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2-Blocking Sets in PG$(4, q)$, q Square

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Abstract. We show that the three smallest minimal point sets of PG$(4, q)$, q square, $q > 9$, that meet all planes are the set of points of a plane, the set of points in a Baer cone and the set of points in a Baer subgeometry PG$(4, \sqrt{q})$. This implies that PG$(4, \sqrt{q})$ is the unique smallest example of a set of points of PG$(4, q)$ that meets every plane and contains no line. It also implies that PG$(4, \sqrt{q})$ is the unique smallest minimal set of points of PG$(4, q)$ that meets all planes and generates PG$(4, q)$.

1. Introduction

Let $\Sigma = \mathrm{PG}(N, q)$ be the projective space of dimension N over the finite field GF(q).

A *t-blocking set* B in PG(N, q), with $N \geq t + 1$, is a set B of points such that any $(N - t)$-dimensional subspace intersects B. A t-blocking set is called *trivial* when it contains a t-dimensional subspace. A 1-blocking set in PG$(2, q)$ is simply called a *blocking set*.

The smallest non-trivial t-blocking sets have been characterized by the work of Beutelspacher [2] and Heim [4]. They proved that the smallest non-trivial t-blocking sets in PG(N, q) are cones with a $(t - 2)$-dimensional vertex and with base a 1-blocking set of minimum cardinality in a plane, skew to the vertex, of PG(N, q).

For q square, this means that this smallest non-trivial example is a cone with $(t - 2)$-dimensional vertex and with base a Baer subplane PG$(2, \sqrt{q})$ in a plane PG$(2, q)$ skew to the vertex. For $t = 1$, the smallest 1-blocking sets are the smallest blocking sets in a plane [3].

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For the particular case of $t = 2$ and q square, this is a *Baer cone*, that is, a cone with vertex a point and with base a Baer subplane PG$(2, \sqrt{q})$ in a plane PG$(2, q)$, having cardinality $q^2 + q\sqrt{q} + q + 1$.

In [5], Heim introduced the problem of finding the size of the second smallest non-trivial minimal t-blocking sets. This problem was studied by Storme and Weiner [6] who proved:

(1) in PG(n, q^2), $q = p^h$, $h \geq 1$, p prime, $p > 3$, $n \geq 3$, the second smallest non-trivial minimal 1-blocking sets are the second smallest non-trivial minimal blocking sets, with respect to lines, in a plane of PG(n, q), $n \geq 3$;

(2) in PG(n, p^3), $p = p_0^h$, $h \geq 1$, p_0 prime, $p_0 \geq 5$, $p \neq 5$, $n \geq 3$, the smallest non-trivial minimal 1-blocking sets are: (a) a Baer subplane of order $p^3 + p^{3/2} + 1$ when p is square, (b) a minimal blocking set of cardinality $p^3 + p^2 + 1$ in a plane, (c) a minimal blocking set of cardinality $p^3 + p^2 + p + 1$ in a plane, and (d) a subgeometry PG$(3, p)$.

Turning the attention to 2-blocking sets in PG$(4, q)$, q square, a subgeometry PG$(4, \sqrt{q})$ in a 4-dimensional subspace of PG(N, q) is a 2-blocking set of cardinality $q^2 + q\sqrt{q} + q + \sqrt{q} + 1$.

We will show that this example, whose cardinality is $\sqrt{q}$ larger than the size of the Baer cone, is the second smallest non-trivial minimal 2-blocking set in PG(N, q), q square, $q > 9$, $N \geq 4$.

To obtain this goal, the following theorems will be proved.

Theorem 1.1. *Every set of at most $q^2 + q\sqrt{q} + q + \sqrt{q} + 1$ points of $PG(3, q)$, q a square and $q > 9$, that meets every line contains a plane or a cone over a Baer subplane.*

Theorem 1.2. *Suppose B is a minimal set of points of $PG(4, q)$, q a square and $q > 9$, that meets every plane. If $|B| \leq q^2 + q\sqrt{q} + q + \sqrt{q} + 1$, then either B is the point set of a plane or B is a cone over a Baer subplane or B is the point set of a subgeometry $PG(4, \sqrt{q})$.*

This will imply the following result for all dimensions $N \geq 4$.

