Asymptotics of Cross Sections for Convex Bodies†

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Abstract. For normed isotropic convex bodies in $\mathbb{R}^n$ we investigate the behaviour of the $(n-1)$-dimensional volume of intersections with hyperplanes orthogonal to a fixed direction, considered as a function of the distance of the hyperplane to the origin. It is a conjecture that for arbitrary normed isotropic convex bodies and random directions this function – with high probability – is close to a Gaussian density, for large dimension n. This would be a kind of central limit theorem. We determine this function explicitly for several families of convex bodies and several directions and obtain results concerning the asymptotic behaviour supporting the conjecture.

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Introduction

The main topic of the present paper is a version of the central limit theorem in the geometric context of convex bodies.

A *normed convex body* $K \subseteq \mathbb{R}^n$ is a convex compact set of volume 1 whose centre of inertia is at 0. A normed convex body is *isotropic* if its ellipsoid of inertia is a Euclidean ball.

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The radius of this ball will be denoted by L_K (following the notation of [10]); thus

$$L_K^2 = \int_K (x \cdot u)^2 dx,$$

independently of u in the unit sphere S^{n-1} of $\mathbb{R}^n$. Note that for each convex body K with nonempty interior there exists an affine transformation $f\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $f(K)$ is normed and isotropic. For a direction $u \in S^{n-1}$ we define

$$\varphi_{K,u}(t) := \lambda_{n-1}(\{x \in K;\, x \cdot u = t\}) \quad (t \in \mathbb{R}),$$

where λ_{n-1} denotes the $(n-1)$-dimensional volume.

For a number of situations we show that $\varphi_{K,u}$ tends to a Gaussian density, for $n \to \infty$. It appears to be a known conjecture among specialists that this is a general phenomenon: For large dimensions the function $\varphi_{K,u}$ should be close to a Gaussian density for all isotropic normed convex bodies K and for 'most' directions $u \in S^{n-1}$. More precisely, the density corresponding to K should be the Gaussian with variance L_K^2.

In Section 1, we define several versions of the central limit property for subsets of the set of isotropic normed convex bodies. The only result of a general nature we have so far is an estimate asserting that the mean value of $\varphi_{K,u}(0)$ over S^{n-1} is bounded from below by the value $\frac{1}{\sqrt{2\pi}L_k}$ of the corresponding Gaussian density $g_{L_K^2}$ at zero, asymptotically for $n \to \infty$ (see Proposition 1.3).

In Sections 2 and 3 we prove versions of the central limit property for cubes and for balls in $\mathbb{R}^n$, respectively.

In Section 4 we show that for the $|\cdot|_1$-ball in $\mathbb{R}^n$, i.e., the cross polytope X_n, normed to volume 1, and $\omega := \frac{1}{\sqrt{n}}(1, \ldots, 1)$, the functions $\varphi_{X_n,\omega}$ tend to the appropriate Gaussian density.

In Section 5 we derive results for the regular simplex Δ_n. In this case we show that $\varphi_{\Delta_n,u}$ converges to the appropriate Gaussian density on a certain discrete set of directions $u \in S^{n-1}$. We show that the set of exceptional u's is small in an appropriate sense. This example may be of particular interest since it shows that our considerations are not restricted to centrally symmetric sets.

The computational results described above should be considered as evidence supporting the conjecture that the central limit property holds generally. Moreover, we feel that the explicit expressions as well as the methods presented in this paper are of independent interest and importance.

The starting point of this paper was a question, addressed to the second-named author by Peter Stollmann, concerning the $(n-1)$-dimensional volume of cross sections of the cube $[0, 1]^n$ orthogonal to the direction ω given above. Motivated by the stochastic interpretation of the resulting explicit expressions and by the explicit computations for the ball the first-named author formulated conjectures which served as a guide line for our further investigations.

The contents of the present paper are related to results and ideas which developed starting from Milman's proof [9] of Dvoretzky's theorem. We refer to [11], [15] for references and further developments.

After finishing the first version of the present paper the authors became aware of the preprint [1] where a central limit property is proved for a certain subclass of isotropic convex bodies. Likewise, the second-named author [16] obtained a version of the central limit property for a subclass containing the Euclidean balls, cubes, cross polytopes and regular simplices. In these papers, the closeness of the marginal distributions to the corresponding Gaussian distribution is described by uniform convergence of the distribution functions and by convergence in law, respectively. In contrast, in the results of the present paper we obtain closeness of the densities in the L_1- and L_∞-norms.

1. The central limit property

In this section we define the 'central limit property'. This definition is motivated by the results which are sketched in the introduction and proved in the subsequent sections. In order to formulate these properties we first introduce some notation.

The set of all isotropic normed convex bodies in $\mathbb{R}^n$ will be denoted by $\mathcal{K}_0^n$.

Let $K \in \mathcal{K}_0^n$. Geometrically, $\varphi_{K,u}(t)$ as defined in the introduction is the $(n-1)$-dimensional volume of the intersection of K with the hyperplane $\{x \in \mathbb{R}^n;\ x \cdot u = t\}$. Note that the Brunn-Minkowski theorem (cf. [12; p. 3], [14; p. 309]) states that the function $\varphi_{K,u}(\cdot)^{\frac{1}{n-1}}$ is concave on its support. It is one of the principle objectives of this paper to investigate the behaviour of $\varphi_{K,u}(t)$ as a function of t for large n and 'typical' u.

We investigate whether for large dimension n the function $\varphi_{K,u}$ is close to the Gaussian density function for all directions u with the exception of a small set of vectors (in the sense of measure). A convenient formulation for this is to use the expected value of the norm (L_∞-norm or L_1-norm) of the difference, for random unit vectors u.

We denote the Gaussian density by g_{σ^2},

$$g_{\sigma^2}(t) \quad := \quad \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{t^2}{2\sigma^2}} = \frac{1}{\sigma} g_1\left(\frac{t}{\sigma}\right),$$

where $t \in \mathbb{R}$, $\sigma > 0$. We recall that the volume of the unit ball in $\mathbb{R}^n$ is

$$\omega_n = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)},$$

and that the $(n-1)$-dimensional volume of the unit sphere S^{n-1} is

$$\sigma_{n-1} = n\omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}.$$

By μ_{n-1} we denote the surface measure on S^{n-1}, normed to a probability measure.

Definition 1.1. *Let $\mathcal{T} \subseteq \bigcup_{n\in\mathbb{N}} \mathcal{K}_0^n$. We say that $\mathcal{T}$ satisfies a central limit property if one of the following properties holds:*

(a) $$\sup_{K\in\mathcal{T}\cap\mathcal{K}_0^n} \mathbf{E}\left(\sup_{t\in\mathbb{R}} \left|\varphi_{K,u}(t) - g_{L_K^2}(t)\right|\right) \longrightarrow 0 \quad \textit{for } n \to \infty,$$

(b) $$\sup_{K\in\mathcal{T}\cap\mathcal{K}_0^n} \mathbf{E}\left(\int_{-\infty}^{\infty}\left|\varphi_{K,u}(t)-g_{L_K^2}(t)\right|dt\right) \longrightarrow 0 \quad \textit{for } n\to\infty.$$

Here, $\mathbf{E}$ *denotes the expectation with respect to the probability measure* μ_{n-1}.

Note that Proposition 2.5 proved below shows that (a) implies (b) provided that $\sup_{K\in\mathcal{T}} L_K < \infty$.

We conjecture that $\mathcal{T}=\bigcup_{n\in\mathbb{N}}\mathcal{K}_0^n$ satifies the central limit property in both forms given above. If this conjecture could be shown to be true, e.g., in the form given in Definition 1.1 (a) then it would follow that

$$\sup_{K\in\mathcal{K}_0^n}\sup_{t\in\mathbb{R}}\left|\int_{S^{n-1}}\varphi_{K,u}(t)d\mu_{n-1}(u)-g_{L_K^2}(t)\right| \longrightarrow 0 \quad \text{for } n\to\infty.$$

The only result we can show in the general context is a one-sided bound at $t=0$ for this convergence. In order to show this we need an expression for the mean of $\varphi_{K,u}(t)$ over $u\in S^{n-1}$ which will be derived next.

Lemma 1.2. *Let* $n\geqslant 2$, $K\in\mathcal{K}_0^n$. *Then*

$$\varphi_K(t) := \int_{S^{n-1}}\varphi_{K,u}(t)d\mu_{n-1}(u) = \frac{1}{\sqrt{\pi}}\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}\int_{\{x\in K;\ |x|\geqslant|t|\}}\left(1-\left(\frac{t}{|x|}\right)^2\right)^{\frac{n-3}{2}}\frac{1}{|x|}dx$$

for all $t\in\mathbb{R}$.

Proof. We define the distribution function

$$\Phi_{K,u}(t) := \lambda_n(\{x\in K;\ x\cdot u\leqslant t\}),$$

for $u\in S^{n-1}$, $t\in\mathbb{R}$. Fubini's theorem implies

$$\begin{aligned}\int_{S^{n-1}}\Phi_{K,u}(t)d\mu_{n-1}(u) &= \int_{\{(u,x)\in S^{n-1}\times K;\ x\cdot u\leqslant t\}} dx\,d\mu_{n-1}(u)\\ &= \int_K \mu_{n-1}(\{u\in S^{n-1};\ x\cdot u\leqslant t\})dx.\end{aligned}$$

Now

$$\begin{aligned}\mu_{n-1}(\{u\in S^{n-1};\ x\cdot u\leqslant t\}) &= \mu_{n-1}(\{u\in S^{n-1};\ \frac{x}{|x|}\cdot u\leqslant\frac{t}{|x|}\})\\ &= \begin{cases} 0 & \text{for } t<-|x|,\\ \frac{\sigma_{n-2}}{\sigma_{n-1}}\int_{-\frac{\pi}{2}}^{\arcsin\frac{t}{|x|}}\cos^{n-2}\varphi\,d\varphi & \text{for } |t|\leqslant|x|,\\ 1 & \text{for } t>|x|.\end{cases}\end{aligned}$$

In order to get the average density function we differentiate the average distribution function with respect to the variable t and thus get

$$\int\limits_{S^{n-1}} \varphi_{K,u}(t)d\mu_{n-1}(u) = \frac{\sigma_{n-2}}{\sigma_{n-1}} \int\limits_{\{x\in K;\, |x|\geqslant|t|\}} \left(1-\left(\frac{t}{|x|}\right)^2\right)^{\frac{n-3}{2}} \frac{1}{|x|}dx. \qquad \square$$

Proposition 1.3. *For all $n \geqslant 2$, $K \in \mathcal{K}_0^n$, one has*

$$\varphi_K(0) \geqslant \gamma_n g_{L_K^2}(0),$$

with

$$\gamma_n = \frac{\sqrt{2}}{\sqrt{n}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}.$$

Moreover, $\gamma_n \to 1$ $(n \to \infty)$.

