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A Construction for Periodically-Cyclic Gale $2m$-Polytopes

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Abstract. For each v, k and m such that $v \geq k \geq 2m + 2 \geq 8$, we construct a periodically-cyclic Gale $2m$-polytope with v vertices and the period k. For such a polytope, there is a complete description of each of its facets based upon a labelling (total ordering) of the vertices so that every subset of k successive vertices generates a cyclic $2m$-polytope.

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1. Introduction

Our motivation for this study is threefold, and the motives may be described as practical, theoretical and exploratory.

Practical. The increased use of (convex) polytopes, as models for problems in areas such as economics, operations research and theoretical chemistry, emphasises the importance of well-understood examples for which all the facets are described.

It is, in general, very difficult to determine if there is a realizable polytope $Q(d, v, f)$ of a prescribed dimension d with given numbers of vertices v and facets f, and so with this point of view, we present a simple way of generating describable examples of $Q(d, v, f)$ for certain ranges of d, v and f.

Theoretical. The construction of new classes of polytopes is an important direction of research in the study of convex polytopes. To paraphrase an expert in the field: Work on several important problems (for example; the d-step conjecture and the characterization of f-vectors of non-simplicial polytopes) in the combinatorial theory of polytopes has not been able

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to progress due to the lack of general constructions of classes of high-dimensional polytopes beyond those provided by simplicial polytopes or by iterative constructions beginning with low-dimensional polytopes and using operations like products, pyramids and so forth.

From this point of view, we present a new and non-trivial class of even-dimensional non-simplicial polytopes. The class has the special property that each member is combinatorially describable and possesses a predetermined set of subpolytopes.

Exploratory. The importance of cyclic polytopes is well known and well documented. The polytopes may be a) constructed from any dth order curve, and b) described by Gale's Evenness Condition. They have important combinatorial properties and many applications in various branches of mathematics and science. Thus, they are a class of polytopes that one seeks to generalize in order to obtain other important and useful combinatorial properties. Their generalization to neighbourly polytopes exemplifies this goal. Neighbourly polytopes are simplicial and possess important extremal properties, and it is doubtful if there is a "better" simplicial generalization of cyclic polytopes.

When considering non-simplicial generalizations, we are led to the properties a) and b) above, and specifically, to the idea of choosing vertices from a generalized dth order curve. The difficulty is that there is no workable definition of such curves for $d > 3$. While the $d = 3$ case enabled us to determine a generalization for $(2m + 1)$-space in [1], the best approach for $2m$-space seems to be the following: (with $d = 2m$) a generalized dth order curve should be locally of dth order. Let us assume now that we are choosing vertices as we move along the curve. Then the successive vertices that are in a dth order neighbourhood determine a cyclic polytope. We simplify our approach further by assuming that a given fixed number k (the period), and no more, of successive vertices are always in some such neighbourhood, and we have the model for a periodically-cyclic d-polytope. The Gale condition not only preserves the spirit of the generalization but also makes it easier to completely describe the facial structure of such polytopes.

From the combinatorial point of view, we are done. Let $x_1, x_2, \dots, x_v$ denote the vertices (chosen in that order) of a periodically-cyclic Gale $2m$-polytope of period k. Then $\operatorname{conv}\{x_1, x_2, \dots, x_k\}$ and $\operatorname{conv}\{x_2, \dots, x_k, x_{k+1}\}$ are cyclic $2m$-polytopes and we apply the Gale condition to determine their "join" $\operatorname{conv}\{x_1, x_2, \dots, x_k, x_{k+1}\}$. From the latter and $\operatorname{conv}\{x_3, \dots, x_{k+1}, x_{k+2}\}$, we determine $\operatorname{conv}\{x_1, x_2, \dots, x_{k+1}, x_{k+2}\}$ and so on.

From the practical point of view, the process is complicated by the fact that the joins are not unique. Thus, choices are made at the joinings and each choice yields a different combinatorial type of polytope, which may or may not be realizable. What we present here is the result of one sequence of choices, each of which made with the aim of obtaining a discernable pattern of facets (combinatorial goal) of a realizable polytope (geometric goal).

Referring to Section 2, the polytope $P_1^v(k)$ was determined combinatorially and then its realizability was verified. As it happens, the proof of the realizability reduces to a simple construction that does not reflect the complicated combinatorial nature of $P_1^v(k)$, and so, in relation to the above, our presentation is from back to front, or from simple to complicated.

The inductive nature of the construction reflects only our model of choosing vertices on a locally dth order curve; that is, vertices are chosen in a proscribed sequence and the choice of given vertex depends upon the previously chosen vertices. This model also explains why the definition of the period is not modulo k with respect to a cyclic labelling. It should be noted that while a combinatorial definition and description of such "modular-cyclic polytopes" are possible, the combinatorial structures do not seem to be realizable.

In Theorems 2 and 3, we verify the complicated combinatorial nature of $P_1^v(k)$. While the proof of Theorem 2 is unavoidably long and complicated because of the explicit description of the facets, the reader can easily ascertain the method of the proof and the accuracy of the facial description by working through a couple of examples.

In summary, we present what we believe to be natural non-simplicial generalization of a cyclic $2m$-polytope. It is the first such generalization and it is one that is realizable, completely describable and has a predetermined set of cyclic subpolytopes. (The only other attempt in this direction is I. Shemer's search for cyclic subpolytopes in [5].) The construction is an algorithm for assigning coordinates to the vertices, and the description makes it possible to determine the number of i-faces for $1 \leq i \leq d-2$, and thus, the set of f-vectors of a non-trivial class of non-simplicial $2m$-polytopes.

