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# On Matrix Rings over Unit-Regular Rings*

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**Abstract.** In this paper, we prove that a unit-regular ring $R$ is isomorphic to a Busque ring $S$ if its matrix rings $M_n(R)$ and $M_n(S)$ are isomorphic. This gives a partial answer to a matrix isomorphism question for unit-regular rings proposed in the text of K. Goodearl.

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## 1. Introduction

A longstanding open problem of Von Neumann regular rings is the matrix-isomorphism problem for the class of unit-regular rings (cf. [5, open problem 47]): If $R$ and $S$ are unit-regular rings such that $M_n(R) \cong M_n(S)$ for some positive integer $n$, is $R \cong S$? This problem, up to the present moment, has not yet been completely answered, except Goodearl himself pointed out implicitly in 1982 [6] that the answer to this problem is positive if the rings are directly finite right $\aleph_0$-continuous regular rings. It is natural to ask whether the answer is still positive if the rings are unit-regular without any restrictions or conditions imposed on them. In this connection, Goodearl [8] conjectured that the answer to this problem might be negative in general as he has constructed a unit-regular ring $R$ whose Grothendieck group $K_0(R)$ contains some torsion elements, so that $K_0(R)$ of $R$ is not necessarily unpreforated.

On the other hand, Busque [3] has introduced the concept of directly finite $\aleph_0$-complete regular rings in 1980. For the sake of brevity, we just call the directly finite $\aleph_0$-complete

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non-simple regular rings the Busque rings and we also name the directly finite $\aleph_0$-continuous regular rings the Goodearl rings, named after Goodearl [6]. It is observed that the structure of Busque rings is quite close to Goodearl rings because they are both unit-regular, but they are independent to each other.

In this paper, some properties of Busque rings will be investigated, in particular, we show that if the lattice $L_R(R)$ of the principal right ideals of a unit-regular ring $R$ is $\aleph_0$-complete then the $n$-cancellation law ($n > 1$) holds in $L_R(R)$. By using this property, we show that if $\phi : M_n(R) \to M_n(S)$ is an isomorphism and $S$ is a Busque ring then $R \cong S$. Thus the matrix-isomorphism problem is answered positively for the class of Busque rings. In fact, this result can be regarded as a parallel result of Goodearl for right $\aleph_0$-continuous regular rings, although Busque regular rings and Goodearl rings are independent. It is noted by Ara, O'Meara and Tyukavkin [2] that cancellation property also holds for projective modules over regular rings with comparability, and all matrix rings $M_n(R)$ satisfy the condition of $s$-comparability if the base $R$ is itself a regular ring with $s$-comparability. Further properties of regular rings with $s$-comparability have been recently obtained by Kutami (see [9] and [10]). Based on the properties of regular rings with comparability, one hopes that the matrix-isomorphism problem might also have a positive answer for some classes of directly finite non-simple regular rings with comparability because the Busque rings can be viewed as its special case. A replacement lemma is introduced here and this lemma may be helpful in solving the proposed question of Goodearl [5].

Throughout this paper, all rings $R$ are associative rings with identity and all $R$-modules are unitary $R$-modules.

For the sake of completeness, we list here the following notations and results which will be frequently referred in the paper:

**Notations.** Let $A$, $B$ be $R$-modules and $k$ any cardinal number. Then we give some notations as follows:

$A \lesssim B$; $B$ has a submodule isomorphic to $A$.

$A \prec B$; $B$ has a proper submodule isomorphic to $A$.

$A \precsim_\oplus B$; $B$ has a direct summand isomorphic to $A$.

$kA$; the direct sum of $k$-copies of $A$.

$FP(R)$; the family of all finitely generated projective $R$-modules.

$V(R)$; the monoid of isomorphic classes of finitely generated projective $R$-modules.

The following are some useful definitions:

**Definition 1.1.** *A ring $R$ is called regular if for any $x \in R$, there exists $y \in R$ such that $xyx = x$.*

A ring $R$ is called unit-regular if for any $x \in R$, there exists a unit $u$ of $R$ such that $xux = x$. The properties of unit-regular rings have been widely studied by Goodearl [8], Busque [3], Kutami [9], [10] and others.

**Definition 1.2.** *Let $L_R(R)$ be the lattice of principal right ideals of a ring $R$ which is partially ordered by set inclusion. Then $R$ is said to be $\aleph_0$-complete if any countable subset of $L_R(R)$ has a supremum and an infimum in $L_R(R)$.*

**Definition 1.3.** *A regular ring $R$ is said to be separative if for any finitely generated projective modules $A_R$ and $B_R$ satisfying $2A \cong A \oplus B \cong 2B$ implies $A \cong B$.*

It has been recently noted in [1] that any unit-regular ring is separative.