Theorem 1.3. *Suppose B is a minimal 2-blocking set of $PG(N, q)$, $N \geq 4$, q a square and $q > 9$. If $|B| \leq q^2 + q\sqrt{q} + q + \sqrt{q} + 1$, then either B is the point set of a plane or B is a cone over a Baer subplane or B is the point set of a subgeometry $PG(4, \sqrt{q})$.*

For $q = 4$ and $q = 9$, the problems remain open.

2. Proof of Theorem 1.1

Suppose that B is a set of at most $q^2 + q\sqrt{q} + q + \sqrt{q} + 1$ points of PG$(3, q)$, q a square and $q > 9$, that meets every line. We suppose that B does not contain a plane and we shall show in a series of lemmas that B contains a cone over a Baer subplane. A line contained in B will be called a *B-line*. A line meeting B in exactly one point will be called a *tangent*. It suffices to prove Theorem 1.1 for the minimal sets B that meet every line. Therefore we shall assume that B is a minimal set of points meeting every line. This implies that every point of B lies on a tangent.

Lemma 2.1. *If the plane $\pi = PG(2, q)$ of $PG(4, q)$, q square, $q > 9$, meets B in less than $q + 2\sqrt{q} + 1$ points, then $\pi \cap B$ either contains a line or a Baer subplane.*

Proof. It is known that every (non-trivial) blocking set of PG$(2,q)$, q a square, $q > 9$, with less than $q + 2\sqrt{q} + 1$ points contains a Baer subplane, see [1]. □

Lemma 2.2. a) *Every plane meets B in at most $q\sqrt{q} + q + \sqrt{q} + 1$ points.*
b) *Every plane contains at most $\sqrt{q} + 1$ B-lines.*

Proof. a) Let π be a plane. Since we assumed that B contains no plane, there exists a point $P \in \pi \setminus B$. Each of the q^2 lines on P that does not lie in π meets B. Hence there exist at least q^2 points in B that are not in π. Therefore $|\pi \cap B| \leq |B| - q^2$.
b) Any $\sqrt{q} + 2$ lines in a plane cover at least

$$(\sqrt{q} + 2)(q + 1) - \binom{\sqrt{q} + 2}{2}$$

points. Since this number is bigger than the one in part a), the assertion follows. □

Lemma 2.3. *Every point of B lies on at least one B-line.*

Proof. Let P be a point and assume that P does not lie on a B-line. Consider a tangent t on P. Lemma 2.1 implies that every plane π on t meets B in at least $q + \sqrt{q} + 1$ points with equality if and only if $\pi \cap B$ is a Baer subplane. Since t lies on $q + 1$ planes, this gives $|B| \geq 1 + (q+1)(q+\sqrt{q})$. Since $|B| \leq 1 + (q+1)(q+\sqrt{q})$, it follows that $|B| = 1 + (q+\sqrt{q})(q+1)$ and that every plane on t meets B in a Baer subplane. It follows that every line on P is a tangent or meets B in precisely $\sqrt{q} + 1$ points. Since every plane on P that contains a tangent on P meets B in a Baer subplane, it also follows that each plane on P either meets B in a Baer subplane, and thus in $q + \sqrt{q} + 1$ points or does not contain a tangent and then meets B in exactly $1 + (q + 1)\sqrt{q}$ points.

Consider a line h on P with $\sqrt{q}+1$ points in B, let b be the number of planes on h which have $q + \sqrt{q} + 1$ points in B and let c be the number of planes on h which have $q\sqrt{q} + \sqrt{q} + 1$ points in B. Then $b + c = q + 1$ and $bq + cq\sqrt{q} = |B| - \sqrt{q} - 1 = q^2 + q\sqrt{q} + q$. This gives $c(\sqrt{q} - 1) = \sqrt{q}$. But $q \neq 4$, a contradiction. □

Lemma 2.4. *If l is a B-line, then some point of l does not lie on a second B-line.*