Let Y be a set of points in R^d. Then conv Y and aff Y denote, respectively, the convex hull and the affine hull of Y. If $Y = \{y_1, \dots, y_n\}$ is a finite set of n points then $|Y| = n$, and we set

$$[y_1, \dots, y_n] = \operatorname{conv} Y = [Y].$$

Let $Q \subset R^d$ be a (convex) d-polytope. For $-1 \leq i \leq d$, let $\mathcal{F}_i(Q)$ denote the set of i-faces of Q and $f_i(Q) = |\mathcal{F}_i(Q)|$. For convenience, $\mathcal{F}(Q) = \mathcal{F}_{d-1}(Q)$ denotes the set of facets of Q.

We assume familiarity with the basic definitions and concepts concerning polytopes (cf. [3] and [8]) and we cite three results necessary for our presentation.

Lemma 1. *Let Q and Q^* be d-polytopes in R^d such that $Q^* = \operatorname{conv}(Q \cup \{y\})$ for some point $y \in R^d \backslash Q$. Let G be a face of Q and $\mathcal{F}(G, Q) = \{F \in \mathcal{F}(Q) \mid G \subseteq F\}$. Then*

a) *G is a face of Q^* if, and only if, y is beneath some $F \in \mathcal{F}(G, Q)$, and*

b) *$G^* = \operatorname{conv}(G \cup \{y\})$ is a face of Q^* if, and only if, either $y \in \operatorname{aff} G$ or for some F' and F'' in $\mathcal{F}(G, Q)$, y is beneath F' and beyond F''* (cf [3], p. 78).

Lemma 2. *If a facet system of one d-polytope is contained in a facet system of another d-polytope then the two d-polytopes are combinatorially equivalent* (cf. [3] and [8], p. 71).

Lemma 3. *Let $C \subset R^{2m}$ be a cyclic $2m$-polytope with the vertex array (see below) $y_1 < y_2 < \cdots < y_v$, $m \geq 2$. Then each $2m$-subpolytope of C is cyclic with respect to the vertex array induced by $y_1 < y_2 < \cdots < y_v$* (cf. [6]).

To explain the concept of a vertex array, let $V = \mathcal{F}_0(Q) = \{y_1, y_2, \dots, y_v\}$. We set $y_i \leq y_j$ if, and only if, $i \leq j$, and call $y_1 < y_2 < \cdots < y_v$ a *vertex array* of Q. We call y_i and y_{i+1} *successive* vertices, and say that y_j *separates* y_i and y_k if $y_i < y_j < y_k$.

Let $Y \subset V$ and assume that $y_1 < y_2 < \cdots < y_v$. We say that Y is an *even* set (in the array) if it is the union of mutually disjoint subsets $\{y_i, y_{i+1}\}$; otherwise, Y is an *odd* set. Next, Y is a *Gale* set (in the array) if any two points of $V \backslash Y$ are separated by an even number of points of Y. We extend these definitions in the obvious manner to the facets of Q, and note that even facets are Gale and that an odd Gale facet contains y_1 or y_v.

As a rule, S_n denotes an even set of $n \geq 0$ vertices with, of course, S_0 denoting the empty set. Also, $\mathcal{F}^e(Q)(\mathcal{F}^0(Q))$ denotes the set of even (odd) facets of Q. Thus,

$$\mathcal{F}(Q) = \mathcal{F}^e(Q) \cup \mathcal{F}^0(Q).$$

Next, we say that Q is a *Gale* d-polytope if each facet of Q is Gale with respect to a fixed vertex array of Q. If $y_1 < y_2 < \cdots < y_v$ is the vertex array, we say also that Q is Gale with $y_1 < y_2 < \cdots < y_v$. We remark that cyclic polytopes are Gale. In fact, a d-polytope C is cyclic if with respect to a fixed vertex array of C: $Y \subset \mathcal{F}_0(C)$ determines a facet of C if, and only if, Y is a d-element Gale set (Gale's Evenness condition).

Finally, we say that Q is a *periodically-cyclic* d-polytope if there is a vertex array, say, $y_1 < y_2 < \cdots < y_v$ and an integer k such that $v \geq k \geq d+2$,

$$[y_{i+1}, \dots, y_{i+k}] \quad \text{is a cyclic } d\text{-polytope with} \quad y_{i+1} < y_{i+2} < \cdots < y_{i+k}$$

for $i = 0, \dots, v-k$, and

$$[y_{i+1}, \dots, y_{i+k}, y_{i+k+1}] \quad \text{is not cyclic for any} \quad 0 \leq i \leq v-k-1.$$

We call k, the *period* of Q.

As stated above, we wish to generalize a cyclic d-polytope and a periodically-cyclic Gale d-polytope is such a generalization. In [2], we introduced a class of these polytopes for $d = 6$ and indicated how to construct them. Presently, we extend the construction to R^{2m} for $m > 3$ and show that it yields a class of realizable periodically-cyclic Gale $2m$-polytopes.

Before introducing the construction, we present one last preliminary result with the reminder that S_n always denotes an even set of n vertices.

Theorem 1. *Let $C \subset R^d$ be a cyclic d-polytope with the vertex array $y_1 < y_2 < \cdots < y_{v-1}$, $v \geq d+2$ and d even. Let $Q \subset R^d$ be a d-polytope with the property that $Q = \operatorname{conv}(C \cup \{y_v\})$, $y_v \notin C$, and*

$$\mathcal{F}^e(C) \cup \{[S_{d-2}, y_{v-1}, y_v] \mid S_{d-2} \subset \{y_1, \dots, y_{v-2}\}\} \subset \mathcal{F}(Q).$$

Then Q is cyclic with $y_1 < \cdots < y_{v-1} < y_v$.