**Definition 1.4.** *An $R$-module $M$ is directly finite provided that $M$ is not isomorphic to a proper direct summand of itself. If $M$ is not directly finite, then $M$ is said to be directly infinite. A ring is directly finite if the $R$-module $R_R$ is directly finite.*

All basic results concerning regular rings can be found in the text of K.R. Goodearl [5].

## 2. Preliminaries

In this section, we first include some basic results of unit-regular rings. Some of these results are crucial results for proving our main results in this paper.

**Definition 2.1.** *Let $R$ be a $\aleph_0$-complete regular ring. Let $\{e_nR\}_{n\geqslant 1}$ be a sequence of principal right ideals of $R$. If $\bigwedge_{m\geqslant 1}(\bigvee_{n\neq m} e_nR) = 0$ in the lattice of principal right ideals $L_R(R)$ of $R$, then the sequence $\{e_nR\}_{n\geqslant 1}$ is called a strongly independent sequence of principal right ideals.*

The following lemma of Busque [3] is crucial in proving that a directly finite $\aleph_0$-complete regular ring is unit-regular.

**Lemma 2.2.** ([3, Theorem 3.4]). *Let $\{e_nR\}_{n\geqslant 1}$ and $\{f_nR\}_{n\geqslant 1}$ be two sequences of principal right ideals in $L_R(R)$ such that $e_nR \bigcap f_nR = 0$ for all $n \in \mathbb{N}$. If $\{e_nR \oplus f_nR\}_{n\geqslant 1}$ is strongly independent with $e_nR \cong f_nR$ for all $n \in \mathbb{N}$, then $\bigvee_{n\geqslant 1} e_nR \cong \bigvee_{n\geqslant 1} f_nR$.*

Hereafter, we just call the directly finite $\aleph_0$-complete non-simple regular rings the Busque rings.

The following lemma is a well known result of regular rings since it relates with the cancellation of small projectives. This result is particularly useful because it can be used to characterize the unit-regular rings.

**Lemma 2.3.** *A regular ring $R$ is unit-regular if and only if the following cancellation property holds in $R$:*

$$A \oplus B \cong A \oplus C \Rightarrow B \cong C, \text{ provided that } A, B \text{ and } C \in FP(R). \qquad (*)$$

**Definition 2.4.** *Let $G_R$ be a right $R$-module with $f, g \in End_R(G)$. Then $f$ and $g$ are called equivalent morphisms if there exist $\alpha, \beta \in Aut(G)$ such that $\alpha f \beta = g$.*

The above concept of equivalent morphism in $End_R(G)$ is useful for studying regular rings, and we notice that this concept is different from the usual concept of equivalence between matrices.

The following lemma of Ara, Goodearl, O'Meara and Pardo [1] gave a criterion for the equivalence of morphisms.

**Lemma 2.5.** (see [1, Lemma 4.1]). *Let $M_R$ be a finitely generated projective right $R$-module. Let $f, g \in End_R(M)$. Then $f$ is equivalent to $g$ if and only if they have the same isomorphic kernels, images and cokernels.*

The following theorem can be found in [1]. (cf. [Theorem 4.5]).

**Theorem 2.6.** *Let $R$ be a unit-regular ring. Then every square matrix over $R$ is equivalent to a diagonal matrix.*

We now modify the above theorem in the following form.

**Theorem 2.7.** *Let $R$ be a unit-regular ring with $f \in M_n(R)$. Then $f$ is equivalent to a diagonal matrix with idempotent entries.*

*Proof.* Since $R$ is a unit-regular ring, $R$ is separative. Also, by Theorem 2.6 and the results in [1], we know that there are $\alpha, \beta \in GL_n(R)$ such that

$$\alpha f \beta = \begin{pmatrix} a_1 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & a_n \end{pmatrix},$$

where $GL_n(R)$ is a general linear group over $R$. Since $R$ is a unit-regular ring, for each $i = 1, 2, \ldots, n$, there exist units $u_i \in R$ such that $a_i u_i a_i = a_i$ for $i = 1, 2, \ldots, n$. Let

$$\beta_1 = \begin{pmatrix} u_1 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & u_n \end{pmatrix}.$$

Then it is obvious that $\beta_1 \in GL_n(R)$ and $(a_i u_i)^2 = a_i u_i$ for $i = 1, 2, \cdots, n$. Thus,

$$\alpha f \beta \beta_1 = \begin{pmatrix} a_1 u_1 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & a_n u_n \end{pmatrix}$$

is the required matrix. □

To deal with the infinite sum of finitely generated projective modules, we first cite a well known result of finitely generated projective modules over unit-regular rings given by Goodearl in [5].