Proof. We first show that a B-line meets less than $2q$ other B-lines. Assume on the contrary that this is not true, so that there exist B-lines l_i for $i = 1, \ldots, 2q$, that meet l. We shall get a contradiction by showing that these cover more points than there are in B. Since every plane contains at most $\sqrt{q} + 1$ B-lines, the lines l_i will cover the smallest number of points, if there are $2\sqrt{q}$ planes π on l that all contain exactly $\sqrt{q}$ of the lines l_i and if no three lines l_i of one plane π will meet in the same point. Consider such a plane π. In $\pi \setminus l$, the $\sqrt{q}$ lines l_i of π will cover

$$\sqrt{q}q - \binom{\sqrt{q}}{2}$$

points of B. Since there are $2\sqrt{q}$ planes π, this gives

$$|B| \geq q + 1 + 2\sqrt{q}\left(\sqrt{q}q - \binom{\sqrt{q}}{2}\right)$$

which is a contradiction.

Thus l meets less than $2q$ other B-lines. It follows that some point P of l lies on at most one more B-line. If there is no other B-line on P then we are done. Therefore assume that there exists a unique second B-line l' on P. We shall obtain a contradiction.

The point P lies on a tangent t. Consider first the case that t lies in the plane π spanned by l and l'. Since l and l' are the only B-lines on P, no plane on t different from π will contain a line. Thus these q planes meet B in a non-trivial blocking set and contain therefore at least $q+\sqrt{q}+1$ points of B. This gives rise to $q(q+\sqrt{q})$ points in $B \setminus \pi$. Since π contains at least $2q+1$ points of B, it follows that $|B| \geq 2q+1+q(q+\sqrt{q})$, which is a contradiction.

Thus t is not in π and similarly, no tangent on P lies in π. Consider the $q-1$ planes τ on t that do not contain l nor l'. These planes τ contain no B-line, since t is a tangent and since P only lies on the B-lines l and l'. Thus these planes τ meet B in at least $q+\sqrt{q}+1$ points. Counting the number of points of B using the $q+1$ planes through t, we also see that at most $2\sqrt{q}$ of the planes τ contain more than $q+\sqrt{q}+1$ points of B. Thus at least $q-1-2\sqrt{q}$ planes on t meet B in exactly $q+\sqrt{q}+1$ points, which must be the points of a Baer subplane. Since π contains no tangent on P, it follows that for each of these $q-1-2\sqrt{q}$ planes τ, the line $\tau \cap \pi$ meets B in a Baer subline. The remaining $2\sqrt{q}$ planes on t also do not meet π in a tangent, so they meet π in a line with at least two points in B. This shows that π has at least

$$1 + 2q + 2\sqrt{q} + (q - 1 - 2\sqrt{q})\sqrt{q} = q\sqrt{q} + \sqrt{q} + 1$$

points. Thus at most $q^2 + q$ points of B lie outside this plane π.

It is not possible that all lines of π on P have at least $\sqrt{q}+1$ points in B. Otherwise we could improve the above bound to

$$|B \cap \pi| \geq 1 + 2q + (q-1)\sqrt{q}$$

which contradicts Lemma 2.2, since $q > 9$.

Consider a line h of π on P that contains less than $\sqrt{q}+1$ points in B. Consider also a plane τ on t for which $\tau \cap B$ is a Baer subplane. Then there are $q-\sqrt{q}$ planes σ on h that meet τ in a tangent. These planes contain therefore no B-lines. Thus $\sigma \cap B$ either has at least $q+2\sqrt{q}+1$ points in B or contains a Baer subplane. In both cases, it follows that $\sigma \setminus h$ contains at least $q+\sqrt{q}$ points of B. Now, every plane on h different from π contains at least q points of B that do not lie in π and at least $q-\sqrt{q}$ of these contain at least $q+\sqrt{q}$ points of B that lie outside π. This gives rise to at least

$$q^2 + (q - \sqrt{q})\sqrt{q}$$

points of B that do not lie in π. But we have seen that $|B \setminus \pi| \leq q^2 + q$, a contradiction. $\square$

Lemma 2.5. *All B-lines meet in a common point.*

Proof. Let l be a B-line. We know that l has a point P lying on no other B-line. First we show the following:

If t is a tangent on P, then the plane $\langle l, t\rangle$ contains at most $q+1+\sqrt{q}$ points of B.

