \phi \tag{4.1}$$

is a function of bounded variation along C_0^+. This is equivalent to both θ and ϕ being functions of bounded variation along C_0^+. The total variation of ϕ along C_0^+ is equal to the integral

$$\int_0^\infty |\theta|\frac{dr}{r}$$

along C_0^+. In terms of the coordinates (t,s) defined in Section 1 we have

$$\text{T.V.}[\phi]_{t=0} = \int_{-\infty}^{\infty} |\theta(0,s)|ds. \tag{4.2}$$

Thus $\alpha_{t=0}$ being a function of bounded variation is equivalent to $\theta_{t=0}$ being a function of bounded variation which is integrable on the real line. Here, we shall characterize initial conditions in terms of the function $\theta_{t=0} = \vartheta$.

Suppose then that ϑ belongs to the set $\mathcal{E}$. Then according to Theorems 2.1 and 3.1, we have

$$I(t) \to \vartheta(0) \text{ as } t \to \infty, \tag{4.3}$$

and with

$$h(s) = \sqrt{\frac{1}{s}\int_0^s (e^{s'}\vartheta(s') - \vartheta(0))^2 ds'}$$

$(s > 0)$,

$$\limsup_{s\to 0+} \left\{\frac{h(s)}{g(s)}\right\} < \infty, \tag{4.4}$$

where $g(s)$ is the function defined in Theorem 3.1. Let f_1 be a nonnegative integrable function on the real line, vanishing on $(-\infty, 0)$, whose restriction to $[0, \infty)$ is absolutely continuous and

$$\lim_{s\to 0+} f_1(s) = 1. \tag{4.5}$$

Let also f_2 be a nonnegative integrable absolutely continuous function on the real line, vanishing on $(-\infty, 0]$, such that

$$\limsup_{s\to 0+} \left\{\frac{1}{g(s)}\sqrt{\frac{1}{s}\int_0^s e^{2s'} f_2^2(s')ds'}\right\} = \infty. \tag{4.6}$$

For example, we may define f_2 on $(0, 1)$ so that

$$\frac{1}{s}\int_0^s e^{2s'} f_2^2(s')ds' = g(s) \quad : \quad \text{for all } \ s \in (0, 1);$$

that is,

$$f_2(s) = e^{-s}\sqrt{\frac{d(sg(s))}{ds}} \quad : \quad \text{for all } \ s \in (0, 1).$$

Given real parameters λ_1, λ_2 we then consider the initial data given by:

$$\tilde{\vartheta}_{(\lambda_1,\lambda_2)} = \vartheta + \lambda_1 f_1 + \lambda_2 f_2. \tag{4.7}$$

Since the restrictions of $\tilde{\vartheta}_{(\lambda_1,\lambda_2)}$ and ϑ to the interval $(-\infty, 0)$ coincide, the corresponding solutions coincide in the interior of C_0^- (domain of dependence) and define the same functions $\gamma(t)$ and $I(t)$. Since by (4.3) and (4.5)

$$\tilde{\vartheta}_{(\lambda_1,\lambda_2)}(0) = \lim_{t\to\infty} I(t) + \lambda_1 \tag{4.8}$$

if $\lambda_1 \neq 0$, then Theorem 2.1 applies so that $\tilde{\vartheta}_{(\lambda_1,\lambda_2)} \notin \mathcal{E}$, while if $\lambda_1 = 0$, $\lambda_2 \neq 0$, then by (4.4) and (4.6),

$$\limsup_{s\to 0+} \left\{\frac{\tilde{h}_{(\lambda_1,\lambda_2)}(s)}{g(s)}\right\} = \infty.$$

Hence Theorem 3.1 applies and again $\tilde{\vartheta}_{(\lambda_1,\lambda_2)} \notin \mathcal{E}$. Therefore the 2-dimensional linear subspace Π_ϑ of the space of initial data defined by

$$\Pi_\vartheta = \{\tilde{\vartheta}_{(\lambda_1,\lambda_2)} \ : \ (\lambda_1, \lambda_2) \in \Re^2\} \tag{4.9}$$

intersects $\mathcal{E}$ at only one point, the point corresponding to $(\lambda_1, \lambda_2) = (0, 0)$, that is ϑ itself.

Suppose next that $\vartheta, \vartheta' \in \mathcal{E}$. Then

$$\Pi_\vartheta \bigcap \Pi_{\vartheta'} = \emptyset \quad \text{unless } \vartheta = \vartheta'.$$

For, if

$$\tilde{\vartheta}_{(\lambda_1,\lambda_2)} = \tilde{\vartheta}'_{(\lambda_1',\lambda_2')} \tag{4.10}$$

then ϑ and ϑ' coincide on $(-\infty, 0)$; thus they define the same functions $\gamma(t), I(t)$ and $g(s)$, and the same functions $f_1(s), f_2(s)$. By (4.3),

$$\vartheta(0) = \vartheta'(0) = \lim_{t\to\infty} I(t).$$

Equality (4.10) at $s = 0$ then yields

$$\lambda_1 = \lambda_1'.$$

Thus (4.10) becomes

$$\vartheta - \vartheta' = (\lambda_2' - \lambda_2) f_2$$

and since by (4.4)

$$\limsup_{s\to 0+} \left\{ \frac{h(s)}{g(s)} \right\} < \infty, \quad \limsup_{s\to 0+} \left\{ \frac{h'(s)}{g(s)} \right\} < \infty$$

we obtain, in view of (4.6),

$$\lambda_2' = \lambda_2, \quad \vartheta = \vartheta'.$$

We have thus proved:

THEOREM 4.1. *Consider the exceptional set $\mathcal{E}$ in the space of initial data $BV \bigcap L^1$ on the real line. Then for each $\vartheta \in \mathcal{E}$ there is a 2-dimensional linear subspace Π_ϑ such that*

$$(\Pi_\vartheta \setminus \{\vartheta\}) \bigcap \mathcal{E} = \emptyset.$$

Moreover, if ϑ,$\vartheta' \in \mathcal{E}$, then

$$\Pi_\vartheta \bigcap \Pi_{\vartheta'} = \emptyset$$

unless ϑ and ϑ' coincide. We may therefore say that $\mathcal{E}$ has positive codimension in the space of initial data.

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