**Lemma 2.8.** (i) *Let $R$ be an unit-regular ring. Suppose $B$, $A_1$, $A_2, \ldots, A_n, \ldots$ is a sequence of modules in $V(R)$. If $A_1 \oplus A_2 \oplus \cdots \oplus A_n \lesssim B$ for all $n$, then $\oplus_{n \geqslant 1} A_n \lesssim B$.*

(ii) *Let $A$, $B$ be projective right modules over a unit-regular ring. If $A_1 \leqslant A_2 \leqslant \cdots \leqslant A_n \leqslant \cdots$ is a chain of submodules of $A$ and $A_n \lesssim B$ for all $n$. Then $\bigcup_{n \geqslant 1} A_n \lesssim B$.*

## 3. Replacement Lemma and Cancellation Theorem

In this section, we introduce the replacement lemma and we will apply it to study the Busque rings. By using this lemma, we can show that if we have an independent sequence of principal right ideals in a Busque ring then we can possibly replace this sequence by another strongly independent sequence in the sense that each term in the new sequence is isomorphic to the corresponding term in the original sequence. By using this replacement lemma, we are able to establish a cancellation theorem for modules in Busque rings.

**Lemma 3.1. (Replacement Lemma)** *Let $B$ be a principal ideal of a Busque ring $R$. If there are sequences of principal right ideals $\{B_i\}_{i\in\mathbb{N}}$ and $\{B'_i\}_{i\in\mathbb{N}}$ contained in $B$ satisfying $B = B_1 \oplus B_2 \oplus \cdots \oplus B_k \oplus B'_k$ (internal direct sum) with $B'_i = B_{i+1} \oplus B'_{i+1}$, then there are sequences of principal right ideals $\{E_i\}_{i\in\mathbb{N}}$ and $\{C_i\}_{i\in\mathbb{N}}$ contained in $B$ satisfying $B \geqslant E_1 \oplus E_2 \oplus \cdots \oplus E_k \oplus C_k$ with $E_i \cong B_i$, $C_k = E_{k+1} \oplus C_{k+1}$ and $\bigwedge_{n\geqslant 1} C_n = 0$. Moreover, the sequence $\{E_n\}_{n\geqslant 1}$ is strongly independent.*

*Proof.* We first construct two sequences of principal right ideals $\{E_i\}_{i\in\mathbb{N}}$ and $\{C_i\}_{i\in\mathbb{N}}$ satisfying the conditions $B \geqslant E_1 \oplus E_2 \oplus \cdots \oplus E_k \oplus C_k$ with $E_i \cong B_i$, $C_k = E_{k+1} \oplus C_{k+1}$ and $\bigwedge_{n\geqslant 1} C_n = 0$ respectively. If $\bigwedge_{n\geqslant 1} B'_n = 0$ then there is nothing to prove because the sequences $\{B_k\}_{k\geqslant 1}$ and $\{B'_k\}_{k\geqslant 1}$ are already the desired sequences. We now assume that $\bigwedge_{n\geqslant 1} B'_n \neq 0$. Since $R$ is a Busque ring, $R$ is $\aleph_0$-complete. We may assume that $\bigwedge_{n\geqslant 1} B'_n = zR$ for some $z \neq 0$ in $R$. We proceed to construct the desired sequence by using induction on $k$. Obviously, $zR \subseteq B'_k$ for all natural numbers $k$. Since $B = C \oplus zR = B_1 \oplus B'_1$ for some principal right ideal $C$, by using the modular law, we immediately have $B'_1 = (C \cap B'_1) \oplus zR$. Now, let $C_1 = C \cap B'_1$ and choose a principal right ideal $E_1$ of $R$ such that $E_1 \oplus C_1 = C$. Then, by substitution, we obtain the following expression:

$$E_1 \oplus C_1 \oplus zR = B = B_1 \oplus B'_1 = B_1 \oplus C_1 \oplus zR.$$

Since the ring $R$ is unit-regular, it is of course cancellative, by Lemma 2.2, we obtain $E_1 = B_1$ and so it leads to $C_1 \oplus zR = B'_1 = B_2 \oplus B'_2$. Because $zR \subseteq B'_2$ and by the modular law, we obtain $B'_2 = C_2 \oplus zR$, where $C_2 = E_2 \oplus C_2$. Now, choose a principal right ideal $E_2$ of $R$ such that $C_1 = E_2 \oplus C_2$. Then by applying the cancellative law of the unit-regular ring $R$ again, we have $E_2 \cong B_2$.