This can be seen as follows. Since P lies only on the B-line l, each of the q planes on t different from $\langle t, l\rangle$ meets B in at least $q+\sqrt{q}+1$ points. This gives rise to at least $q(q+\sqrt{q})$ points of B that do not lie in $\langle l, t\rangle$. Thus $\langle l, t\rangle$ meets B in at most $|B|-q(q+\sqrt{q}) \leq q+1+\sqrt{q}$ points.

Now fix a tangent t on P, put $\pi := \langle t, l\rangle$ and denote by $\pi_1, \ldots, \pi_q$ the other q planes on t. Since each of the planes π_i meets B in at least $q+\sqrt{q}+1$ points, the above counting argument also shows that each of the planes π_i meets B in at most $q+2\sqrt{q}+1$ points. The preceding argument also shows that not all planes π can have more than $q+\sqrt{q}+1$ points in B, so that we may assume that π_1 meets B in exactly $q+\sqrt{q}+1$ points. And it also shows that if a plane π_i meets B in exactly $q+2\sqrt{q}+1$ points, then the plane $\langle l, t\rangle$ only shares the line l with B. Then l must intersect all B-lines. So, from now on, assume that all planes π_i share less than $q+2\sqrt{q}+1$ points with B.

By Lemma 2.1, the set $\pi_i \cap B$ contains a Baer subplane B_i. The point P belongs to B_i, because t is a tangent of π_i on P.

Since π_1 has exactly $q+\sqrt{q}+1$ points in B, we have $B_1 = \pi_1 \cap B$. Thus P lies on $q-\sqrt{q}$ tangents of π_1. Hence, there exist $q-\sqrt{q}$ planes on l that meet π_1 in a tangent; these planes will be called τ below. Moreover there are $\sqrt{q}+1$ planes on l that meet π_1 in a line g for which $g \cap B$ is a Baer subline of B_1; these planes will be called σ below.

Now we consider first the case that each of the $q-\sqrt{q}$ planes τ meets B in at most $q+\sqrt{q}$ points, that is, apart from the $q+1$ points of l, in at most $\sqrt{q}-1$ further points in B. Then for all $i = 1, \ldots, q$ and for all planes τ, the line $\tau \cap \pi_i$ contains at most $\sqrt{q}$ points in B. Hence, the $\sqrt{q}+1$ Baer sublines of B_i on P must be contained in the $\sqrt{q}+1$ planes σ. This shows that each plane σ meets each Baer subplane B_i in a Baer subline. Consequently, each plane σ meets B in at least $q+1+q\sqrt{q}$ points.

Now we consider the case that some plane τ on l contains at least $q+\sqrt{q}+1$ points of B. We may assume that π is this plane. Then the counting argument at the beginning of the proof shows $|\pi \cap B| = q+\sqrt{q}+1$ and also that each plane π_i meets B in exactly $q+\sqrt{q}+1$ points. Thus $\pi_i \cap B$ is equal to the Baer subplane B_i.

The $q-\sqrt{q}$ planes τ on l that meet π_1 in a tangent have at most $q+\sqrt{q}+1$ points in B. Thus each plane τ can meet at most one of the Baer subplanes B_i in a Baer subline on P. Since there are only $q-\sqrt{q}$ different planes τ but q different planes π_i, it follows that there are at least $\sqrt{q}$ planes π_i for which the Baer subplane $B_i = \pi_i \cap B$ is contained in the $\sqrt{q}+1$ planes σ on l.

Thus, each of the planes σ meets B in the $q+1$ points of l and in at least $\sqrt{q}$ different Baer sublines on P. Thus each plane σ has at least $2q+1$ points in B. As we have seen in the beginning of the proof, this implies that σ contains no tangent on P. Thus each Baer subplane $B_i = \pi_i \cap B$ must meet σ in a Baer subline. As before it follows that each of the $\sqrt{q}+1$ planes σ meets B in at least $q+1+q\sqrt{q}$ points.