Proof. We recall that

$$L_K^2 = \int\limits_K (x\cdot u)^2 dx = \frac{1}{n}\int\limits_K |x|^2 dx \quad (u \in S^{n-1})$$

and

$$g_{L_K^2}(0) = \frac{1}{\sqrt{2\pi}L_K} = \frac{\sqrt{n}}{\sqrt{2\pi}\left(\int\limits_K |x|^2dx\right)^{\frac{1}{2}}}.$$

Hölder's inequality, for $p=3$, $q=\frac{3}{2}$, implies

$$1 = \int\limits_K \frac{|x|^{\frac{2}{3}}}{|x|^{\frac{2}{3}}}dx \leqslant \left(\int\limits_K |x|^2dx\right)^{\frac{1}{3}} \left(\int\limits_K \frac{1}{|x|}dx\right)^{\frac{2}{3}},$$

$$\varphi_K(0) = \frac{1}{\sqrt{\pi}}\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}\int\limits_K \frac{1}{|x|}dx \geqslant \frac{1}{\sqrt{\pi}}\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}\frac{1}{\left(\int\limits_K |x|^2dx\right)^{\frac{1}{2}}} = \gamma_n g_{L_K^2}(0).$$

The convergence $\gamma_n \to 1$ $(n\to\infty)$ is a consequence of Stirling's formula which we will state subsequently. $\square$

The version of Stirling's formula used in the present article is the inequality

$$\left(\frac{x}{e}\right)^x \sqrt{2\pi x} \leqslant \Gamma(x+1) \leqslant \left(\frac{x}{e}\right)^x \sqrt{2\pi x}e^{\frac{\vartheta(x)}{12x}},$$

valid for all $x>0$, where $0 \leqslant \vartheta(x) \leqslant 1$. The left hand side inequality is a consequence of an equality of Binet (see [7; sec. 1.5, (63)]), whereas the right hand side inequality follows from Stirling's series (see [7; sec. 1.5, (66)]).

Remarks 1.4. (a) In [16], a subclass of $\bigcup_{n\in\mathbb{N}}\mathcal{K}_0^n$ is introduced for which the asymptotic formula $\lim_{\dim K\to\infty}\frac{\varphi_K(0)}{g_{L_K^2}(0)}=1$ is shown.

(b) In [6], [4], a two-sided estimate for $\varphi_{K,u}(0)$ in terms of L_K is given, namely

$$c_1\frac{1}{L_K}\leqslant\varphi_{K,u}(0)\leqslant c_2\frac{1}{L_K},$$

which holds independently of $n\in\mathbb{N}$, $K\in\mathcal{K}_0^n$, $u\in S^{n-1}$. Besides boundedness, no asymptotic properties, for $n\to\infty$, can be derived from these bounds.

2. Results for the cube

We consider the cube $C^n:=\left[-\frac{1}{2},\frac{1}{2}\right]^n$ in $\mathbb{R}^n$. The main result of this section is the fact that the set of cubes satisfies the central limit property either sense as stated in Definition 1.1.

Theorem 2.1. *There exists a function* $\alpha\colon(0,1]\to(0,\infty)$,

$$\alpha(a)=O(a^2),\tag{2.1}$$

such that, for all $n\in\mathbb{N}$, $u\in S^{n-1}$,

$$\left\|\varphi_{C^n,u}-g_{\frac{1}{12}}\right\|_\infty\leqslant\alpha(|u|_\infty).\tag{2.2}$$

As a consequence, one obtains

$$\int\limits_{S^{n-1}}\left\|\varphi_{C^n,u}-g_{\frac{1}{12}}\right\|_\infty d\mu_{n-1}(u)=O\left(\frac{1+\ln n}{n}\right).\tag{2.3}$$

Theorem 2.2. *There exists a function* $\beta\colon(0,1]\to(0,2]$,

$$\beta(a)=O(a^2(1-\ln a)^{\frac{1}{2}}),\tag{2.4}$$

such that, for all $n\in\mathbb{N}$ $u\in S^{n-1}$,

$$\left\|\varphi_{C^n,u}-g_{\frac{1}{12}}\right\|_1\leqslant\beta(|u|_\infty).\tag{2.5}$$

As a consequence, one obtains

$$\int\limits_{S^{n-1}}\left\|\varphi_{C^n,u}-g_{\frac{1}{12}}\right\|_1 d\mu_{n-1}(u)=O\left(\frac{(1+\ln n)^{\frac{3}{2}}}{n}\right).\tag{2.6}$$

The proof of these theorems will require some preparations.

Let $X_1,\dots,X_n$ be independent random variables which are uniformly distributed on $[-\frac{1}{2},\frac{1}{2}]$. Let $u\in S^{n-1}$. Then $\varphi_{C^n,u}$ is the density of the distribution of $u_1X_1+\ldots+u_nX_n$, $\varphi_{C^n,u}=\varphi_{u_1}*\varphi_{u_2}*\ldots*\varphi_{u_n}$, where $\varphi_{u_j}=\frac{1}{|u_j|}1_{[-\frac{|u_j|}{2},\frac{|u_j|}{2}]}$, and the Fourier transform $\widehat{\varphi_{C^n,u}}(\xi)=\int_{\mathbb{R}}\varphi_{C^n,u}(t)e^{-it\xi}dt$ satisfies

$$\widehat{\varphi_{C^n,u}}=\widehat{\varphi_{u_1}}\widehat{\varphi_{u_2}}\cdots\widehat{\varphi_{u_n}},$$

with $\widehat{\varphi_{u_j}}(\xi)=\frac{2}{\xi u_j}\sin\frac{\xi u_j}{2}$.

Proposition 2.3. *There exists a function* $\alpha_1\colon (0,\frac{1}{2}] \to (0,\infty)$,

$$\alpha_1(a) = O(a^2), \tag{2.7}$$

such that

$$\left\|\widehat{\varphi_{C^n,u}} - \widehat{g_{\frac{1}{12}}}\right\|_1 \leqslant \alpha_1(|u|_\infty) \tag{2.8}$$

for all $n \in \mathbb{N}$, $u \in S^{n-1}$ *with* $|u|_\infty \leqslant \frac{1}{2}$. *(Recall that* $g_{\frac{1}{12}}(t) = \sqrt{\frac{6}{\pi}}e^{-6t^2}$*; thus* $\widehat{g_{\frac{1}{12}}}(\xi) = e^{-\frac{1}{3!}\left(\frac{\xi}{2}\right)^2}$*.)*

Proof. First we note the inequalities

$$1 - \frac{1}{3!}\xi^2 \leqslant \frac{1}{\xi}\sin\xi \leqslant e^{-\frac{1}{3!}\xi^2},$$

for $\xi \in \mathbb{R}$, $|\xi| \leqslant \pi$, and

$$\left(\frac{\sin\xi}{\xi}\right)^2 \leqslant \frac{1}{1+\frac{1}{3}\xi^2},$$

for all $\xi \in \mathbb{R}$. Choose $0 < a \leqslant \frac{1}{2}$, and let $u \in S^{n-1}$, $|u|_\infty \leqslant a$. In order to estimate $\int_{|\xi|\leqslant\frac{2\sqrt{6}}{a}} |\widehat{\varphi_{C^n,u}}(\xi) - \widehat{g_{\frac{1}{12}}}(\xi)|\,d\xi$ we use the inequality

$$\prod_{j=1}^n\left(1-\frac{1}{3!}\left(\frac{\xi u_j}{2}\right)^2\right) \leqslant \prod_{j=1}^n \frac{2}{\xi u_j}\sin\frac{\xi u_j}{2} \leqslant \prod_{j=1}^n e^{-\frac{1}{3!}\left(\frac{\xi}{2}\right)^2 u_j^2} = e^{-\frac{1}{3!}\left(\frac{\xi}{2}\right)^2},$$

valid for $|\xi| \leqslant \frac{2\sqrt{6}}{a}$. Looking for $u \in S^{n-1}$ satisfying $|u|_\infty \leqslant a$ for which the left hand side of the last inequality becomes smallest one shows that this is the case for $u_1 = \ldots = u_k = a$, where $k = [\frac{1}{a^2}]$. Therefore the left hand side is estimated below by

$$\begin{aligned}
&\left(1-\frac{(\xi a)^2}{24}\right)^{\left[\frac{1}{a^2}\right]}\left(1-\frac{\xi^2}{24}\left(1-\left[\frac{1}{a^2}\right]a^2\right)\right)\\
&=\left(1-\frac{(\xi a)^2}{24}\right)^{\left[\frac{1}{a^2}\right]}\left(1-\frac{(\xi a)^2}{24}\left(\frac{1}{a^2}-\left[\frac{1}{a^2}\right]\right)\right)\\
&\geqslant\left(1-\frac{(\xi a)^2}{24}\right)^{\frac{1}{a^2}}.
\end{aligned}$$

This finally implies

$$\int\limits_{|\xi|\leqslant\frac{2\sqrt{6}}{a}} |\widehat{\varphi_{C^n,u}}(\xi) - \widehat{g_{\frac{1}{12}}}(\xi)|\,d\xi \leqslant \int\limits_{|\xi|\leqslant\frac{2\sqrt{6}}{a}} \left(e^{-\frac{\xi^2}{24}} - \left(1-\frac{\xi^2}{24}a^2\right)^{\frac{1}{a^2}}\right)d\xi.$$

In order to estimate $\int_{\frac{2\sqrt{6}}{a}}^{\infty} |\widehat{\varphi_{C^n,u}}(\xi)|d\xi$ we use

$$|\widehat{\varphi_{C^n,u}}(\xi)|^2 \leqslant \prod_{j=1}^{n} \frac{1}{1+\frac{1}{3}\left(\frac{\xi u_j}{2}\right)^2}.$$

A similar argument as above shows that the right hand side is estimated by

$$\frac{1}{\left(1+\frac{\xi^2}{12}a^2\right)^{\left[\frac{1}{a^2}\right]}} \cdot \frac{1}{\left(1+\frac{\xi^2}{12}a^2\left(\frac{1}{a^2}-\left[\frac{1}{a^2}\right]\right)\right)} \leqslant \frac{1}{\left(1+\frac{\xi^2}{12}a^2\right)^{\frac{1}{a^2}}}.$$

Summarizing we obtain inequality (2.8) with the function

$$\alpha_1(a) := 2\left(\int_0^{\frac{2\sqrt{6}}{a}} \left(e^{-\frac{\xi^2}{24}} - \left(1-\frac{\xi^2}{24}a^2\right)^{\frac{1}{a^2}}\right)d\xi + \int_{\frac{2\sqrt{6}}{a}}^{\infty}\left(\left(\frac{1}{1+\frac{\xi^2}{12}a^2}\right)^{\frac{1}{2a^2}} + \widehat{g_{\frac{1}{12}}}(\xi)\right)d\xi\right).$$

Finally we estimate the function α_1. We split the first term on the right hand side as

$$\int_0^{\frac{2\sqrt{6}}{a}} \left(e^{-\frac{\xi^2}{24}} - \left(1-\frac{\xi^2}{24}a^2\right)^{\frac{1}{a^2}}\right)d\xi$$

$$= \frac{2\sqrt{6}}{a}\int_0^a \left(e^{-\frac{\eta^2}{a^2}} - \left(1-\eta^2\right)^{\frac{1}{a^2}}\right)d\eta + \frac{2\sqrt{6}}{a}\int_a^1 \left(e^{-\frac{\eta^2}{a^2}} - \left(1-\eta^2\right)^{\frac{1}{a^2}}\right)d\eta$$

It is elementary (but somewhat tedious) to show that the integrand in the first of these integrals takes its maximal value for $\eta = a$, and therefore this integral is bounded by

$$\frac{2\sqrt{6}}{a}a\left(e^{-1} - \left(1-a^2\right)^{\frac{1}{a^2}}\right) = O(a^2).$$