Suppose that we have constructed $B_1 \geqslant E_1 \oplus E_2 \oplus \cdots \oplus E_{k-1} \oplus C_{k-1}$ with $E_i \cong B_i$, $C_i = E_{i+1} \oplus C_{i+1}$ and $C_i \oplus zR = B'_i = B_{i+1} \oplus B'_{i+1}$ for $i = 1, 2, \ldots, k-1$. Then, in order to finish the proof by induction, we need to construct $C_k$ and $E_k$. In fact, by using the modular law again, we have $B'_k = B'_k \cap C_{k-1} \oplus zR$. By writing $C_k = B'_k \cap C_{k-1}$ and choosing a principal right ideal $E_k$ such that $C_{k-1} = C_k \oplus E_k$, we obtain

$$B_k \oplus C_k \oplus zR = B_k \oplus B'_k = C_{k-1} \oplus zR = C_k \oplus E_k \oplus zR.$$

By the unit-regularity of the ring $R$ again, we can cancel $C_k \oplus zR$ and obtain $B_k \cong E_k$.

We still need to verify that $\bigwedge_{n\geqslant 1} C_n = 0$. This part is easy since $\bigwedge_{n\geqslant 1} C_n \leqslant \bigwedge_{n\geqslant 1} B'_n = zR$ and hence $zR \cap C_i = 0$ for all $i$. Thus, $\bigwedge_{n\geqslant 1} C_n = 0$.

Finally, we show that the sequence of ideals $\{E_n\}_{n\geqslant 1}$ is strongly independent. For this purpose, we first observe that $\bigvee_{n\neq m} E_n \leqslant E_1 \oplus \cdots \oplus E_{m-1} \oplus C_m$. Now, we let $V_m =$

$E_1 \oplus \cdots \oplus E_{m-1} \oplus C_m$. Then, to show $\{E_n\}_{n\geqslant 1}$ is strongly independent, it suffices to show that $\bigwedge V_m = 0$. We first let $x \in V_m$ for all $m \geqslant 1$. We claim that $x \in C_m$ by induction on $m$. Clearly $x \in V_1 = C_1$. Suppose that $x \in C_j$ for all $j \leqslant k$. Then we can show that $x \in C_{k+1}$. This is because $x \in V_{k+1}$, we can express $x$ by $x = e_1 + e_2 + \cdots + e_k + c$ with $e_i \in E_i$ and $c \in C_{k+1}$. This leads to $x = e_1 + (e_2 + \cdots + e_k + c) \in E_1 \oplus C_1$. By the unique representation of direct sums and by our induction hypothesis that $x \in C_1$, we obtain $e_1 = 0$. Suppose that we have already shown that $e_1 = e_2 = \cdots = e_j = 0$ for all $j \leqslant k-1$. Then, we can write $x = e_{j+1} + \cdots + e_k + c$. Since $C_j = C_{j+1} \oplus E_{j+1}$ and $x \in C_{j+1}$. we can easily see that $e_{j+1} = 0$ by the uniqueness of direct sum representation. This shows that $e_i = 0$ for all $i = 1, 2, \ldots, n$ and thereby $x = c \in C_{k+1}$. Thus, our claim is established, by induction. This implies that $x \in \bigwedge V_m \subseteq \bigwedge_{n\geqslant 1} c_n = 0$, and whence, the sequence $\{E_n\}_{n\geqslant 1}$ is strongly independent. This finishes the proof. □

The following lemma, due to Goodearl in [6], is used to prove that $\aleph_0$-continuous regular rings have n-cancellation property over modules (cf. [6, Theorem 14.30]). We cite his lemma below because it can also be applied to Busque rings.

**Lemma 3.2.** [Goodearl] *Suppose $R$ is a unit-regular ring with $neR \lesssim nfR$. Then there exist decompositions $eR = A_1 \oplus A_2 \oplus \cdots \oplus A_k \oplus A'_k$ and $fR = B_1 \oplus B_2 \oplus \cdots \oplus B_k \oplus B'_k$, respectively with $A_i = A_{i+1} \oplus A'_{i+1}$ and $B'_i = B_{i+1} \oplus B'_{i+1}$ satisfying $A_i = B_i$ and $(n + k - 1)A'_k \lesssim neR$. We observe here that all the direct summands in the above decompositions are the principal right ideals of $R$.*

The following lemma due to Busque in [3] is to show that a Busque regular ring is unit-regular.