Proof. We note that there is a point $y^* \in R^d$ such that $y^* \notin C$ and $C^* = \operatorname{conv}(C \cup \{y^*\}\}$ is cyclic with $y_1 < \cdots < y_{v-1} < y^*$. By Gale's Evenness Condition, $\mathcal{F}(C^*) = \mathcal{F}^e(C^*) \cup \mathcal{F}^0(C^*)$ with

$$\begin{aligned}
\mathcal{F}^e(C^*) &= \{[S_d] \mid S_d \subset \{y_1, \dots, y_{v-1}, y^*\}\} \\
&= \{[S_d] \mid S_d \subset \{y_1, \dots, y_{v-1}\}\} \cup \{[S_{d-2}, y_{v-1}, y^*\} \mid S_{d-2} \subset \{y_1, \dots, y_{v-2}\}\} \\
&= \mathcal{F}^e(C) \cup \{[S_{d-2}, y_{v-1}, y^*] \mid S_{d-2} \subset \{y_1, \dots, y_{v-2}\}\}
\end{aligned}$$

and $\mathcal{F}^0(C) = \{[y_1, S_{d-2}, y^*] \mid S_{d-2} \subset \{y_2, \dots, y_{v-1}\}\}$. Thus, if

$$\{[y_1, S'_{d-2}, y_v] \mid S'_{d-2} \subset \{y_2, \dots, y_{v-1}\}\} \subset \mathcal{F}(Q),$$

then a facet system of C^* is contained in a facet system of Q, C^* and Q are equivalent by Lemma 2, and the assertion follows.

Let $S'_{d-2} \subset \{y_2, \dots, y_{v-1}\}$. Since $v \geq d+2$, there is a least (greatest) vertex y_i (y_j) such that $y_2 \leq y_i < y_j \leq y_{v-1}$ and $\{y_i, y_j\} \cap S'_{d-2} = \emptyset$. We note that

$$\begin{aligned} &\text{either} \quad y_2 = y_i \quad \text{or} \quad \{y_2, \dots, y_{i-1}\} \subset S'_{d-2}, \\ \text{and} \quad &\text{either} \; y_j = y_{v-1} \quad \text{or} \quad \{y_{j+1}, \dots, y_{v-1}\} \subset S'_{d-2}. \end{aligned}$$

In all cases, it follows that

$$[y_1, y_i, S'_{d-2}] \in \mathcal{F}^e(C) \quad \text{and} \quad [y_1, S'_{d-2}, y_j] \in \mathcal{F}^0(C).$$

Let

$$F^e = [y_1, , y_i, S'_{d-2}] \quad \text{and} \quad F^0 = [y_1, S'_{d-2}, y_j].$$

Then $F^e \in \mathcal{F}(Q)$ and

$$[y_1, S'_{d-2}] = F^e \cap F^0 \quad \text{is a } (d-2)\text{-face of } C.$$

Since y_v is beneath F^e, we obtain from Lemma 1 that if y_v is beyond F^0 (that is, $F^0 \notin \mathcal{F}(Q)$) then $[y_1, S'_{d-2}, y_v] \in \mathcal{F}(Q)$.

We recall that $y_{v-1} \in S'_{d-2} \cup \{y_j\}$ and note that

$$S'_{d-2} \cup \{y_j\} = S_{d-2} \cup \{y_{v-1}\} \quad \text{for some} \quad S_{d-2} \subset \{y_2, \dots, y_{v-2}\}.$$

In particular, if $y_j = y_{v-1}$ then $S_{d-2} = S'_{d-2}$, and if $y_j \neq y_{v-1}$ then $S_{d-2} = (S'_{d-2} \backslash \{y_{v-1}\}) \cup \{y_j\}$. Thus by the hypothesis,

$$[S_{d-2}, y_{v-1}, y_v] \in \mathcal{F}(Q).$$

Since $S_{d-2} \subset \{y_2, \dots, y_{v-2}\}$ and $v \geq d+2$, there is a greatest vertex y_r such that $y_r \leq y_{v-2}$ and $y_r \notin S_{d-2}$. Again,

$$\text{either} \quad y_r = y_{v-2} \quad \text{or} \quad \{y_{r+1}, \dots, y_{v-2}\} \subset S_{d-2},$$

and $[S_{d-2}, y_r, y_{v-1}] \in \mathcal{F}^e(C) \subset \mathcal{F}(Q)$.

We have just shown that the two facets of Q, that contain $[S_{d-2}, y_{v-1}] = [S'_{d-2}, y_j]$, do not contain y_1. Hence, $F^0 = [y_1, S'_{d-2}, y_j] \notin \mathcal{F}(Q)$. $\square$

2. The construction

Let $m \geq 3$, $d = 2m$ and $v \geq k \geq d+2$. We present a construction of a d-polytope $P_1^v(k)$ in R^d that is Gale and periodically-cyclic with the vertex array $x_1 < x_2 < \cdots < x_v$ and the period k.

Step 1. Let $P_1^k(k)$ be a cyclic d-polytope in R^d with the vertex array $x_1 < x_2 < \cdots < x_k$.

Step 2. Assume that $v > k$ and that $P_1^{v-1}(k) = [x_1, \dots, x_k, \dots, x_{v-1}]$ is a d-polytope in R^d constructed in the prescribed manner.

Step 3. Choose $x_v \in R^d$ so that

- $x_v \in \operatorname{aff}\{x_1, x_{v-k+1}, x_{v-k+2}, x_{v-1}\}$
- x_v is beyond each $F \in \mathcal{F}(P_1^{v-1}(k))$ with the property that $F \cap [x_1, x_{v-k+1}, x_{v-1}] = [x_1, x_{v-1}]$; and
- x_v is beneath each $F \in \mathcal{F}(P_1^{v-1}(k))$ with the property that $[x_1, x_{v-k+1}, x_{v-k+2}, x_{v-1}] \not\subset F$ and $F \cap [x_1, x_{v-k+1}, x_{v-1}] \neq [x_1, x_{v-1}]$.

Henceforth, we assume that $P_1^v(k) = [x_1, \dots, x_k, \dots, x_v]$ is a d-polytope in R^d constructed in the prescribed manner. In order to verify that the construction is valid and that it results in a polytope with the desired properties, we need to describe $P_1^v(k)$ explicitly.