We estimate the second integral by

$$\frac{2\sqrt{6}}{a}\int_a^1 \frac{\eta}{a}\left(e^{-\frac{\eta^2}{a^2}} - \left(1-\eta^2\right)^{\frac{1}{a^2}}\right)d\eta = -\sqrt{6}\left(e^{-\frac{\eta^2}{a^2}} - \frac{1}{1+a^2}\left(1-\eta^2\right)^{\frac{1}{a^2}+1}\right)\Bigg|_a^1$$

$$\leqslant \sqrt{6}\left(e^{-1} - \left(1-a^2\right)^{\frac{1}{a^2}}\right) + \sqrt{6}\left(1-\frac{1-a^2}{1+a^2}\right)\left(1-a^2\right)^{\frac{1}{a^2}} = O(a^2).$$

The second term in the expression of α_1 given above can be estimated by

$$\int_{\frac{2\sqrt{6}}{a}}^{\infty} \frac{1}{\left(1+\frac{\xi^2}{12}a^2\right)^{\frac{1}{2a^2}}}d\xi \leqslant \int_{\frac{2\sqrt{6}}{a}}^{\infty} \frac{1}{\left(\frac{a}{2\sqrt{3}}\right)^{\frac{1}{a^2}}\xi^{\frac{1}{a^2}}}d\xi$$

$$= \left(\frac{2\sqrt{3}}{a}\right)^{\frac{1}{a^2}}\left(1-\frac{1}{a^2}\right)^{-1}\xi^{1-\frac{1}{a^2}}\Bigg|_{\frac{2\sqrt{6}}{a}}^{\infty} = \left(\frac{1}{\sqrt{2}}\right)^{\frac{1}{a^2}}\left(\frac{a^2}{1-a^2}\right)\frac{2\sqrt{6}}{a} = O(a^2),$$

whereas the third term is estimated by

$$\frac{a}{2\sqrt{6}} \int_{\frac{2\sqrt{6}}{a}}^{\infty} \xi e^{-\frac{\xi^2}{24}} d\xi = \sqrt{6} a e^{-\frac{1}{a^2}} = O(a^2). \qquad \square$$

Lemma 2.4. *If* $0 < a \leqslant 1$ *then*

$$\mu_{n-1}(\{u \in S^{n-1};\ |u|_\infty \geqslant a\}) \leqslant n(1-a^2)^{\frac{n-1}{2}}.$$

Proof. Note that

$$\mu_{n-1}(\{u \in S^{n-1};\ |u|_\infty \geqslant a\}) \leqslant 2n\mu_{n-1}(\{u \in S^{n-1};\ u_1 \geqslant a\}),$$

$$\mu_{n-1}(\{u \in S^{n-1};\ u_1 \geqslant a\}) = $$
$$= \frac{\int_0^{\arccos a} \sin^{n-2}\varphi d\varphi}{2\int_0^{\frac{\pi}{2}} \sin^{n-2}\varphi d\varphi} = \frac{\int_0^{\sqrt{1-a^2}} \frac{t^{n-2}}{\sqrt{1-t^2}} dt}{2\int_0^1 \frac{t^{n-2}}{\sqrt{1-t^2}} dt} \leqslant \frac{\int_0^{\sqrt{1-a^2}} t^{n-2} dt}{2\int_0^1 t^{n-2} dt} = \frac{(1-a^2)^{\frac{n-1}{2}}}{2}.$$

(In order to prove the inequality note that $t^{n-2} > 0$ and that $\frac{1}{\sqrt{1-t^2}}$ is monotone increasing on $(0,1)$; invert the fraction and subdivide the integral $\int_0^1 \cdot$.) $\square$

Proof of Theorem 2.1. First we note that Ball [2] has shown the remarkable equality

$$\sup_{t \in \mathbb{R}, u \in S^{n-1}, n \in \mathbb{N}} \varphi_{C^n,u}(t) = \sqrt{2}.$$

Defining $\alpha := \frac{1}{2\pi}\alpha_1$ on $(0, \frac{1}{2})$ and $\alpha := \sqrt{2} + \sqrt{\frac{6}{\pi}}$ on $(\frac{1}{2}, 1]$ we obtain (2.1) and (2.2) from Proposition 2.3.

In order to show (2.3) we use Lemma 2.4 and obtain

$$\mu_{n-1}\left(\left\{u \in S^{n-1};\ \left\|\varphi_{C^n,u} - g_{\frac{1}{12}}\right\|_\infty > \alpha(a)\right\}\right)$$
$$\leqslant \mu_{n-1}\left(\left\{u \in S^{n-1};\ |u|_\infty > a\right\}\right) \leqslant n\left(1-a^2\right)^{\frac{n-1}{2}}.$$

This implies

$$\int_{S^{n-1}} \left\|\varphi_{C^n,u} - g_{\frac{1}{12}}\right\|_\infty d\mu_{n-1}(u) \leqslant \alpha(a) + \left(\sqrt{2} + \sqrt{\frac{6}{\pi}}\right)\left(1-a^2\right)^{\frac{n-1}{2}},$$

for $0 < a \leqslant 1$. For $a = \sqrt{\frac{4\ln n}{n-1}}$ we obtain

$$n\left(1-a^2\right)^{\frac{n-1}{2}} = n\left(1 - \frac{4\ln n}{n-1}\right)^{\frac{n-1}{4\ln n}\frac{4\ln n}{2}} \leqslant n e^{-\frac{4\ln n}{2}} = \frac{1}{n}, \tag{2.9}$$

$$\int_{S^{n-1}} \left\|\varphi_{C^n,u} - g_{\frac{1}{12}}\right\|_\infty d\mu_{n-1}(u) \leqslant \alpha\left(\sqrt{\frac{4\ln n}{n-1}}\right) + \frac{\sqrt{2} + \sqrt{\frac{6}{\pi}}}{n} = O\left(\frac{1+\ln n}{n}\right). \qquad \square$$

Next we show that smallness of $\left\|\varphi_{C^n,u} - g_{\frac{1}{12}}\right\|_\infty$ implies smallness of $\left\|\varphi_{C^n,u} - g_{\frac{1}{12}}\right\|_1$.

Proposition 2.5. *There exists a function* $\beta_1\colon (0,\infty) \to (0,2]$,

$$\beta_1(\delta) = O(\delta(-\ln\delta)^{\frac{1}{2}}), \qquad (0 < \delta \leqslant \frac{1}{\sqrt{2\pi}}) \tag{2.10}$$

such that

$$\|\varphi - g_{\sigma^2}\|_1 \leqslant \beta_1\left(\sigma \|\varphi - g_{\sigma^2}\|_\infty\right) \tag{2.11}$$

for all $\varphi \in L_1(\mathbb{R}) \cap L_\infty(\mathbb{R})$, $\varphi \geqslant 0$, $\int \varphi(t)dt = 1$, $\sigma > 0$.

Proof. It is sufficient to treat the case $\sigma = 1$; we let $g := g_1$. Let φ be as just described, $0 < \delta := \|\varphi - g\|_\infty \leqslant \frac{1}{\sqrt{2\pi}}$. Then

$$\begin{aligned}
\|\varphi - g\|_1 &\leqslant \int\limits_{|t|\geqslant r} |\varphi(t) - g(t)|dt + \int\limits_{-r}^{r} \delta dt \leqslant 2r\delta + \int\limits_{|t|\geqslant r} g(t)dt + 1 - \int\limits_{|t|\leqslant r} \varphi(t)dt \\
&\left(\int\limits_{|t|\leqslant r} \varphi(t)dt \geqslant \int\limits_{|t|\leqslant r} g(t)dt - 2r\delta \right) \\
&\leqslant 4r\delta + \int\limits_{|t|\geqslant r} g(t)dt + 1 - \int\limits_{|t|\leqslant r} g(t)dt \\
&= 4r\delta + 2\int\limits_{|t|\geqslant r} g(t)dt = 4\left(r\delta + \frac{1}{\sqrt{2\pi}} \int\limits_{r}^{\infty} e^{-\frac{t^2}{2}}dt \right).
\end{aligned}$$

The last expression attains its minimum for $r = (-\ln(\delta^2 2\pi))^{\frac{1}{2}}$, which shows that (2.11) is satisfied with

$$\beta_1(\delta) := 4\left(\delta\left(-\ln\left(\delta^2 2\pi\right)\right)^{\frac{1}{2}} + \frac{1}{\sqrt{2\pi}} \int\limits_{(-\ln(\delta^2 2\pi))^{\frac{1}{2}}}^{\infty} e^{-\frac{t^2}{2}}dt \right).$$

The first term in the function β_1 is bounded by

$$\delta(-2\ln\delta - \ln(2\pi))^{\frac{1}{2}} = O(\delta(-\ln\delta)^{\frac{1}{2}}).$$

The second term is bounded by

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi}} \frac{1}{\left(-\ln\left(\delta^2 2\pi\right)\right)^{\frac{1}{2}}} \int\limits_{(-\ln(\delta^2 2\pi))^{\frac{1}{2}}}^{\infty} te^{-\frac{t^2}{2}}dt = \frac{1}{\sqrt{2\pi}} \frac{1}{\left(-\ln\left(\delta^2 2\pi\right)\right)^{\frac{1}{2}}} e^{-\frac{(-\ln(\delta^2 2\pi))}{2}} \\
&= \frac{1}{\sqrt{2\pi}} \frac{\sqrt{2\pi}\delta}{\left(-\ln\left(\delta^2 2\pi\right)\right)^{\frac{1}{2}}} = O(\delta(-\ln\delta)^{\frac{1}{2}}),
\end{aligned}$$

for $0 < \delta \leqslant \frac{1}{2\sqrt{2\pi}}$. □

Proof of Theorem 2.2. By Theorem 2.1 and Proposition 2.5 there exist constants $a_0 > 0$, $c > 0$ such that

$$\beta_1(\frac{1}{\sqrt{12}}\alpha(a)) \leqslant ca^2(1-\ln a)^{\frac{1}{2}} \qquad \text{for} \quad 0 < a \leqslant a_0,$$
$$2 \leqslant ca^2(1-\ln a)^{\frac{1}{2}} \qquad \text{for} \quad a_0 < a \leqslant 1.$$

Choosing $\beta := \beta_1 \circ (\frac{1}{\sqrt{12}}\alpha)$ on $(0, a_0]$ and β as 2 on $(a_0, 1]$ one obtains (2.4) and (2.5). As in the proof of Theorem 2.2 we now use Lemma 2.4,

$$\mu_{n-1}\left(\left\{u \in S^{n-1};\ \left\|\varphi_{C^n,u} - g_{\frac{1}{12}}\right\|_1 > \beta(a)\right\}\right)$$
$$\leqslant \mu_{n-1}\left(\left\{u \in S^{n-1};\ |u|_\infty > a\right\}\right) \leqslant n\left(1-a^2\right)^{\frac{n-1}{2}}.$$

This implies

$$\int_{S^{n-1}} \left\|\varphi_{C^n,u} - g_{\frac{1}{12}}\right\|_1 d\mu_{n-1}(u) \leqslant \beta(a) + 2n\left(1-a^2\right)^{\frac{n-1}{2}},$$

for $0 < a \leqslant 1$. For $a = \sqrt{\frac{4\ln n}{n-1}}$ we now find, observing (2.9),

$$\int_{S^{n-1}} \left\|\varphi_{C^n,u} - g_{\frac{1}{12}}\right\|_1 d\mu_{n-1}(u) \leqslant \beta\left(\sqrt{\frac{4\ln n}{n-1}}\right) + \frac{2}{n} = O\left(\frac{(1+\ln n)^{\frac{3}{2}}}{n}\right). \qquad \square$$

3. Lower dimensional volumes for Euclidean balls

In this section we show results which are similar to those of Section 2 but stronger.