**Lemma 3.3.** [Busque] *Let $R$ be a Busque regular ring. Then $\aleph_0 eR \lesssim fR$ implies $eR = 0$.*

By using the replacement lemma and the above two lemmas, we give the following generalized version of Lemma 2.3.

**Lemma 3.4.** *Let $R$ be a Busque regular ring. Then $\aleph_0 eR \lesssim nfR$ implies $eR = 0$.*

*Proof.* We prove this lemma by induction on $n$. Of course, the lemma holds trivially for $n = 1$ since this is just our lemma 3.3. Also we observe that in the Busque's lemma, $fR$ contains at most finitely many copies of $eR$. In this case, we may assume that $eR$ cannot be embedded in $eR$. Thus, by eliminating out the principal ideal $eR \cap fR$ via cancellation, we can assume that $eR$ and $fR$ are both independent right ideals. Suppose that the lemma is still true for all $k \leqslant n-1$. We need to show that it is true for $k = n$. In fact, since $\aleph_0 eR \lesssim nfR$, we have $neR \lesssim nfR$. By Goodearl's lemma (Lemma 3.2), we obtain the following decompositions $eR = A_1 \oplus A_2 \oplus \cdots \oplus A_k \oplus A'_k$ and $fR = B_1 \oplus B_2 \oplus \cdots \oplus B_k \oplus B'_k$ with $A_i = A_{i+1} \oplus A'_{i+1}$ and $B'_i = B_{i+1} \oplus B'_{i+1}$ satisfying $A_i = B_i$ and $(n + k - 1)A'_k \lesssim (n-1)eR$. We now claim that $\bigvee_n A_n = eR$. Since $R$ is a Busque regular ring, $R$ is accordingly a $\aleph_0$-complete regular ring and hence $\bigvee_n A_n$ is a principal right ideal of $R$. Suppose on the contrary, that there is a direct principal ideal $J$ of $R$ such that $\bigvee_n A_n \oplus J = eR$. Then, for any integer $k$, there is a summand $D_k$ satisfying $\bigvee_n A_n \oplus J = A_1 \oplus A_2 \oplus \cdots \oplus A_k \oplus D_k$. Thus, we have

$$\bigvee\nolimits_n A_n = A_1 \oplus \cdots \oplus A_k \oplus D_k \oplus J = eR = A_1 \oplus \cdots \oplus A_k \oplus A'_k.$$

By using the cancellation property of the unit-regular ring $R$, we obtain $\oplus D_k \oplus J = A'_k$. This shows that $J \leqslant A'_k$ for all $k$. Consequently, we deduce immediately that $(n+k-1)J \leqslant (n+k-1)A'_k \leqslant (n-1)eR$, for all $k \geqslant 1$. By Lemma 2.8 (i), we have $\aleph_0 J \leqslant (n-1)eR$. Now by the induction hypothesis, we conclude that $J = 0$ and whence, $\bigvee_n A_n = eR$. Thus, our claim is established.

We now show that $\bigwedge A'_k = 0$. Suppose, if possible, that $\bigwedge A'_k \neq 0$. Then we have $0 \neq z \in \bigwedge A'_k$ and so $\bigwedge A'_k = zR$. Consequently, $zR \subseteq A'_k$ for all $k \geqslant 1$. This leads to $\aleph_0 zR \leqslant (n-1)eR$, by applying the same arguments in the above paragraph. Thus, $z = 0$ by induction hypothesis and so $\bigwedge A'_k = 0$.

By applying our replacement Lemma 3.1, we can find some strongly independent sequences of principal right ideals $\{E_n\}_{n\geqslant 1}$ and $\{C_n\}_{n\geqslant 1}$ satisfying the conditions $fR \geqslant E_1 \oplus E_2 \oplus \cdots \oplus E_k \oplus C_k$ with $E_i \cong B_i$ and $C_k = E_{k+1} \oplus C_{k+1}$ with $\bigwedge_n C_n = 0$. Now we need to show that $\{A_n \oplus E_n\}_{n\geqslant 1}$ is a strongly independent sequence, that is, to show that $\bigwedge_{m\geqslant 1} \bigvee_{n\neq m} (A_n \oplus E_n) = 0$. We first observe that

$$\bigvee_{n\neq m} (A_n \oplus E_n) \leqslant (A_1 \oplus E_1) \oplus \cdots \oplus (A_{m-1} \oplus E_{m-1}) \oplus (A'_m \oplus C_m).$$