Firstly, we set

$$\begin{aligned} x_i &= x_1 \quad \text{for} \quad i < 1, \\ P_i^j(k) &= [x_i, x_{i+1}, \dots, x_j] \quad \text{for} \quad 1 \le i < j \le v, \quad \text{and} \\ P_i^j &= P_i^j(k) \quad \text{when there is no danger of confusion.} \end{aligned}$$

Next, we let

$$\begin{aligned} X_i &= \bigcup_{j=2}^{m} \{[x_1, x_{i+5-2j}, \dots, x_{i+1}, x_{i+2}, S_{d-2j}, x_{i+k+3-2j}, \dots, x_{i+k-1}, x_{i+k}] \mid \\ &\qquad\qquad S_{d-2j} \subset \{x_{i+4}, \dots, x_{i+k+1-2j}\}\}, \end{aligned} \tag{1}$$

$$Y_i = \{[S_d] \mid S_d \subset \{x_{i+1}, \dots, x_{i+k-1}\}\}, \tag{2}$$

$$Z_i = Z_i' \cup Z_i'' \quad \text{where} \tag{3}$$

$$\begin{aligned} Z_i' &= \bigcup_{j=2}^{m} \{[x_1, x_{i+6-2j}, \dots, x_{i+2}, x_{i+3}, S_{d-2j}, x_{i+k+4-2j}, \dots, x_{i+k}] \mid \\ &\qquad\qquad S_{d-2j} \subset \{x_{i+4}, \dots, x_{i+k+2-2j}\}\} \end{aligned}$$

and

$$\begin{aligned} Z_i'' &= \bigcup_{j=3}^{m} \{[x_1, x_{i+7-2j}, \dots, x_{i+1}, x_{i+2}, x_{i+3}, x_{i+4}, S_{d-2j}, x_{i+k+5-2j}, \dots, x_{i+k-1}, x_{i+k}] \mid \\ &\qquad\qquad S_{d-2j} \subset \{x_{i+5}, \dots, x_{i+k+3-2j}\}\}, \end{aligned}$$

and

$$W_i = \{[x_1, S_{d-2}, x_{i+k}] \mid S_{d-2} \subset \{x_{i+3}, \dots, x_{i+k-1}\}\}. \tag{4}$$

In order to make X_i and Z_i more understandable, we observe that

$$[x_1, x_{i+1}, x_{i+2}, x_{i+k-1}, x_{i+k}] = \bigcap_{F \in X_i} F, \tag{5}$$

$$[x_i, x_{i+2}, x_{i+3}, x_{i+k}] = \bigcap_{F \in Z_i} F \tag{6}$$

and that, in general,

$$\begin{aligned} &\{[x_s, S_{d-2r}, x_t] \mid S_{d-2r} \subset \{x_{s+1}, , x_{t-1}\}\} \\ &\quad = \bigcup_{j=r}^{m} \{[x_s, S_{d-2j}, x_{t+2r-2j}, \dots, x_t] \mid S_{d-2j} \subset \{x_{s+1}, \dots, x_{t+2r-2j-2}\}\}. \end{aligned} \tag{7}$$

It is now easy to check that

$$\left.\begin{aligned} |X_i| &= |\{x_1, x_{i+1}, x_{i+2}, S_{d-4}, x_{i+k-1}, x_{i+k}] \mid S_{d-4} \subset \{x_{i+4}, \dots, x_{i+k-2}\}\}|, \\ |Z_i'| &= |\{[x_1, x_{i+2}, x_{i+3}, S_{d-4}, x_{i+k}] \mid S_{d-4} \subset \{x_{i+4}, \dots, x_{i+k-1}\}\}| \quad \text{and} \\ |Z_i''| &= |\{[x_1, x_{i+1}, \dots, x_{i+4}, S_{d-6}, x_{i+k-1}, x_{i+k}] \mid S_{d-6} \subset \{x_{i+5}, \dots, x_{i+k-2}\}\}|. \end{aligned}\right\} \tag{8}$$

Theorem 2. $\mathcal{F}(P_1^v(k)) = \left(\bigcup_{i=0}^{v-k} X_i\right) \cup \left(\bigcup_{i=0}^{v-k+1} Y_i\right) \cup Z_{v-k} \cup W_{v-k}$.

Proof. By Step 1, $P_1^k = [x_1, x_2, \dots, x_k]$ is a cyclic d-polytope with $x_1 < x_2 < \dots < x_k$. Hence, $\mathcal{F}(P_1^k) = \mathcal{F}^e(P_1^k) \cup \mathcal{F}^0(P_1^k)$ with

$$\begin{aligned}
\mathcal{F}^e(P_1^k) &= \{[S_d] \mid S_d \subset \{x_1, \dots, x_k\}\} \\
&= \{[S_d] \mid S_d \subset \{x_1, \dots, x_{k-1}\}\} \cup \{[S_d] \mid S_d \subset \{x_2, \dots, x_k\}\} \\
&\quad \cup \{[x_1, x_2, S_{d-4}, x_{k-1}, x_k] \mid S_{d-4} \subset \{x_3, \dots, x_{k-2}\}\} \\
&= Y_0 \cup Y_1 \cup \{[x_1, x_2, S_{d-4}, x_{k-1}, x_k] \mid S_{d-4} \subset \{x_4, \dots, x_{k-2}\}\} \\
&\quad \cup \{[x_1, x_2, x_3, x_4, S_{d-6}, x_{k+1}, x_k] \mid S_{d-6} \subset \{x_5, \dots, x_{k-2}\}\} \\
&= Y_0 \cup Y_1 \cup X_0 \cup Z_0''
\end{aligned}$$

by (1), (2), (3), (7) and (8), and

$$\begin{aligned}
\mathcal{F}^0(P_1^k) &= \{[x_1, S_{d-2}, x_k] \mid S_{d-2} \subset \{x_2, \dots, x_{k-1}\}\} \\
&= \{[x_1, S_{d-2}, x_k] \mid S_{d-2} \subset \{x_3, \dots, x_{k-1}\}\} \cup \{[x_1, x_2, x_3, S_{d-4}, x_k] \mid \\
&\qquad\qquad S_{d-4} \subset \{x_4, \dots, x_{k-1}\}\} \\
&= W_0 \cup Z_0'
\end{aligned}$$

by (3), (4), (7) and (8).