Let $B_n \subseteq \mathbb{R}^n$ be the Euclidean ball of volume 1, i.e., of radius

$$r_n := \omega_n^{-\frac{1}{n}} = \frac{\Gamma\left(\frac{n}{2}+1\right)^{\frac{1}{n}}}{\sqrt{\pi}}.$$

Recall that $\omega_n \to 0$ for $n \to \infty$ and, more strongly, that

$$r_n = \frac{\Gamma\left(\frac{n}{2}+1\right)^{\frac{1}{n}}}{\sqrt{\pi}} \approx \frac{\left(\frac{n}{2e}\right)^{\frac{1}{2}}\sqrt{2\pi\frac{n}{2}}^{\frac{1}{n}}}{\sqrt{\pi}} \approx \sqrt{\frac{n}{2\pi e}} \longrightarrow \infty$$

(from Stirling's formula) for $n \to \infty$.

For $0 \leqslant m \leqslant n$ we denote by $G(n,m)$ the set of all m-dimensional subspaces on $\mathbb{R}^n$. For $U \in G(n,m)$, $x \in U$ we define

$$\varphi_{B_n,U}(x) := \lambda_{n-m}\left((x+U^\perp) \cap B_n\right).$$

Moreover, we shall use the notation

$$g_{\sigma^2,m}(x) := \frac{1}{(2\pi\sigma^2)^{\frac{m}{2}}} e^{-\frac{|x|^2}{2\sigma^2}},$$

for $x \in \mathbb{R}^n$, $m \in \mathbb{N}$, $\sigma > 0$.

Before stating the main result we note that

$$L_{B_n}^2 = \frac{1}{n} \int\limits_{B_n} |x|^2 dx = \frac{1}{n} \sigma_{n-1} \int\limits_0^{r_n} r^{n+1} dr =$$

$$= \omega_n \frac{r_n^{n+2}}{n+2} = \frac{\Gamma\left(\frac{n}{2}+1\right)^{\frac{2}{n}}}{(n+2)\pi} \approx \frac{\left(\frac{n}{2e}\right)\sqrt{2\pi\frac{n}{2}}^{\frac{2}{n}}}{(n+2)\pi} \longrightarrow \frac{1}{2\pi e},$$

for $n \to \infty$.

We define $s_n := L_{B_n}$.

Theorem 3.1. *There exists $c > 0$ such that for all $1 \leqslant m \leqslant n-4$ and all $U \in G(n,m)$ one has*

$$\int\limits_{x \in U} \left|\varphi_{B_n,U}(x) - g_{s_n^2,m}(x)\right| dx \leqslant c\,\frac{m}{n}\,.$$

Proof. The Lebesgue measure on B_n can be decomposed as $\frac{n}{r_n^n}\int_0^{r_n} \mu_{n-1,r} r^{n-1} dr$, where $\mu_{n-1,r}$ denotes the normed surface measure on rS^{n-1}. Let $1 \leqslant m \leqslant n-4$, $U \in G(n,m)$; without restriction $U = \mathbb{R}^m$ $(= \mathbb{R}^m \times \{0\})$. Then [3; Theorem (2)] implies

$$\left\|\varphi_{B_n,U} - \frac{n}{r_n^n}\int_0^{r_n} g_{\frac{r^2}{n},m} r^{n-1} dr\right\|_{L_1(U)} \leqslant 2\frac{m+3}{n-m-3}. \tag{3.1}$$

The next step is obtaining an estimate for

$$\left\|\frac{n}{r_n^n}\int_0^{r_n} g_{\frac{r^2}{n},m} r^{n-1} dr - g_{\frac{r_n^2}{n},m}\right\|_1 \leqslant \frac{n}{r_n^n}\int_0^{r_n} \left\|g_{\frac{r^2}{n},m} - g_{\frac{r_n^2}{n},m}\right\|_1 r^{n-1} dr. \tag{3.2}$$

In order to achieve this we note

$$\left\|g_{\frac{r^2}{n},m} - g_{\frac{r_n^2}{n},m}\right\|_1 \leqslant \sum_{j=1}^{m} \int\limits_{x_j \in \mathbb{R}} \left|g_{\frac{r^2}{n}}(x_j) - g_{\frac{r_n^2}{n}}(x_j)\right| dx_j$$

$$= m\left\|g_{\frac{r^2}{n}} - g_{\frac{r_n^2}{n}}\right\|_1 = m\left\|g_{(\frac{r}{r_n})^2} - g_1\right\|_1.$$

From the fact that $(t,x) \mapsto g_t(x)$ solves the heat equation one obtains that there is a constant c' such that

$$\|g_t - g_1\|_1 \leqslant c'|t-1| \tag{3.3}$$

for all $0 < t \leqslant 1$. Therefore we can continue inequality (3.2) by

$$\begin{aligned} &\leqslant \frac{mn}{r_n^n} \int_0^{r_n} \left\| g_{(\frac{r}{r_n})^2} - g_1 \right\|_1 r^{n-1} dr \\ &= \frac{mn}{r_n^n} \int_0^1 \left\| g_{s^2} - g_1 \right\|_1 s^{n-1} ds \qquad (3.4) \\ &\leqslant c' mn \int_0^1 \left(1 - s^2\right) s^{n-1} ds = c' \frac{2m}{n+2}. \end{aligned}$$

It remains to obtain an estimate for

$$\left\| g_{\frac{r_n^2}{n},m} - g_{s_n^2,m} \right\|_1 \leqslant m \left\| g_{\frac{r_n^2}{n}} - g_{s_n^2} \right\|_1 . \qquad (3.5)$$

Obviously $s_n^2 \leqslant \frac{r_n^2}{n}$, and Stirling's formula implies

$$\frac{n(\pi n)^{\frac{1}{n}}}{2e(n+2)\pi} \leqslant s_n^2 \leqslant \frac{r_n^2}{n} \leqslant \frac{(\pi n)^{\frac{1}{n}}}{2e\pi} e^{\frac{1}{6n}}.$$

Therefore, by the argument leading to (3.3) we obtain $c'', c''' > 0$ such that

$$\begin{aligned} \left\| g_{\frac{r_n^2}{n}} - g_{s_n^2} \right\|_1 &\leqslant c'' \left| \frac{r_n^2}{n} - s_n^2 \right| \\ &\leqslant c'' \frac{(\pi n)^{\frac{1}{n}}}{2\pi e} \left(e^{\frac{1}{6n}} - \frac{n}{n+2} \right) \leqslant c''' \frac{1}{n}. \qquad (3.6) \end{aligned}$$

Putting the inequalities (3.1), (3.2), (3.4), (3.5), (3.6) together one obtains

$$\left\| \varphi_{B_n,U} - g_{s_n^2,m} \right\|_1 \leqslant 2 \left(\frac{m+3}{n-m-3} + c' \frac{m}{n+2} + c''' \frac{m}{n} \right)$$

which amounts to an estimate as asserted. □

We note that Theorem 3.1 implies that smallness of $\frac{m}{n}$ implies closeness of the marginal densities $\varphi_{B_n,U}$ to the corresponding Gaussian density with respect to the L_1-norm. For the L_∞-norm the statement turns out to be weaker, as stated in the following theorem. In this case, we aim for closeness of $\varphi_{B_n,U}$ to $g_{\frac{1}{2\pi e},m}$.

Theorem 3.2. *For each $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\sup_{x \in U} \left| \varphi_{B_n,U}(x) e^{-\frac{m}{2}} - e^{-\pi e |x|^2} \right| < \varepsilon.$$

for all $n > m \geqslant 1$ satisfying $\frac{m^2}{n} < \delta$ and all $U \in G(n,m)$.

Proof. (i) First we calculate $\varphi_{B_n,U}(x)$. Since $(x+U^\perp)\cap B_n$ is an $(n-m)$-dimensional ball of radius $(r_n^2-|x|^2)^{\frac{1}{2}}$ we obtain

$$\varphi_{B_n,U}(x)=\omega_{n-m}\left(r_n^2-|x|^2\right)^{\frac{n-m}{2}}=\frac{\Gamma\left(\frac{n}{2}+1\right)^{\frac{n-m}{n}}}{\Gamma\left(\frac{n-m}{2}+1\right)}\left(1-|x|^2\frac{\pi}{\Gamma\left(\frac{n}{2}+1\right)^{\frac{2}{n}}}\right)^{\frac{n-m}{2}}$$

for $|x|\leqslant r_n$, and zero otherwise.

(ii) Next we show that $\varphi_{B_n,U}(0)e^{-\frac{m}{2}}$ is close to 1 if $\frac{m^2}{n}$ is close to zero.
We have, by Stirling's formula,

$$\begin{aligned}\varphi_{B_n,U}(0)e^{-\frac{m}{2}} &= \frac{\Gamma\left(\frac{n}{2}+1\right)^{\frac{n-m}{n}}}{\Gamma\left(\frac{n-m}{2}+1\right)}e^{-\frac{m}{2}}\approx\frac{\left(\frac{n}{2e}\right)^{\frac{n-m}{2}}\sqrt{2\pi\frac{n}{2}}^{\frac{n-m}{n}}}{\left(\frac{n-m}{2e}\right)^{\frac{n-m}{2}}\sqrt{2\pi\frac{n-m}{2}}}e^{-\frac{m}{2}}\\ &=\left(\frac{n}{n-m}\right)^{\frac{n-m}{2}}(2\pi)^{-\frac{m}{2n}}\left(\frac{n}{n-m}\right)^{\frac{1}{2}}\left(\frac{n}{2}\right)^{-\frac{m}{2n}}e^{-\frac{m}{2}}.\end{aligned}$$

Now, $\left(\frac{n}{n-m}\right)^{\frac{n-m}{2}}e^{-\frac{m}{2}}=\left(1+\frac{m}{n-m}\right)^{\frac{n-m}{m}\cdot\frac{m}{2}}e^{-\frac{m}{2}}$ is close to 1 if $\frac{m^2}{n}$ is close to zero (take the logarithm and develop), and it is easy to see that the other terms are close to 1. (Also, the inequality in Stirling's formula can be used to derive suitable estimates instead of the '$\approx$'-sign.)