This now implies that all B-lines meet in a common point. Because we know that each point of B lies on at least one B-line. Hence, there exist at least $|B|/(q+1) > \sqrt{q}+1$ B-lines.

Consider one B-line l. By the previous arguments, either l intersects all B-lines or l lies on $\sqrt{q}+1$ planes σ_i, $i = 0, \ldots, \sqrt{q}$, that meet B each in at least $q\sqrt{q}+q+1$ points. In the union of the σ_i, there are at least $q+1+(\sqrt{q}+1)q\sqrt{q} = q^2+q\sqrt{q}+q+1 \geq |B| - \sqrt{q}$ points of B. So there are at most $\sqrt{q}$ points left that can lie outside of one of the planes σ_i. This implies that every B-line lies inside one of the planes σ_i. Hence l meets every other B-line.

We have shown that the B-lines mutually meet. Thus all B-lines pass through a common point or all B-lines lie in a common plane. The second case is however not possible, since by Lemma 2.2, every plane contains at most $\sqrt{q}+1$ B-lines. □

Now we are ready to complete the proof. Let V be the point belonging to every B-line. If r is the number of B-lines, then $|B| = 1 + rq$, since every point of B lies on a B-line. Since $|B| \leq q^2+q\sqrt{q}+q+\sqrt{q}+1$, it follows that $r \leq q+\sqrt{q}+1$. If we take a tangent to a point P of B with $P \neq V$, then P lies on only one B-line. As in Lemma 2.5, it follows that a tangent t on P lies in a plane π that meets B in the points of a Baer subplane. It follows that $V \notin \pi$. Since a Baer subplane has $q+\sqrt{q}+1$ points, we obtain that $r = q+\sqrt{q}+1$ and that B is a cone with vertex V over a Baer subplane. This completes the proof of Theorem 1.1.

3. A characterization of PG$(4, \sqrt{q})$ in PG$(4, q)$

In this section, we assume that B is a set of at most $q^2+q\sqrt{q}+q+\sqrt{q}+1$ points in PG$(4, \sqrt{q})$, where q is a square and $q > 9$. We assume that B does not contain a plane or a Baer cone. We shall show that B consists of the points in a subgeometry PG$(4, \sqrt{q})$.

For a point P outside B, we will consider the projection of B in solids S not containing P. The image is a set of points in S that meets every line of S. By the result of the previous section, this image will contain a plane or a Baer cone of S. Our first lemma says that the first case cannot occur.

Lemma 3.1. *If $P \notin B$, then the projection of B in a solid not on P contains no plane.*

Proof. Suppose the statement is not true. That means that the projection contains a plane, that is, PG$(4, q)$ has a solid S on P with the property that every line of S on P meets B. In particular, the solid S contains at least q^2+q+1 points of S. Outside of S there are at most $q\sqrt{q}+\sqrt{q}$ points in B. Consider a point P' with $P' \notin S \cup B$.

First we consider the case that the projection of B from P' onto S contains a plane. Then there exists a solid S' on P' that also meets B in at least q^2+q+1 points. Since S contains all but at most $q\sqrt{q}+\sqrt{q}$ points of B, it follows that the plane $\pi := S \cap S'$ meets B in at least $q^2+q+1-q\sqrt{q}-\sqrt{q} \geq q^2-q\sqrt{q}$ points. Since B contains no plane, there exists a point $X \in \pi \setminus B$. Then X lies on q^4 planes that meet π only in X. Each of these planes has a point in B and every point of $B \setminus \pi$ lies in exactly q^2 of these planes. This shows that at least q^2 points of B lie outside of π. Since π meets B in at least $q^2-q\sqrt{q}$ points, we obtain $|B| \geq 2q^2-q\sqrt{q}$, which is a contradiction.

Now we consider the case that the projection of B from P' onto S contains a Baer cone C. Thus, each of the lines $P'X$ with $X \in C$ meets B. Notice that $|C| = 1+(q+\sqrt{q}+1)q \geq |B| - \sqrt{q}$. Hence, it follows that at least $q^2+q+1-\sqrt{q}$ points of C belong to B since the points of S are fixed under the projection.