We assume that $v > k$ and that the assertion is true for P_1^{v-1}. Thus,

$$\mathcal{F}(P_1^{v-1}) = \left(\bigcup_{i=0}^{v-k-1} X_i\right) \cup \left(\bigcup_{i=0}^{v-k} Y_i\right) \cup Z_{v-k-1} \cup W_{v-k},$$

with

$$\begin{aligned}
X_{v-k-1} &= \bigcup_{j=2}^{m} \{[x_1, x_{v-k+4-2j}, \dots, x_{v-k+1}, S_{d-2j}, x_{v+2-2j}, \dots, x_{v-1}] \mid \\
&\qquad\qquad S_{d-2j} \subset \{x_{v-k+3}, \dots, x_{v-2j}\}\}, \\
Z'_{v-k-1} &= \bigcup_{j=2}^{m} \{[x_1, x_{v-k+5-2j}, \dots, x_{v-k+2}, S_{d-2j}, x_{v+3-2j}, \dots, x_{v-1}] \mid \\
&\qquad\qquad S_{d-2j} \subset \{x_{v-k+3}, \dots, x_{v+1-2j}\}\}, \\
Z''_{v-k-1} &= \bigcup_{j=3}^{m} \{[x_1, x_{v-k+6-2j}, \dots, x_{v-k+3}, S_{d-2j}, x_{v+4-2j}, \dots, x_{v-1}] \mid \\
&\qquad\qquad S_{d-2j} \subset \{x_{v-k+4}, \dots, x_{v+2-2j}\}\}, \\
W_{v-k-1} &= \{[x_1, S_{d-2}, x_{v-1}] \mid S_{d-2} \subset \{x_{v-k+2}, \dots, x_{v-2}\}\} \\
&= \{F \in \mathcal{F}(P_1^{v-1}) \mid F \cap [x_1, x_{v-k+1}, x_{v-1}] = [x_1, x_{v-1}]\}
\end{aligned}$$

and $[x_1, x_{v-k+1}, x_{v-k+2}, x_{v-1}] = \bigcap_{F^* \in Z_{v-k-1}} F^*$.

Thus by Step 3, $x_v \in \operatorname{aff} F^*$ for each $F^* \in Z_{v-k-1}$ and x_v is beyond only the facets of P_1^{v-1} in W_{v-k-1}. By Lemma 1, it follows that

$$\left(\bigcup_{i=0}^{v-k-1} X_i\right) \cup \left(\bigcup_{i=0}^{v-k} Y_i\right) \cup A' \cup A'' \subset \mathcal{F}(P_1^v) \tag{*}$$

where

$$A' = \{\text{conv}\,(F' \cup \{x_v\}) \mid F' \in Z'_{v-k-1}\}$$

and

$$A'' = \{\text{conv}\,(F'' \cup \{x_v\}) \mid F'' \in Z''_{v-k-1}\},$$

and that we obtain the remaining facets of P_1^v from the $(d-2)$-faces of the $F \in W_{v-k-1}$. Let $F \in W_{v-k-1}$ and $x_t \in F$. Then $F = [x_1, S_{d-2}, x_{v-1}]$ for some $S_{d-2} \subset \{x_{v-k+2}, \dots, x_{v-2}\}$, $G_t = [(\{x_1\} \cup S_{d-2} \cup \{x_{v-1}\}) \backslash \{x_t\}]$ is a $(d-2)$-face of P_1^{v-1} and

$$G_t = F \cap F_t \quad \text{for some} \quad F_t \in \mathcal{F}(P_1^{v-1}).$$

By Lemma 1,

$$\text{conv}\,(G_t \cup \{x_v\}) \in \mathcal{F}(P_1^v) \quad \text{if, and only if,} \quad F_t \notin Z_{v-k-1} \cup W_{v-k-1}.$$

Case 1. $x_t = x_1$.

Since $S_{d-2} \cup \{x_{v-1}\} \subset \{x_{v-k+2}, \dots, x_{v-1}\}$ and $k \geq d+2$, it is clear that there is an $S_d^* \subset \{x_{v-k+1}, \dots, x_{v-1}\}$ such that $S_{d-2} \cup \{x_{v-1}\} \subset S_d^*$. As $[S_d^*] \in Y_{v-k}$, we have that $F_1 = [S_d^*]$, and so,

$$B = \{[S_{d-2}, x_{v-1}, x_v] \mid S_{d-2} \subset \{x_{v-k+2}, \dots, x_{v-2}\}\} \subset \mathcal{F}(P_1^v). \tag{*}$$

Case 2. $x_t = x_{v-1}$.

Let x_s be the greatest vertex in S_{d-2}. If $x_s \leq x_{k-1}$ then $\{x_1\} \cup S_{d-2} \subset S'_d$ for some $S'_d \subset \{x_1, \dots, x_{k-1}\}$. Now, $[S'_d] \in Y_0$ implies that $F_{v-1} = [S'_d]$ and $[x_1, S_{d-2}, x_v] \in \mathcal{F}(P_1^v)$.