(iii) In order to treat the factor $\left(1-|x|^2\frac{\pi}{\Gamma\left(\frac{n}{2}+1\right)^{\frac{2}{n}}}\right)^{\frac{n-m}{2}}$ in $\varphi_{B_n,U}(x)$ we first calculate

$$\frac{\Gamma(\frac{n}{2}+1)^{\frac{2}{n}}}{\frac{n}{2}}\approx\frac{\frac{n}{2e}\sqrt{2\pi\frac{n}{2}}^{\frac{2}{n}}}{\frac{n}{2}}\longrightarrow\frac{1}{e},$$

and also

$$\frac{\Gamma(\frac{n}{2}+1)^{\frac{2}{n}}}{\frac{n}{2}}\leqslant\frac{1}{e}\,(\pi n)^{\frac{1}{n}}\,e^{\frac{1}{3n^2}}.$$

This implies

$$\left(1-|x|^2\frac{\pi e}{e\,\Gamma(\frac{n}{2}+1)^{\frac{2}{n}}}\right)^{\frac{n-m}{2}}\leqslant\left(\left(1-\frac{|x|^2\pi e}{e\,\Gamma(\frac{n}{2}+1)^{\frac{2}{n}}}\right)^{e\Gamma(\frac{n}{2}+1)^{\frac{2}{n}}}\right)^{\tilde\alpha}\leqslant e^{-\pi e|x|^2\tilde\alpha}$$

for $|x|\leqslant r_n$, where

$$\tilde\alpha:=\inf\left\{\frac{n-m}{n}(\pi n)^{-\frac{1}{n}}e^{-\frac{1}{3n^2}};\ n\geqslant 2,\ m\leqslant\frac{n}{2}\right\}>0$$

This shows that there is a constant c, such that $\varphi_{B_n,U}(x)$ is bounded by $ce^{-\tilde\alpha\pi e|x|^2}$, for $m\leqslant\frac{n}{2}$. Again by Stirling's formula, for any $R\geqslant 0$, $\varphi_{B_n,U}(x)e^{-\frac{m}{2}}$ is close to $e^{-\pi e|x|^2}$ uniformly for $|x|\leqslant R$ if $\frac{m^2}{n}$ is close to zero. Therefore we obtain the statement of the theorem. □

Corollary 3.3. *For all $m \in \mathbb{N}$ one has, for $n \to \infty$,*

$$\sup_{U \in G(n,m)} \sup_{x \in U} \left| \varphi_{B_n,U}(x) - g_{\frac{1}{2\pi e},m}(x) \right| \longrightarrow 0,$$

$$\sup_{U \in G(n,m)} \int_U \left| \varphi_{B_n,U}(x) - g_{\frac{1}{2\pi e},m}(x) \right| dx \longrightarrow 0.$$

Remark 3.4. A corresponding statement, for $m = n-1$, for the sequence of spheres $\sqrt{n}S^{n-1}$ is well-established; in [8] this observation is attributed to Poincaré [13]. (It was pointed out to the authors that this was noticed earlier by Maxwell; see also the discussion in [3] concerning the history of this property.) In this case the limiting Gaussian density has variance one. The occurence of the different variance in our case is explained by the fact that the mass of the ball B_n is 'concentrated' in a suitable spherical shell around the sphere $(\sqrt{n}L_{B_n})S^{n-1}$. This latter phenomenon is discussed in [16] for more general sets in $\mathcal{K}_0^n$.

4. $(n-1)$-dimensional volumes for the cross polytope

Our next example is the normed isotropic cross polytope X_n, i.e., a $|\cdot|_1$-ball. The volume of the $|\cdot|_1$-unit ball in $\mathbb{R}^n$ is $2^n\frac{1}{n!}$. Thus, in order to normalize to volume 1 we consider the $|\cdot|_1$-ball

$$X_n := \left\{ x \in \mathbb{R}^n;\ \sum_{j=1}^n |x_j| \leqslant \frac{n!^{\frac{1}{n}}}{2} \right\}.$$

It is obvious that, for normed isotropic sets, there may be exceptional directions where closeness to a Gaussian distribution fails. For example, taking the cube C^n and the direction $u = e_1$, one simply obtains $1_{[-\frac{1}{2},\frac{1}{2}]}$ for all n. Similarly, for $u = e_1$ and X_n, an application of Stirling's formula yields the convergence of $\varphi_{X_n,u}$ to $ee^{-2e|t|}$, uniformly and in $L_1(\mathbb{R})$.

The main purpose of this section is to derive explicit formulas and estimates for $\varphi_{X_n,\omega}$, where the direction is given by $\omega = \frac{1}{\sqrt{n}}(1,\dots,1)$.

Theorem 4.1. *Let $d_n := \frac{2\sqrt{n}}{n!^{\frac{1}{n}}}$. For $|t| \leqslant \frac{1}{d_n}$ we have*

$$\varphi_{X_n,\omega}(t) = \frac{nd_n}{2^{2n-1}} \sum_{k=0}^{n-1} \binom{n-1}{k}^2 (1+d_nt)^{n-1-k}(1-d_nt)^k,$$

and $\varphi_{X_n,\omega}(t) = 0$ otherwise.

Proof. Instead of treating X_n directly we first treat the $|\cdot|_1$-unit ball and renormalize later. For this purpose we denote by $c_n(r)$ the $(n-1)$-dimensional volume of the intersection of the $|\cdot|_1$-unit ball with the hyperplane $\{x \in \mathbb{R}^n;\ x \cdot \omega = \frac{r}{\sqrt{n}}\}$. For $-1 < r < 1$, this hyperplane intersects $2^n - 2$ faces of the surface of the $|\cdot|_1$-unit ball. These faces are $(n-1)$-dimensional simplices in a 2^n-tant having k negative and $n-k$ positive coordinates, for $1 \leqslant k \leqslant n-1$, the vertices of the simplex being k negative and $(n-k)$ positive unit vectors. Without restriction we assume the vertices to be $e_1, \dots, e_{n-k}, -e_{n-k+1}, \dots, -e_n$; e_j denoting the unit vectors. The $(n-2)$-dimensional intersection of this face with the hyperplane $\{x \in \mathbb{R}^n;\, x \cdot \omega = \frac{r}{\sqrt{n}}\}$ is the orthogonal product of an $(n-k-1)$-dimensional simplex in $\mathbb{R}^{n-k}$ and a $(k-1)$-dimensional simplex in $\mathbb{R}^k$. The volumes of these simplices are

$$\frac{1}{2^{n-k-1}}(1+r)^{n-k-1}\frac{\sqrt{n-k}}{(n-k-1)!} \text{ and } \frac{1}{2^{k-1}}(1-r)^{k-1}\frac{\sqrt{k}}{(k-1)!}.$$

Thus the $(n-2)$-dimensional volume is

$$\frac{1}{2^{n-2}}(1+r)^{n-k-1}(1-r)^{k-1}\frac{\sqrt{(n-k)k}}{(n-k-1)!(k-1)!}.$$

The height of the corresponding pyramid is obtained as the convex combination of the respective heights for $r = \pm 1$, which are the distances

$$\begin{aligned} \left|\frac{1}{n-k}(\underbrace{1,\dots,1}_{n-k},0,\dots,0) - \frac{1}{n}(1,\dots,1)\right| &= \left(\frac{k}{(n-k)n}\right)^{\frac{1}{2}}, \\ \left|\frac{1}{k}(0,\dots,0,\underbrace{-1,\dots,-1}_{k}) - \frac{1}{n}(-1,\dots,-1)\right| &= \left(\frac{n-k}{kn}\right)^{\frac{1}{2}} : \end{aligned}$$

$$\frac{1}{2}(1+r)\left(\frac{k}{(n-k)n}\right)^{\frac{1}{2}} + \frac{1}{2}(1-r)\left(\frac{n-k}{kn}\right)^{\frac{1}{2}}.$$

Therefore the $(n-1)$-dimensional volume of the corresponding pyramid is

$$\begin{aligned} &\tfrac{1}{n-1}\tfrac{1}{2^{n-2}}(1+r)^{n-k-1}(1-r)^{k-1}\tfrac{\sqrt{(n-k)k}}{(n-k-1)!(k-1)!}\cdot \\ &\quad\cdot\tfrac{1}{2}\left((1+r)\left(\tfrac{k}{(n-k)n}\right)^{\frac{1}{2}} + (1-r)\left(\tfrac{n-k}{kn}\right)^{\frac{1}{2}}\right) \\ &= \tfrac{1}{2^{n-1}\sqrt{n}(n-1)(n-k-1)!(k-1)!}\cdot \\ &\quad\cdot\left(k(1+r)^{n-k}(1-r)^{k-1} + (n-k)(1+r)^{n-k-1}(1-r)^k\right). \end{aligned}$$

Using the fact that there are $\binom{n}{k}$ of these 2^n-tants and summing everything up one obtains

$$\begin{aligned} c_n(r) &= \frac{1}{2^{n-1}\sqrt{n}(n-1)}\sum_{k=1}^{n-1}\binom{n}{k}\frac{1}{(n-k-1)!(k-1)!}\cdot \\ &\quad\cdot\left(k(1+r)^{n-k}(1-r)^{k-1} + (n-k)(1+r)^{n-k-1}(1-r)^k\right) \\ &= \frac{\sqrt{n}}{2^{n-1}(n-1)!}\sum_{k=0}^{n-1}\binom{n-1}{k}^2(1+r)^{n-k-1}(1-r)^k, \end{aligned}$$

where the last equality involves an elementary computation.

Using the relation

$$\varphi_{X_n,\omega}(t) = \left(\frac{n!^{\frac{1}{n}}}{2}\right)^{n-1} c_n\left(d_n t\right) = \frac{(n-1)!\sqrt{n}d_n}{2^n} c_n(d_n t)$$

we finally obtain the indicated formula. □

In order to discuss the behaviour of $\varphi_{X_n,\omega}(t)$ for $n \to \infty$ we need an expression of $\varphi_{X_n,\omega}(t)$ in terms of powers of t.

Lemma 4.2. *For $n \in \mathbb{N}$, $x \in \mathbb{R}$ one has*

$$\sum_{k=0}^{n} \binom{n}{k}^2 (1+x)^{n-k}(1-x)^k = \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{2(n-k)}{n}\binom{n}{k}\left(-x^2\right)^k.$$

For the proof of Lemma 4.2 we need the following preparation.

Lemma 4.3. *For all $n \in \mathbb{N}$, $x, y \in \mathbb{C}$ one has*

$$\sum_{j=0}^{n}\sum_{k=0}^{n} \binom{n}{j}\binom{n}{k}(1+x)^j(1-x)^k y^{2n-j-k} = \sum_{j=0}^{n}\sum_{k=0}^{2(n-j)} \binom{n}{j}\binom{2(n-j)}{k}\left(-x^2\right)^j y^k.$$

Proof. The two expressions are obtained by applying the binomial formula to

$$((1+x)+y)^n((1-x)+y)^n = \left((1+y)^2 - x^2\right)^n.$$ □

Lemma 4.4. *For all $n \in \mathbb{N}$, $m \in \mathbb{Z}$ with $-n \leqslant m \leqslant n$, $x \in \mathbb{C}$ one has*

$$\sum_{k=m}^{n} \binom{n}{k}\binom{n}{k-m}(1+x)^{k-m}(1-x)^{n-k} = \sum_{j=0}^{\left[\frac{n-m}{2}\right]} \binom{2(n-j)}{n+m}\binom{n}{j}\left(-x^2\right)^j.$$

Proof. Compare the coefficients of y^{n+m} in Lemma 4.3. □

Proof of Lemma 4.2. Set $m = 0$ in Lemma 4.4. □

As a consequence of Theorem 4.1 and Lemma 4.2 we get the representation

$$\varphi_{X_n,\omega}(t) = \frac{n d_n}{2^{2n-1}} \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{2(n-k-1)}{n-1}\binom{n-1}{k}\left(-\left(d_n t\right)^2\right)^k.$$

Theorem 4.5. *We have*

$$\varphi_{X_n,\omega}(t) \longrightarrow \frac{e}{\sqrt{\pi}} e^{-e^2 t^2} = g_{\frac{1}{2e^2}}(t)$$

for $n \to \infty$, uniformly for $t \in \mathbb{R}$. Also, $\varphi_{X_n,\omega} \longrightarrow g_{\frac{1}{2e^2}}$ in $L_1(\mathbb{R})$, for $n \to \infty$.