Since we assumed in the beginning of this section that B does not contain a Baer cone, there exists a point $Y \in C$ with $Y \notin B$. Consider a solid T that does not contain Y and project B from Y into this solid T. Then the image of $C \setminus \{Y\}$ of this projection is either a Baer subplane (if Y is the vertex of the cone) or the union of $\sqrt{q}+1$ concurrent lines (if Y is not the vertex of the cone). In any case, the image of the cone has at most $q\sqrt{q}+q+1$ points. Thus the image of $C \cap B$ under this projection has at most $q\sqrt{q}+q+1$ points. Thus, if $x := |C \cap B|$, then the image of B under the projection has at most $|B| - x + q\sqrt{q}+q+1$ points.

But the image blocks every line of T and has therefore at least q^2+q+1 points. It follows that $|B| - x + q\sqrt{q}+q+1 \geq q^2+q+1$, that is, $x \leq |B| - q^2 + q\sqrt{q} \leq 2q\sqrt{q}+q+\sqrt{q}+1$. But we have seen that $x \geq q^2+q+1-\sqrt{q}$. Hence $q^2 \leq 2q\sqrt{q}+2\sqrt{q}$ and thus $q < 9$. □

Lemma 3.2. *If $P \notin B$ and if S is a solid not containing P, then the projection of B from P on S contains a Baer cone. If T is a set consisting of t points of B, then the image of T under this projection contains at least $t - \sqrt{q}$ points of this Baer cone.*

Proof. By the previous lemma and Theorem 1.1, the image of the projection contains a Baer cone C of S. We have $|C| = 1 + (q+\sqrt{q}+1)q \geq |B| - \sqrt{q}$, which implies the second statement. □

Lemma 3.3. *Every plane meets B in at most $q+\sqrt{q}+1$ points.*

Proof. Consider a plane π. It contains a point P not in B. Apply the previous lemma to P.□

Lemma 3.4. *If $P \notin B$, then one line on P has $\sqrt{q}+1$ points in B and these form a Baer subline, and every other line on P meets B in at most one point.*

Proof. Consider the projection of B from P to a solid S not containing P. This projection contains a Baer cone C. Let V be the vertex of the Baer cone. We may assume that we have chosen S in such a way that $V \in B$.

Let l be a line contained in C (then $V \in l$). The plane $\pi := \langle l, P\rangle$ contains a point of B on each line through P and thus π meets B in at least $q+1$ points. We claim that $\pi \cap B$ meets every line of π. Assume to the contrary that π has a line not containing a point of B. Let P' be the intersection of this line and the line PV. Then $P' \notin B$ so $P' \neq V$ and thus $P' \notin S$. Projecting B from P' onto S, we obtain a point set containing a Baer cone C'. Since π has at least $q+1$ points in B, Lemma 3.2 shows that the line $l = \pi \cap S$ contains at least $q+1-\sqrt{q}$ points in C'. Since each line of S meets a Baer cone of S in 1, $\sqrt{q}+1$ or $q+1$ points and since $q+1-\sqrt{q} > \sqrt{q}+1$, it follows that l is a line of C'. But then every line of π on P' meets B, a contradiction.

This shows that $\pi \cap B$ meets every line of π. It follows from Lemma 3.3 and Lemma 2.1 that $\pi \cap B$ either contains a B-line or that $\pi \cap B$ is a Baer subplane of π.

If for one of the lines $l \subseteq C$, the plane $\pi = \langle l, P\rangle$ meets B in a Baer subplane, then P lies on a Baer subline of this Baer subplane and the application of Lemma 3.2 proves the claim.