Let $x_s \geq x_k$. Let x_r be the greatest vertex in S_{d-2} such that $x_{r-1} \notin S_{d-2}$. Then

$$x_{s+3-d} \leq x_r \leq x_{s-1}, \quad r = s+3-2j \quad \text{for some} \quad 2 \leq j \leq m = d/2,$$

and
$$S_{d-2} = S_{d-2j} \cup \{x_{s+3-2j}, \dots, x_{s-1}, x_s\}$$
with
$$S_{d-2j} = S_{d-2} \cap \{x_{v-k+2}, \dots, x_{s+1-2j}\}.$$

Since $k \leq s \leq v-2 = k + (v-k-2)$, it follows that

$$s = k + i \quad \text{for some } 0 \leq i \leq v-k-2.$$

Since $v-k+2 \geq i+4$, we have shown that

$$S_{d-2} = S_{d-2j} \cup \{x_{i+k+3-2j}, \dots, x_{i+k-1}, x_{i+k}\}$$
and
$$S_{d-2j} \subset \{x_{i+4}, \dots, x_{i+k+1-2j}\}.$$

Since $[x_1, x_{i+5-2j}, \dots, x_{i+1}, x_{i+2}, S_{d-2}] \in X_i$ for this i and j, we have that $F_{v-1} \in X_i$ and

$$D = \{[x_1, S_{d-2}, x_v] \mid S_{d-2} \subset \{x_{v-k+2}, \dots, x_{v-2}\}\} \subset \mathcal{F}(P_1^v). \tag{*}$$

Case 3. $x_t \in S_{d-2}$.

Firstly, we note that $\{x_1, x_{v-1}\} \subset G_t = F \cap F_t$. Hence, $v - 1 \geq k$ implies that $F_t \in X_{v-k-1} \cup Z_{v-k-1} \cup W_{v-k-1}$, and as a consequence,

$$\text{conv}\,(G_t \cup \{x_v\}) \in \mathcal{F}(P_1^v) \quad \text{if, and only if,} \quad F_t \in X_{v-k-1}.$$

Next, let

$$S_{d-2} = \{y_1, \ldots, y_{d-2}\} \quad \text{with} \quad y_1 < y_2 < \cdots < y_{d-2}.$$

Then for $1 \leq i \leq m-1$, $y_{2i} = x_s$ implies that $y_{2i-1} = x_{s-1}$.

Let $x_{v-k+2} = y_1$. If $x_t \neq y_1$ then $x_{v-k+2} \in F_t$. As $x_{v-k+2} \notin F$ for any $F \in X_{v-k-1}$, it follows that $F_t \notin X_{v-k-1}$. If $x_t = y_1$ then $S_{d-2} \backslash \{x_t\} \subset \tilde{S}_{d-2}$ for some $\tilde{S}_{d-2} \subset \{x_{v-k+3}, \ldots, x_{v-2}\}$ and $F_t = [x_1, \tilde{S}_{d-2}, x_{v-1}] \in W_{v-k-1}$. Thus, no facets of P_1^v are generated in this case, and we may assume that $x_{v-k+2} < y_1$.

If $x_t \in \{y_2, y_4, \ldots, y_{d-2}\}$ then it is clear that

$$S_{d-2} \backslash \{x_t\} \subset \tilde{S}_{d-2} \quad \text{for some} \quad \tilde{S}_{d-2} \subset \{x_{v-k+2}, \ldots, x_{v-2}\}$$

and, as a consequence, $F_t = [x_1, \tilde{S}_{d-2}, x_{v-1}] \in W_{v-k-1}$ again. Hence, we may assume that

$$x_{v-k+2} < y_1 \quad \text{and} \quad x_t \in \{y_1, y_3, \ldots, y_{d-3}\}.$$

If $y_{d-2} < x_{v-2}$ then, arguing as above, we obtain that $F_t \in W_{v-k-1}$. So let $y_{d-2} = x_{v-2}$. Then $y_{d-3} = x_{v-3}$ and

$$S_{d-2} = S_{d-4} \cup \{x_{v-3}, x_{v-2}\} \quad \text{with} \quad S_{d-4} \subset \{x_{v-k+3}, \ldots, x_{v-4}\}.$$

Now, $x_t = y_{d-3}$ yields that

$$F_{v-3} = [x_1, x_{v-k}, x_{v-k+1}, S_{d-4}, x_{v-2}, x_{v-1}] \in X_{v-k-1},$$

and

$$E_2 = \{[x_1, S_{d-4}, x_{v-2}, x_{v-1}, x_v] \mid S_{d-4} \subset \{x_{v-k+3}, \ldots, x_{v-4}\}\} \subset \mathcal{F}(P_1^v). \qquad (*)$$

Hence, we may assume that

$$\{y_{d-3}, y_{d-2}\} = \{x_{v-3}, x_{v-2}\} \quad \text{and} \quad x_t \in \{y_1, y_3, \ldots, y_{d-5}\}.$$

If $y_{d-4} < x_{v-4}$ then we obtain again that $F_t \in W_{v-k-1}$. With $y_{d-4} = x_{v-4}$, we have that $y_{d-5} = x_{v-5}$ and

$$S_{d-2} = S_{d-6} \cup \{x_{v-5}, x_{v-4}, x_{v-3}, x_{v-2}\} \quad \text{with} \quad S_{d-6} \subset \{x_{v-k+3}, \ldots, x_{v-6}\}.$$

Now, $x_t = y_{d-5}$ yields that

$$F_{v-5} = [x_1, x_{v-k-2}, \ldots, x_{v-k+1}, S_{d-6}, x_{v-4}, x_{v-3}, x_{v-2}, x_{v-1}] \in X_{v-k-1},$$

and

$$E_3 = \{[x_1, S_{d-6}, x_{v-4}, x_{v-3}, x_{v-2}, x_{v-1}, x_v] \mid S_{d-6} \subset \{x_{v-k-3}, \dots, x_{v-6}\}\} \subset \mathcal{F}(P_1^v). \qquad (*)$$

Hence, we may assume that

$$\{y_{d-5}, \dots, y_{d-2}\} = \{x_{v-5}, \dots, x_{v-2}\} \quad \text{and} \quad x_t \in \{y_1, y_3, \dots, y_{d-7}\}.$$