Proof. (i) Denote by $a_{n,k}$ the coefficient of $(-t^2)^k$ in $\varphi_{X_{n,\omega}}$,

$$a_{n,k} = \begin{cases} \frac{n}{2^{2n-1}}\binom{2(n-k-1)}{n-1}\binom{n-1}{k} d_n^{2k+1} & \text{for } k \leqslant \left[\frac{n-1}{2}\right], \\ 0 & \text{otherwise.} \end{cases}$$

In order to determine $\lim_{n\to\infty} a_{n,k}$ we note

$$\lim_{n\to\infty} \frac{1}{n^k}\binom{n-1}{k} = \frac{1}{k!},$$
$$\lim_{n\to\infty} \frac{\sqrt{n}}{2^{2(n-k-1)}}\binom{2(n-k-1)}{n-1} = \frac{1}{\sqrt{\pi}},$$
$$\lim_{n\to\infty} \sqrt{n} d_n = 2e,$$

where the last two limits are obtained using Stirling's formula. Therefore

$$\begin{aligned} a_{n,k} &= \frac{\sqrt{n}}{2^{2(n-k-1)}}\binom{2(n-k-1)}{n-1}\frac{1}{n^k}\binom{n-1}{k}\left(\frac{\sqrt{n}d_n}{2}\right)^{2k+1} \\ &\longrightarrow \frac{1}{\sqrt{\pi}k!}e^{2k+1} \quad (n\to\infty). \end{aligned}$$

(ii) Now we show that, for $k \in \mathbb{N}_0$, the sequence $(a_{n,k})_n$ is increasing. In order to do this we show

$$\frac{a_{n+1,k}}{a_{n,k}} \geqslant 1 \text{ for } n \geqslant 2k+1.$$

Using the estimates in Stirling's formula we obtain

$$\begin{aligned} \frac{a_{n+1,k}}{a_{n,k}} &= \left(\frac{n+1}{n}\right)^{k+\frac{3}{2}} \frac{n-k-\frac{1}{2}}{n-2k}\left(\frac{n!^{\frac{1}{n}}}{(n+1)!^{\frac{1}{n+1}}}\right)^{2k+1} \\ &\geqslant \left(\frac{n+1}{n}\right)^{k+\frac{3}{2}} \frac{n-k-\frac{1}{2}}{n-2k}\left(\frac{n}{n+1}\right)^{2k+1}\left(\frac{(2\pi n)^{\frac{1}{2n}}}{(2\pi(n+1))^{\frac{1}{2(n+1)}} e^{\frac{1}{12(n+1)^2}}}\right)^{2k+1} \\ &\geqslant \left(\frac{n}{n+1}\right)^{k-\frac{1}{2}} \frac{n-k-\frac{1}{2}}{n-2k}\left(\frac{n^{\frac{1}{2n}}}{(n+1)^{\frac{1}{2(n+1)}}} e^{-\frac{1}{12(n+1)^2}}\right)^{2k+1}. \end{aligned}$$

For $k \geqslant 1$, $x \geqslant 2k+1$, we define

$$f(x) := \left(\frac{x}{x+1}\right)^{k-\frac{1}{2}} \frac{x-k-\frac{1}{2}}{x-2k}.$$

Then $f(x) \to 1$ $(x \to \infty)$, and

$$f'(x) = \left(k-\frac{1}{2}\right)\frac{x^{k-\frac{3}{2}}}{(x+1)^{k+\frac{1}{2}}(x-2k)^2}\left(-(3k+1)x+2k^2+k-\frac{1}{2}\right) < 0$$

for $x \geqslant 2k+1$; thus

$$\left(\frac{n}{n+1}\right)^{k-\frac{1}{2}} \frac{n-k-\frac{1}{2}}{n-2k} = f(n) > 1.$$

In order to show that the last factor in the previous lower estimate for $\frac{a_{n+1,k}}{a_{n,k}}$ is greater or equal 1 we have to show

$$\begin{aligned} 0 &\leqslant \ln\left(\frac{n^{\frac{1}{2n}}}{(n+1)^{\frac{1}{2(n+1)}}} e^{-\frac{1}{12(n+1)^2}}\right) \\ &= \frac{1}{2n}\ln n - \frac{1}{2(n+1)}\ln(n+1) - \frac{1}{12(n+1)^2}. \end{aligned}$$

Multiplying the last expression by $2n(n+1)$ and estimating,

$$\begin{aligned} (n+1)\ln n - n\ln(n+1) - \frac{n}{6(n+1)} &= \ln n - n\ln(1+\frac{1}{n}) - \frac{n}{6(n+1)} \\ &\geqslant \ln n - 1 - \frac{n}{6(n+1)} \geqslant \ln n - \frac{7}{6}, \end{aligned}$$

we see that the desired inequality holds for $n \geqslant 4$.

This implies the inequality $a_{n+1,k} \geqslant a_{n,k}$ for all $k \geqslant 1$, $n \geqslant 4$. Direct computation shows this inequality also for $k=1$, $n=3$.

In order to treat the case $k=0$ we compute and estimate

$$\begin{aligned} \left(\frac{a_{n+1,0}}{a_{n,0}}\right)^{2n(n+1)} &= \left(\frac{n+1}{n}\right)^{3n(n+1)} \frac{\left(n-\frac{1}{2}\right)^{2n(n+1)}}{n^{2n(n+1)}} \frac{n!^{n+1}}{(n+1)!^n} \\ &= \left(\frac{n+1}{n}\right)^{n(n+1)} \frac{\left((n+1)\left(n-\frac{1}{2}\right)\right)^{2n(n+1)}}{n^{4n(n+1)}} \frac{n!}{(n+1)^n} \\ &\geqslant \left(\frac{n+1}{n}\right)^{n^2} \frac{n!}{n^n} =: b_n. \end{aligned}$$

Then $b_1 = 2$, and the sequence (b_n) is increasing because of

$$\begin{aligned} \frac{b_n}{b_{n-1}} &= \left(\frac{n+1}{n}\right)^{n^2} \frac{n!}{n^n} \left(\frac{n-1}{n}\right)^{(n-1)^2} \frac{(n-1)^{n-1}}{(n-1)!} \\ &= \left(\frac{(n+1)(n-1)}{n^2}\right)^{n^2} \left(\frac{n-1}{n}\right)^{-n} = \frac{\left(1-\frac{1}{n^2}\right)^{n^2}}{\left(1-\frac{1}{n}\right)^n} > 1. \end{aligned}$$

(iii) From (i), (ii) we obtain $a_{n,k} \leqslant \frac{e}{\sqrt{\pi}} \frac{e^{2k}}{k!}$ for all $k \geqslant 0$, $n \geqslant 1$. This implies that the coefficients of the polynomial $\varphi_{X_{n,\omega}}$ are bounded by the coefficients of the power series of $\frac{e}{\sqrt{\pi}} e^{e^2 t^2}$. Since the radius of convergence of this power series is ∞ we can conclude $\varphi_{X_{n,\omega}}(t) \longrightarrow \frac{e}{\sqrt{\pi}} e^{-e^2 t^2}$

uniformly on compact subsets of $\mathbb{R}$. In order to conclude uniform convergence on $\mathbb{R}$ it is now sufficient to observe that $g_{\frac{1}{2e^2}}(t) \longrightarrow 0$ $(t \to \infty)$ and that $\varphi_{X_n,\omega}$ is even and monotone decreasing on $[0, \infty)$. For the latter note that, by the Brunn-Minkowski inequality (cf. [12; p. 3], [14; p. 309]), $(\varphi_{X_n,\omega})^{\frac{1}{n-1}}$ is concave on $[-\frac{1}{d_n}, \frac{1}{d_n}]$.

(iv) The convergence $\varphi_{X_n,\omega} \longrightarrow g_{\frac{1}{2e^2}}$ in $L_1(\mathbb{R})$ follows from Proposition 2.5. □

5. Results for the regular simplex

In this section we prove a discrete version of the central limit property for the regular simplices: We will not take the mean over all directions in S^{n-1} but only over those belonging to partitions of the vertices of the given simplex. As an appropriate weight of these directions we will use the $(n-1)$-dimensional volume of the Dirichlet-Voronoi cells on the unit sphere S^{n-1}.

In order to fix the notation let $\tilde{\Delta}_n$ be the standard n-dimensional regular simplex in $\mathbb{R}^{n+1}$, i.e.,

$$\tilde{\Delta}_n = \operatorname{conv}\{e_1, \dots, e_{n+1}\}.$$

We first compute the desired function for $\tilde{\Delta}_n$ and obtain the final formula by suitable scaling. Let $1 \leqslant k \leqslant n$, $m := n + 1 - k$. We calculate

$$\tilde{\varphi}_u(t) := \lambda_{n-1}(\{x \in \tilde{\Delta}_n;\ x \cdot u = t\})$$

for u pointing from the centre of an $(m-1)$-dimensional face to the centre of a $(k-1)$-dimensional face of $\tilde{\Delta}_n$; without restriction

$$\begin{aligned} u = u_{n,k} :&= \sqrt{\frac{km}{n+1}}\Big(\frac{1}{k}(\underbrace{1, \dots, 1}_{k}, 0, \dots, 0) - \frac{1}{m}(0, \dots, 0, \underbrace{1, \dots, 1}_{m})\Big) \\ &= \frac{1}{\sqrt{(n+1)km}}(\underbrace{m, \dots, m}_{k}, \underbrace{-k, \dots, -k}_{m}). \end{aligned}$$

Noting that $\{x \in \tilde{\Delta}_n;\ x \cdot u = t\}$ is the orthogonal cartesian product of suitably scaled copies of these faces one obtains the formula

$$\tilde{\varphi}_u(t) = \frac{\sqrt{m}}{(m-1)!}\frac{\sqrt{k}}{(k-1)!}\left(\frac{m}{n+1} + \sqrt{\frac{km}{n+1}}t\right)^{m-1}\left(\frac{k}{n+1} - \sqrt{\frac{km}{n+1}}t\right)^{k-1}.$$

Here we have used the expression $\frac{\sqrt{k}}{(k-1)!}$ for the $(k-1)$-dimensional volume of the standard embedded simplex. In order to obtain the regular simplex Δ_n of volume 1 one has to blow up $\tilde{\Delta}_n$ by a factor of $\left(\frac{n!}{\sqrt{n+1}}\right)^{\frac{1}{n}} =: c_n$. Hence, letting

$$\varphi_u(t) := \varphi_{\Delta_n,u}(t) = \lambda_{n-1}\left(\{x \in \Delta_n;\ x \cdot u = t\}\right)$$

we find

$$\begin{aligned}\varphi_u(t) &= c_n^{n-1}\tilde{\varphi}_u\left(\frac{t}{c_n}\right)\\ &= \frac{n!}{(m-1)!(k-1)!}d_{n,k}\left(\frac{m}{n+1}+d_{n,k}t\right)^{m-1}\left(\frac{k}{n+1}-d_{n,k}t\right)^{k-1},\end{aligned}$$

with $d_{n,k} := \frac{1}{c_n}\sqrt{\frac{km}{n+1}}$. We want to show that, for large n, the function φ_u is close to the appropriate Gaussian density for 'most' of the directions u considered above. These directions, however, are not evenly distributed on the unit sphere S^{n-1}. After the proof of the following theorem we will present considerations showing that this result can be interpreted as a discrete version of the central limit property.