Assume therefore that for all choices of the $q+\sqrt{q}+1$ lines $l \subset C$, the plane $\pi = \langle l, P\rangle$ contains a B-line. All these planes π contain the line PV. Hence some point of the line PV must lie on at least two lines that are contained in B. This contradicts Lemma 3.3. □

Lemma 3.5. *No line is contained in B.*

Proof. Assume that the line l is contained in B. Let P be a point of B that is not in l. Then the plane $\pi := \langle P, l\rangle$ meets B in at least $q+2$ points and the set $\pi \cap B$ contains the line l. By Lemma 3.3 we have $|B \cap \pi| \leq q+1+\sqrt{q}$. It follows that some line of π meet B in exactly two points. This contradicts Lemma 3.4. □

Lemma 3.6. *The set B is a subgeometry isomorphic to $PG(4, \sqrt{q})$.*

Proof. We know that every line that has more than one point in B meets B in $\sqrt{q}+1$ points. Consider the incidence structure consisting of the points of B and the lines that meet B in $\sqrt{q}+1$ points. Of course two points of this incidence structure are on a unique line and every line of this incidence structure has $\sqrt{q}+1 \geq 3$ points. It also satisfies the axiom of Pasch (or Veblen and Young). In fact, if l_1, l_2, l_3, l_4 are four lines, no three on a point and any two meet except possibly for l_3 and l_4, then l_3 and l_4 must meet in $PG(4, q)$. But the intersection point lies on two lines that meet B in $\sqrt{q}+1$ points, so the intersection point must be in B (Lemma 3.4). Thus the incidence structure is PG$(4, \sqrt{q})$. □

The preceding lemma completes the proof of Theorem 1.2.

4. 2-Blocking sets in PG(N, q), $N > 4$, q square

Now we prove Theorem 1.3.

Proof. We proceed by induction on N. The theorem is valid for $N = 4$. Assume $N > 4$ and assume that the theorem is true for $N-1$ dimensions.

Suppose there is a point $P \notin B$ lying on a secant. If no such point exists, then B is a plane.

Project from P onto a hyperplane S. Let the projection be B'. Then B' is a 2-blocking set in S of size $|B'| < q^2 + q\sqrt{q} + q + \sqrt{q} + 1$. So, by the induction hypothesis, B' contains a plane or a Baer cone. This all implies that there is a hyperplane π_1 through P containing at least q^2+q+1 points of B.

Project now from a point $P' \notin \pi_1 \cup B$ onto π_1.

Case 1. The projection contains a plane π'.

Then in $\langle \pi', P'\rangle$ lie at least q^2+q+1 points of B and so in the plane $\pi'' = \langle \pi', P'\rangle \cap \pi_1$ lie at least $q^2 - q\sqrt{q} + q - \sqrt{q} + 1$ points.

Suppose this plane π'' does not lie in B. Then take a point $P'' \in \pi'' \setminus B$. If we project from P'', the projection has at most $2q\sqrt{q} + q + 2\sqrt{q} + 1$ points. But the projection must have at least q^2+q+1 points since the smallest 2-blocking set is a plane.

Hence, the plane π'' is contained in B.

Case 2. The projection contains a Baer cone.

At most $\sqrt{q}$ points cannot be projected onto the Baer cone. So at least $q^2+q+1-\sqrt{q}$ points of $\pi_1 \cap B$ lie on a Baer cone C. Assume there is a point Y of the Baer cone not in B.

By using the arguments of the proof of Lemma 3.1, after projection from Y, the cone is projected onto at most $q\sqrt{q} + q + 1$ points.

At most $q\sqrt{q} + 2\sqrt{q}$ points of B do not lie on C. Hence, the projection has at most $2q\sqrt{q} + q + 2\sqrt{q} + 1$ points. This gives the same contradiction as above. This shows that the Baer cone C is contained in B.

Case 3. The projection is a subgeometry PG$(4, \sqrt{q})$.
Then all points of $\pi_1 \cap B$ lie in this subgeometry $\sigma =$ PG$(4, \sqrt{q})$. So B shares at least q^2+q+1 points with σ.

Suppose $Y \in \sigma \setminus B$. Project from Y, then the projection of PG$(4, \sqrt{q})$ contains at most $q\sqrt{q} + q + \sqrt{q} + 1$ points.

At most $q\sqrt{q} + \sqrt{q}$ points of B do not lie in σ; so the projection has at most $2q\sqrt{q} + q + 2\sqrt{q} + 1$ points. This again is false. Hence, $\sigma = B$. □

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