It is now clear that reiterations of the preceding argument yield that

$$E_j = \{[x_1, S_{d-2j}, x_{v+2-2j}, \dots, x_v] \mid S_{d-2j} \subset \{x_{v-k+3}, \dots, x_{v-2}\}\} \subset \mathcal{F}(P_1^v) \qquad (*)$$

for $j = 2, 3, \dots, m$, and

$$\mathcal{F}(P_1^v) = \left(\bigcup_{i=0}^{v-k-1} X_i\right) \cup \left(\bigcup_{i=0}^{v-k} Y_i\right) \cup A' \cup A'' \cup B \cup D \cup \left(\bigcup_{j=2}^{m} E_j\right).$$

We observe that by (7),

$$\bigcup_{j=2}^{m} E_j = \{[x_1, S_{d-4}, x_{v-2}, x_{v-1}, x_v] \mid S_{d-4} \subset \{x_{v-k+3}, \dots, x_{v-3}\}\}$$

and

$$D = \{[x_1, S_{d-2}, x_v] \mid S_{d-2} \subset \{x_{v-k+3}, \dots, x_{v-2}\}\} \cup D'$$

where

$$D' = \{[x_1, x_{v-k+2}, x_{v-k+3}, S_{d-4}, x_{v]} \mid S_{d-4} \subset \{x_{v-k+4}, \dots, x_{v-2}\}\}.$$

It is now easy to check that

$$W_{v-k} = \left(\bigcup_{j=2}^{m} E_j\right) \cup (D \backslash D'), \quad Z'_{v-k} = A'' \cup D', \quad B = Y_{v-k+1} \backslash Y_{v-k}$$

and

$$A' = X_{v-k} \cup Z''_{v-k}. \qquad \square$$

Theorem 3. *$P_1^v(k)$ is a periodically-cyclic d-polytope with $x_1 < x_2 < \dots < x_v$ and the period k.*

Proof. Clearly, we may assume that $v > k$ and that $P_1^{v-1} = P_1^{v-1}(k)$ is a periodically-cyclic d-polytope with $x_1 < x_2 < \dots < x_{v-1}$ and the period k. Thus, P_{v-k}^{v-1} is a cyclic d-polytope with $x_{v-k} < x_{v-k+1} < \dots < x_{v-1}$, and by $d = 2m$ and Lemma 3, P_{v-k+1}^{v-1} is a cyclic d-polytope with $x_{v-k+1} < \dots < x_{v-1}$. From (2) and Theorem 2, we observe that

$$Y_{v-k} \cup Y_{v-k+1} = \mathcal{F}^e(P_{v-k+1}^{v-1}) \cup \{[S_{d-2}, x_{v-1}, x_v] \mid S_{d-2} \subset \{x_{v-k+2}, \dots, x_{v-2}\}\}$$

and $Y_{v-k} \cup Y_{v-k+1} \subset \mathcal{F}(P_{v-k+1}^v)$.

Next, it is easy to check that $X_{v-k} \subset \mathcal{F}(P_1^v)$ yields that

$$A = \{[x_{v-k+1}, x_{v-k+2}, S_{d-4}, x_{v-1}, x_v] \mid S_{d-4} \subset \{x_{v-k+4}, \dots, x_{v-2}\}\} \subset \mathcal{F}(P_{v-k+1}^v),$$

and $Z''_{v-k} \subset \mathcal{F}(P_1^v)$ yields that

$$B = \{[x_{v-k+1}, \dots, x_{v-k+4}, S_{d-6}, x_{v-1}, x_v] \mid S_{d-6} \subset \{x_{v-k+5}, \dots, x_{v-2}\}\} \subset \mathcal{F}(P_{v-k+1}^v)$$

(cf. (7) and (8)). Finally, as

$$\begin{aligned}&\{[S_{d-2}, x_{v-1}, x_v] \mid S_{d-2} \subset \{x_{v-k+1}, \dots, x_{v-2}\}\}\\ =\ &\{[S_{d-2}, x_{v-1}, x_v] \mid S_{d-2} \subset \{x_{v-k+2}, \dots, x_{v-2}\}\} \cup A \cup B,\end{aligned}$$

it follows by Theorem 1 that P^v_{v-k+1} is a cyclic d-polytope with $x_{v-k+1} < x_{v-k+2} < \cdots < x_v$.

Since $d \geq 6$ and Z'_{v-k} contains

$$\{[x_1, x_{v-k}, \dots, x_{v-k+3}, S_{d-6}, x_{v-2}, x_{v-1}, x_v] \mid S_{d-6} \subset \{x_{v-k+4}, \dots, x_{v-4}\}\},$$

it follows that $[x_{v-k}, \dots, x_{v-k+3}, S_{d-6}, x_{v-2}, x_{v-1}, x_v]$ is a facet of P^v_{v-k} for such S_{d-6}. The facet has $(d-6) + 7 = d+1$ vertices and so, P^v_{v-k} is not a cyclic d-polytope. □

3. Remarks

First, $P^v_1(k)$ is clearly a Gale d-polytope with $x_1 < x_2 < \cdots < x_v$ for $v \geq k \geq d+2 = 2m+2 \geq 8$.

Second, as $P^k_1(k)$ is a cyclic d-polytope, it is realizable in R^d. Assuming that $v > k$ and that $P^{v-1}_1(k)$ is realizable in R^d, it is easy to check (cf. [2]) that aff $\{x_1, x_{v-k+1}, x_{v-k+2}, x_{v-1}\}$ is a 3-flat in R^d and that there are points in this 3-flat that satisfy the conditions in Step 3.

Next, in view of the restrictions on d, it is natural to ask what is the result of the construction if $d = 4$ or if d is odd.