Theorem 5.1. *For $k \to \infty$ one has*

$$\begin{aligned}&\sup\{\|\varphi_{\Delta_n,u_{n,k}}(t) - g_{\frac{1}{e^2}}\|_\infty;\ n \in \mathbb{N},\ n \geqslant 2k\} \longrightarrow 0,\\ &\sup\{\|\varphi_{\Delta_n,u_{n,k}}(t) - g_{\frac{1}{e^2}}\|_1;\ n \in \mathbb{N},\ n \geqslant 2k\} \longrightarrow 0.\end{aligned}$$

Remark 5.2. The radius of inertia for Δ_n is

$$L_{\Delta_n} = \frac{(n!)^{\frac{1}{n}}}{\sqrt{(n+1)(n+2)}(n+1)^{\frac{1}{2n}}},$$

and Stirling's formula implies $L_{\Delta_n} \longrightarrow \frac{1}{e}$ $(n \to \infty)$, showing that one may substitute $g_{\frac{1}{e^2}}$ for $g_{L^2_{\Delta_n}}$ in Theorem 5.1.

Proof of Theorem 5.1. In the following, we consider only the cases where $1 \leqslant k \leqslant \frac{n}{2}$. As before, we define $m := n+1-k$.

(i) Stirling's formula yields

$$\begin{aligned}&\varphi_{u_{n,k}}(t) =\\ &= \frac{n(n-1)^{n-1}d_{n,k}}{(m-1)^{m-1}(k-1)^{k-1}\sqrt{2\pi\frac{(k-1)(m-1)}{n-1}}}\cdot\\ &\qquad\cdot\left(\frac{m}{n+1}+d_{n,k}t\right)^{m-1}\left(\frac{k}{n+1}-d_{n,k}t\right)^{k-1}\left(1+O\left(\frac{1}{k}\right)\right)\\ &= C_{n,k}\psi_{u_{n,k}}(t),\end{aligned}$$

with

$$C_{n,k} = \frac{n}{\sqrt{2\pi\frac{(k-1)(m-1)}{n-1}}}d_{n,k}\left(1+O\left(\frac{1}{k}\right)\right),$$

$$
\begin{aligned}
&\psi_{u_{n,k}}(t) = \\
&= \left(\frac{m(n-1)}{(n+1)(m-1)} + d_{n,k}\frac{n-1}{(m-1)}t\right)^{m-1}\left(\frac{k(n-1)}{(n+1)(k-1)} - d_{n,k}\frac{n-1}{k-1}t\right)^{k-1} \\
&= \left(1 + \frac{1-\frac{2m}{n+1} + d_{n,k}(n-1)t}{m-1}\right)^{m-1}\left(1 + \frac{1-\frac{2k}{n+1} - d_{n,k}(n-1)t}{k-1}\right)^{k-1} \\
&= \left(1 + \frac{x}{m-1}\right)^{m-1}\left(1 - \frac{x}{k-1}\right)^{k-1},
\end{aligned}
$$

where, for abbreviation,

$$x = \frac{k-m}{n+1} + d_{n,k}(n-1)t.$$

(ii) Note first

$$\lim_{n\to\infty}\frac{c_n}{n} = \lim_{n\to\infty}\frac{1}{n}\frac{n}{e}\left(\sqrt{2\pi\frac{n}{n+1}}\right)^{\frac{1}{n}} = \frac{1}{e}.$$

For the first factor $C_{n,k}$ we therefore have

$$\lim_{k,m\to\infty} C_{n,k} = \lim_{k,m\to\infty}\frac{n}{c_n}\frac{\sqrt{\frac{km}{n+1}}}{\sqrt{2\pi\frac{(k-1)(m-1)}{n-1}}} = \frac{e}{\sqrt{2\pi}}.$$

(iii) Next we choose $t_0 > 0$ and show

$$\psi_{u_{n,k}}(t) \longrightarrow e^{-\frac{e^2t^2}{2}}$$

for $k, m \to \infty$, uniformly for $-t_0 \leqslant t \leqslant t_0$. We have

$$x = \frac{k-m}{n+1} + \frac{n-1}{c_n}\sqrt{\frac{km}{n+1}}t = O\left(\sqrt{\frac{km}{n}}\right)$$

uniformly for $-t_0 \leqslant t \leqslant t_0$. So

$$
\begin{aligned}
&\ln\psi_{u_{n,k}}(t) = \\
&= (m-1)\frac{x}{m-1} - (k-1)\frac{x}{k-1} - \frac{x^2}{2}\left(\frac{1}{m-1} + \frac{1}{k-1}\right) \\
&+\frac{x^3}{3}\left(\frac{1}{(m-1)^2} - \frac{1}{(k-1)^2}\right) - \dots \\
&= -\frac{x^2}{2}\frac{(n-1)}{(m-1)(k-1)} + O\left(\sqrt{\frac{n}{km}}\right).
\end{aligned}
$$

Now,

$$\begin{aligned}
&\lim_{k,m\to\infty} x^2 \frac{(n-1)}{(m-1)(k-1)} = \\
&= \lim_{k,m\to\infty} \frac{n-1}{(m-1)(k-1)} \cdot \\
&\cdot \left(\frac{(k-m)^2}{(n+1)^2} + 2\frac{(k-m)(n-1)}{(n+1)c_n}\sqrt{\frac{km}{n+1}}t + \frac{km(n-1)^2}{(n+1)c_n^2}t^2 \right) \\
&= e^2 t^2
\end{aligned}$$

uniformly for $|t| \leqslant t_0$, and

$$\lim_{k,m\to\infty} \sqrt{\frac{n}{km}} = 0;$$

therefore

$$\lim_{k,m\to\infty} \psi_{u_{n,k}}(t) = e^{-\frac{e^2t^2}{2}},$$

uniformly for $|t| \leqslant t_0$.

(iv) We now show that, for t_0 large enough, the functions $\varphi_{u_{n,k}}$, outside $[-t_0, t_0]$, are dominated by $\max(\varphi_{u_{n,k}}(-t_0), \varphi_{u_{n,k}}(t_0))$.

Differentiation shows that the function $x \mapsto \left(1 + \frac{x}{m-1}\right)^{m-1} \left(1 - \frac{x}{k-1}\right)^{k-1}$ is increasing on the interval $[-m+1, 0]$ and decreasing on $[0, k-1]$. The maximum at zero of this function is moved to $-\frac{k-m}{n+1}\frac{c_n}{d_{n,k}(n-1)}$ for $\varphi_{u_{n,k}}$, and this quantity tends to zero for $k, m \to \infty$.

(v) Steps (ii), (iii), (iv) show

$$\lim_{k,m\to\infty} \varphi_{u_{n,k}}(t) = g_{\frac{1}{e^2}}(t)$$

uniformly for $t \in \mathbb{R}$. Because of $\|\varphi_{u_{n,k}}\|_1 = \|g_{\frac{1}{e^2}}\|_1 = 1$ this also implies

$$\lim_{k,m\to\infty} \|\varphi_{u_{n,k}} - g_{\frac{1}{e^2}}\|_1 = 0.$$

These statements can be rephrased as in the assertions of the theorem. □

The next part of this section is devoted to determining smallest spherical caps containing the Dirichlet-Voronoi cells belonging to the vectors $u_{n,k}$ $(k = 1, \dots, n)$ defined above. We define D_n as the set of unit vectors in S^n which are, up to a permutation of the components, the vectors $u_{n,k}$ $(k = 1, \dots, n)$; the set D_n consists of $2^{n+1} - 2$ vectors. We now introduce the $(n-1)$-dimensional sphere

$$S_\omega^{n-1} := \{x \in S^n;\, x \cdot \omega = 0\},$$

where, as before, $\omega = \frac{1}{\sqrt{n}}(1, \dots, 1) \in \mathbb{R}^{n+1}$.

With any $u \in D_n$ we associate the Dirichlet-Voronoi *cell* (DV-*cell* for brevity)

$$C_u := \left\{x \in S_\omega^{n-1};\ |x-u| \leqslant \inf\{|x-\tilde{u}|;\ \tilde{u} \in D_n\}\right\}.$$

We want to associate with $u \in D_n$ the $(n-1)$-dimensional volume $\mu_{n-1}(C_u)$ as the appropriate weight.

For fixed $k \in \{1, \dots, n\}$ there are $\binom{n+1}{k}$ vectors arising from $u_{n,k}$ by a permutation of the coordinates; these vectors and the corresponding DV-cells will be called of type k.

For $1 \leqslant k \leqslant \frac{n}{2}$ let $a_{n,k}$ be the largest number such that the set

$$W_n(a_{n,k}) := \{u \in S_\omega^{n-1};\ |u|_\infty \geqslant a_{n,k}\}$$

covers all DV-cells of types k and $n+1-k$. (Here, $|\cdot|_\infty$ denotes the maximum norm on $\mathbb{R}^{n+1}$.)

Proposition 5.3. *The numbers $a_{n,k}$ defined above satisfy*

$$a_{n,k} = \left(2k + \frac{2k}{n+1-k} \sum_{i=k+1}^{[\frac{n+1}{2}]} \left(\sqrt{i(n+1-i)} - \sqrt{(i-1)(n+2-i)}\right)^2\right)^{-\frac{1}{2}},$$

and $W_n(a_{n,k})$ covers all DV*-cells of types* $1, \dots, k, n+1-k, \dots, n$.

Proof. Let $1 \leqslant k \leqslant \frac{n}{2}$. We want to determine the minimum of $|x|_\infty$ for x in a DV-cell of type k or $n+1-k$. Since $|x|_\infty$ is invariant under permutations of the coordinates and under reflection in 0 it is sufficient to consider the DV-cell $C_{u_{n,k}}$. A point $x \in S_\omega^{n-1}$ belongs to $C_{u_{n,k}}$ if and only if

$$x \cdot u_{n,k} \geqslant x \cdot u \quad \text{for all } u \in D_n. \tag{5.1}$$

Exploiting this inequality for the permutations of the coordinates of $u_{n,k}$ we obtain

$$\min\{x_i;\ 1 \leqslant i \leqslant k\} \geqslant \max\{x_i;\ k+1 \leqslant i \leqslant n\}.$$

Since $u_{n,k}$ is invariant under the permutations of the first k and the last m $(= n+1-k)$ coordinates we may finally assume without loss of generality

$$x_1 \geqslant x_2 \geqslant \ldots \geqslant x_{n+1}. \tag{5.2}$$

Exploiting (5.1) for $u = u_{n,s}$, where $s \in \{1, \dots, n\}$, $t := n+1-s$, we obtain (using also $x \cdot \omega = 0$)

$$\begin{aligned} \frac{n+1}{\sqrt{(n+1)km}} \sum_{i=1}^{k} x_i &= \frac{1}{\sqrt{(n+1)km}} \left(m \sum_{i=1}^{k} x_i - k \sum_{i=k+1}^{n+1} x_i\right) = x \cdot u_{n,k} \\ &\geqslant x \cdot u_{n,s} = \frac{1}{\sqrt{(n+1)st}} \left(t \sum_{i=1}^{s} x_i - s \sum_{i=s+1}^{n+1} x_i\right) = \frac{n+1}{\sqrt{(n+1)st}} \sum_{i=1}^{s} x_i; \end{aligned}$$

thus in particular

$$\frac{1}{\sqrt{k(n+1-k)}}\sum_{i=1}^{k} x_i \geqslant \frac{1}{\sqrt{s(n+1-s)}}\sum_{i=1}^{s} x_i \text{ for } s = k+1, \ldots, n+1-k. \tag{5.3}$$

Let $x \in S_\omega^{n-1}$ be such that x satisfies (5.2) and (5.3), and x minimizes $|x|_\infty$ among the elements of S_ω^{n-1} satisfying (5.2) and (5.3). Then clearly $|x|_\infty \leqslant a_{n,k}$. Define $a := |x|_\infty$.