Let $d = 4$. Then $m = 2$, $X_i = \{[x_1, x_{i+1}, x_{i+2}, x_{i+k-1}, x_{i+k}]\}$ and $Z_i = \{[x_1, x_{i+2}, x_{i+3}, x_{i+k}]\}$. We obtain again Theorem 2 and that P^{i+k}_{i+1} is a cyclic 4-polytope with $x_{i+1} < x_{i+2} < \cdots < x_{i+k}$ for $i = 0, \dots, v-k$. Then P^{i+k-1}_{i+1} is a cyclic 4-polytope with $x_{i+1} < x_{i+2} < \cdots < x_{i+k-1}$ and $Y_i = \mathcal{F}^e(P^{i+k-1}_{i+1})$. Thus

$$\begin{aligned}\mathcal{F}(P^{k+2}_1) &= X_0 \cup X_1 \cup X_2 \cup Y_0 \cup Y_1 \cup Y_2 \cup Y_3 \cup W_2 \cup Z_2\\ &\supset \{[x_1, x_2, x_3, x_k, x_{k+1}], [x_1, x_3, x_4, x_{k+1}, x_{k+2}]\} \cup Y_1 \cup Y_2 \cup Y_3\end{aligned}$$

and

$$\begin{aligned}&\mathcal{F}^e(P^k_2) \cup \mathcal{F}^e(P^{k+1}_3) \cup \mathcal{F}^e(P^{k+2}_4) \cup \{[x_2, x_3, x_k, x_{k+1}], [x_3, x_4, x_{k+1}, x_{k+2}]\}\\ =\ &\mathcal{F}^e(P^{k+1}_2) \cup \{[x_i, x_{i+1}, x_{k+1}, x_{k+2}] \mid i = 3, \dots, k-1\} \subset \mathcal{F}(P^{k+2}_2).\end{aligned}$$

It is easy to check that $[x_2, x_3, x_k, x_{k+2}]$ is also a facet of P^{k+2}_2, and hence by Theorem 1, P^{k+2}_2 is a cyclic 4-polytope with $x_2 < x_3 < \cdots < x_{k+2}$. As P^{k+1}_1 is not a cyclic 4-polytope, P^v_1 is not periodically-cyclic for $v \geq k+2$.

It is worthwhile to note that there are periodically-cyclic Gale 4-polytopes but the only known examples are generated by points on generalized trigonometric moment curves; cf. [7].

Let $d \geq 3$ be odd and suppose that P^k_1 is a cyclic d-polytope with $x_1 < x_2 < \cdots < x_k$. Then

$$\mathcal{F}(P^k_1) = \{[x_1, S_{d-1}] \mid S_{d-1} \subset \{x_2, \dots, x_k\}\} \cup \{[S_{d-1}, x_k] \mid S_{d-1} \subset \{x_1, \dots, x_{k-1}\}\},$$

and by Step 2, x_{k+1} is beneath $\tilde{F} = [x_2, \dots, x_d, x_k]$. Thus, $\tilde{F} \in \mathcal{F}(P^{k+1}_1) \cap \mathcal{F}(P^{k+1}_2)$ and, P^{k+1}_2 is neither Gale nor cyclic with $x_2 < \cdots < x_k < x_{k+1}$.

Last, we remark that by Theorem 2,

$$\mathcal{F}(P_1^v(k)) = \left(\bigcup_{i=0}^{v-k} X_i\right) \cup Y_0 \cup \left(\bigcup_{i=1}^{v-k+1} (Y_i \backslash Y_{i-1})\right) \cup Z_{v-k} \cup W_{v-k}$$

with $Y_i \backslash Y_{i-1} = \{[S_{d-2}, x_{i+k-2}, x_{i+k-1}] \mid S_{d-2} \subset \{x_{i+1}, \dots, x_{i+k-3}\}\}$, and that the right-hand sets of facets are mutually disjoint. Next, we recall that

$$\binom{b-a}{2a} = |\{S_{2a} \mid S_{2a} \subset \{x_{i+1}, \dots, x_{i+b}\}\}|,$$

and thus by (2), (4) and (8),

$$|X_i| = \binom{k-m-3}{m-2}, \quad |Y_i| = \binom{k-m-1}{m}, \quad |Y_i \backslash Y_{i-1}| = \binom{k-m-2}{m-1} = |W_i|,$$

and $|Z_i| = |Z_i'| + |Z_1''| = \binom{k-m-2}{m-2} + \binom{k-m-3}{m-2}$.

We recall also that as $P_1^k(k)$ is a cyclic d-polytope with k vertices and $d = 2m$,

$$f_{d-1}(P_1^k(k)) = \frac{k}{k-m}\binom{k-m}{m}.$$

Thus, $v = k$ yields that

$$\begin{aligned} \frac{k}{k-m}\binom{k-m}{m} &= |X_0| + |Y_0| + |Y_1 \backslash Y_0| + |Z_0| + |W_0| \\ &= |X_i| + |Y_0| + |Y_i \backslash Y_{i-1}| + |Z_i| + |W_i|, \end{aligned}$$

and $\binom{a+1}{b+1} = \frac{a+1}{b+1}\binom{a}{b}$ yields that

$$\begin{aligned} f_{d-1}(P_1^v(k)) &= (v-k+1)|X_i| + |Y_0| + (v-k+1)|Y_i \backslash Y_{i-1}| + |Z_{v-k}| + |W_{v-k}| \\ &= (v-k)(|X_i| + |Y_i \backslash Y_{i-1}|) + |X_i| + |Y_0| + |Y_i \backslash Y_{i-1}| + |Z_i| + |W_i| \\ &= (v-k)\left(\binom{k-m-3}{m-2} + \binom{k-m-2}{m-1}\right) + \frac{k}{k-m}\binom{k-m}{m} \\ &= \frac{(v-k)(k-3)}{m-1}\binom{k-m-3}{m-2} + \frac{k}{k-m}\binom{k-m}{m}. \end{aligned}$$

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