First we show $x_1 = \ldots = x_k = a$: (5.3) for $s = n+1-k$ implies $\sum_{i=k+1}^{n+1-k} x_i \leqslant 0$. If $x_k < a$ it is possible to replace $x_1, \ldots, x_k$ by a and $x_{n+1-k}, \ldots, x_{n+1}$ by smaller numbers $\geqslant -a$, such that the resulting vector $\tilde{x}$ still satisfies $\tilde{x} \cdot \omega = 0$ and (5.2). Then (5.3) is also satisfied. The renormed vector $\frac{1}{|\tilde{x}|}\tilde{x}$ still satisfies (5.2) and (5.3) but also $\left|\frac{1}{|\tilde{x}|}\tilde{x}\right|_\infty < |x|_\infty$ contradicting the choice of x.

Let y be the unique vector with $y_1 = \ldots = y_k = a$ satisfying (5.3) with equalities, i.e.,

$$y_i = a\sqrt{\frac{k}{n+1-k}}\left(\sqrt{i(n+1-i)} - \sqrt{(i-1)(n+2-i)}\right)$$

for $k+1 \leqslant i \leqslant n+1-k$, and also satisfying $|y|_\infty = a$ and $y \cdot \omega = 0$, thus $y_{n-k+2} = \ldots = y_{n+1} = -a$. This y also satifies (5.2). We are going to show $x = y$.

Define

$$h_x(s) := \sum_{i=1}^{s} x_i \quad \text{for} \quad s = k, k+1, \ldots, n+1.$$

Then h_x and h_y are concave because of (5.2), $h_x(s) \leqslant h_y(s)$ for all $s = k+1, \ldots, n$, $h_x(k) = h_y(k) = ka$, $h_x(n+1) = h_y(n+1) = 0$. Assume that there exists $s \in \{k+1, \ldots, n\}$ with $h_x(s) < h_y(s)$, and let k_1, k_2 be such that $x_{s-k_1} > x_{s-k_1+1} = \ldots = x_s \geqslant x_{s+1} = \ldots = x_{s+k_2} > x_{s+k_2+1}$. Replacing x_i by $x_i + \frac{\varepsilon}{k_1}$ for $s-k_1+1 \leqslant i \leqslant s$ and by $x_i - \frac{\varepsilon}{k_2}$ for $s+1 \leqslant i \leqslant s+k_2$, with small $\varepsilon > 0$, and retaining the other coordinates, we obtain an element $\tilde{x}$ such that $h_{\tilde{x}}$ is still concave (i.e., $\tilde{x}$ satifies (5.2)), $h_{\tilde{x}} \leqslant h_y$ (i.e. $\tilde{x}$ satifies the inequalities (5.3)), and $\tilde{x} \cdot \omega = 0$. A short computation shows $|\tilde{x}| > |x| = 1$, and therefore we obtain a contradiction as above.

This shows $h_x = h_y$, $x = y$. It is easy to see that for this x the inequalities in (5.3) are also satisfied for $s = 1, \ldots, k, n-k+2, \ldots, n+1$. This means that (5.1) is satisfied for all $u = u_{n,k}$ $(k = 1, \ldots, n)$ and a fortiori for all $u \in D_n$. This means that x belongs in fact to $C_{u_{n,k}}$, and therefore $a_{n,k} = |x|_\infty = a$. From

$$1 = |x|^2 = a^2(2k + \frac{2k}{n+1-k}\sum_{i=k+1}^{\left[\frac{n+1}{2}\right]}(\sqrt{i(n+1-i)} - \sqrt{(i-1)(n+2-i)})^2)$$

we obtain the expression for $a_{n,k}$ as asserted.

In order to show the second statement we have to show that $(a_{n,k})$ is decreasing in k. Since $\left(\frac{k}{n+1-k}\right)$ is increasing in k it is sufficient to show

$$\frac{k}{n+1-k}\left(\sqrt{(k+1)(n-k)} - \sqrt{k(n+1-k)}\right)^2 \leqslant 1.$$

Now, setting $u = k(n+1-k)$, $v = n-2k$ in the inequality

$$\left(\sqrt{u+v}-\sqrt{u}\right)^2 = u\left(\sqrt{1+\frac{v}{u}}-1\right)^2 \leqslant u\left(\frac{v}{2u}\right)^2 = \frac{v^2}{4u}, \tag{5.4}$$

valid for $u > 0$, $v \geqslant 0$, we obtain

$$\frac{k}{n+1-k}\left(\sqrt{(k+1)(n-k)}-\sqrt{k(n+1-k)}\right)^2 \leqslant \frac{k}{n+1-k}\frac{(n-2k)^2}{4k(n+1-k)} \leqslant \frac{1}{4}. \qquad \square$$

Proposition 5.4. *Let (k_n) be a sequence in $\mathbb{N}$ satisfying $k_n = O(n^{1-\varepsilon})$ for some $\varepsilon > 0$. Then $\mu_{n-1}(W_n(a_{n,k_n})) = o(n^{-\alpha})$ for all $\alpha > 0$.*

Proof. Let $0 < a < 1$. Note that $W_n(a)$ is the union of $2(n+1)$ congruent spherical caps. The centre of the $(n-2)$-sphere $\{x \in \mathbb{R}^{n+1};\ |x| = 1,\ x\cdot\omega = 0,\ x_1 = a\}$ is $(a, -\frac{a}{n}, \ldots, -\frac{a}{n})$. Its norm is $a\sqrt{\frac{n+1}{n}}$. Using the inequality in the proof of Lemma 2.4 we get

$$\begin{aligned}\mu_{n-1}(W_n(a)) &\leqslant 2(n+1)\mu_{n-1}\left(\{u \in S_\omega^{n-1};\ u_1 \geqslant a\}\right)\\ &< (n+1)\left(1-a^2\frac{n+1}{n}\right)^{\frac{n-1}{2}}\\ &< (n+1)e^{-a^2\frac{(n+1)(n-1)}{2n}}.\end{aligned}$$

We now derive an upper bound for the sum occuring in the expression for $a_{n,k}$ in Proposition 5.3. Let

$$f(x) := \left(\sqrt{x(n+1-x)}-\sqrt{(x-1)(n+2-x)}\right)^2.$$

Note that f is decreasing and convex for $1 \leqslant x \leqslant \frac{n+2}{2}$. (For the convexity note that the second derivative $g''(x) = -(\frac{n+1}{2})^2 g(x)^{-3}$ of the function $g(x) := \sqrt{x(n+1-x)}$ is monotone increasing on $(0, \frac{n+1}{2})$ and satisfies $g''(\frac{n+1}{2}+x) = g''(\frac{n+1}{2}-x)$; therefore $x \mapsto f(x)^{\frac{1}{2}} = g(x)-g(x-1)$ is convex for $1 \leqslant x \leqslant \frac{n+2}{2}$.) Due to inequality (5.4) we obtain

$$f(x) \leqslant \frac{(n+2-2x)^2}{4(x-1)(n+2-x)} = -1 + \frac{n^2}{4(n+1)(x-1)} + \frac{(n+2)^2}{4(n+1)(n+2-x)}.$$

This implies the estimate

$$\begin{aligned}&\sum_{i=k+1}^{[\frac{n+1}{2}]}\left(\sqrt{i(n+1-i)}-\sqrt{(i-1)(n+2-i)}\right)^2 =\\ &\quad= \sum_{i=k+1}^{[\frac{n+1}{2}]} f(i) \leqslant \int_{k+\frac{1}{2}}^{[\frac{n+1}{2}]+\frac{1}{2}} f(x)dx\end{aligned}$$

$$
\begin{aligned}
&\leqslant \int_{k+\frac{1}{2}}^{\frac{n+2}{2}} \left(-1 + \frac{n^2}{4(n+1)(x-1)} + \frac{(n+2)^2}{4(n+1)(n+2-x)}\right) dx \\
&= k + \frac{1}{2} - \frac{n+2}{2} + \frac{n^2}{4(n+1)} \ln \frac{\frac{n+2}{2} - 1}{k - \frac{1}{2}} + \frac{(n+2)^2}{4(n+1)} \ln \frac{n+2-k-\frac{1}{2}}{\frac{n+2}{2}} \\
&= k - \frac{n+1}{2} + \frac{(n+2)^2}{4(n+1)} \ln \frac{n+\frac{3}{2}-k}{k-\frac{1}{2}} + \ln \frac{k-\frac{1}{2}}{\frac{n+2}{2}} + \frac{n^2}{4(n+1)} \ln \frac{n}{n+2} \\
&\leqslant k - \frac{n+1}{2} + \frac{(n+2)^2}{4(n+1)} \ln \frac{n+\frac{3}{2}-k}{k-\frac{1}{2}}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
a_{n,k} &\geqslant \left(2k + \frac{2k}{n+1-k}\left(k - \frac{n+1}{2} + \frac{(n+2)^2}{4(n+1)} \ln \frac{n+\frac{3}{2}-k}{k-\frac{1}{2}}\right)\right)^{-\frac{1}{2}} \\
&= \left(\frac{2k}{n+1-k}\left(\frac{n+1}{2} + \frac{(n+2)^2}{4(n+1)} \ln \frac{n+\frac{3}{2}-k}{k-\frac{1}{2}}\right)\right)^{-\frac{1}{2}} \\
&\geqslant c_1 (k \ln n)^{-\frac{1}{2}},
\end{aligned}
$$

where the last (rough) estimate is valid with some $c_1 > 0$. Therefore, with suitable $c_2, c_3 > 0$, one has

$$
\begin{aligned}
\mu_{n-1}\left(W_n(a_{n,k})\right) &\leqslant (n+1) e^{-c_1^2 \frac{1}{k \ln n} \frac{n^2-1}{2n}} \leqslant e^{-c_2 \frac{n}{k \ln n} + \ln(n+1)} \\
&\leqslant e^{-c_2 \frac{n}{k \ln n} + c_3 \ln n} = n^{-c_2 \frac{n}{k (\ln n)^2} + c_3}.
\end{aligned}
$$

From this estimate the statement is easily obtained. □

Remarks 5.5. (a) Combining the results of Theorem 5.1 and Proposition 5.4 one finds statements analogous to the central limit property in Definition 1.1 (a), (b): For large dimensions, the measure of points $u \in D_n$ (weighted by the volumes of the corresponding DV-cells) for which $\varphi_{\Delta_n, u}$ is not close to the appropriate Gaussian density is small.

(b) A more careful inspection of the proofs would probably yield a quantitative statement in the spirit of Theorems 2.1 and 2.2.

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