All we need is to show that $\bigwedge_{m\geqslant 1}\{(A_1 \oplus E_1) \oplus \cdots \oplus (A_{m-1} \oplus E_{m-1}) \oplus (A'_m \oplus C_m)\} = 0$. For this purpose, we let $x \in \bigwedge_{m\geqslant 1}\{(A_1 \oplus E_1) \oplus \cdots \oplus (A_{m-1} \oplus E_{m-1}) \oplus (A'_m \oplus C_m)\}$. Then $x \in (A_1 \oplus E_1) \oplus \cdots \oplus (A_{m-1} \oplus E_{m-1}) \oplus (A'_m \oplus C_m)$, for all $m \geqslant 1$. In particular, we have $x \in (A'_1 \oplus C_1)$. Write $x = (x_1, x_2)$ with $x_1 \in A'_1$ and $x_2 \in C_1$. Then, since $x \in (A_1 \oplus E_1) \oplus (A'_2 \oplus C_2)$, by the uniqueness of direct sum representation, we can easily obtain that $x_1 \in A'_2$ and $x_2 \in C_2$. Continue this process, we eventually obtain $x_1 \in A'_i$ and $x_2 \in C_i$ for all $i \geqslant 1$. Since $\bigwedge_n A'_n = 0$ and $\bigwedge_n C_n = 0$, we get $x = 0$. This shows that the sequence $\{A_n \oplus E_n\}_{n\geqslant 1}$ is strongly independent.

Invoking Lemma 2.1, we conclude that $\bigvee_n A_n \cong \bigvee_n E_n \leqslant fR$. Hence, we have $eR \leqslant fR$. However, this is clearly a contradiction since $fR$ is assumed not to contain any copy of $eR$. Thus, our theorem is proved. □

**Corollary 3.5.** *Let $R$ be a Busque regular ring. If $P_R$ is a finitely generated projective right $R$-module over $R$, then $P_R$ contains no infinite direct sum of isomorphic generated submodules of $P_R$.*

*Proof.* Since $P_R$ is a finitely generated projective right $R$-module, there is $n \in \mathbb{N}$ such that $P_R \leqslant nR$. Suppose that $N_R$ is a finitely generated projective right $R$-module with $\aleph_0 N_R \leqslant P_R$. Then we can choose an idempotent $e \in R$ with $eR \leqslant N$. Hence it follows that $\aleph_0 eR \leqslant P_R \leqslant nR$. By Lemma 3.4, we obtain $eR = 0$ which forces $N_R = 0$. Our proof is completed. □

By using the above lemmas, we establish a cancellation theorem for Busque regular rings. This result is helpful in solving the matrix-isomorphism problem.

**Theorem 3.6.** *Let $R$ be a Busque regular ring. If $neR \leqslant nfR$ then $eR \leqslant fR$.*

*Proof.* We may assume that $fR$ contains no isomorphic copy of $eR$ and $eR \cap fR = 0$. Since $neR \leqslant nfR$, we obtain the decompositions $eR = A_1 \oplus A_2 \oplus \cdots \oplus A_k \oplus A'_k$ and

$fR = B_1 \oplus B_2 \oplus \cdots \oplus B_k \oplus B'_k$ with $A_i = A_{i+1} \oplus A'_{i+1}$ and $B'_i = B_{i+1} \oplus B'_{i+1}$ satisfying $A_i = B_i$ and $(n+k-1)A'_k \lesssim (n-1)eR$ for all $k \in \mathbb{N}$, respectively. We need to show that $\bigvee_n A_n = eR$. Suppose on the contrary that $\bigvee_n A_n \subsetneqq eR$. Then we can find a principal right ideal $J$ of $R$ such that $\bigvee_n A_n \oplus J = eR$. Also, for any integer $k$, there exists a principal right ideal $D_k$ such that $A_1 \oplus \cdots \oplus A_k \oplus D_k = \bigwedge_n A_n$. In this way, we obtain the following decompositions for $eR$.

$$A_1 \oplus \cdots \oplus A_k \oplus D_k \oplus J = \bigwedge{}_n A_n \oplus J = eR = A_1 \oplus \cdots \oplus A_k \oplus A'_k.$$

Since $R$ is a Busque regular ring, it is unit-regular and thereby, by Lemma 2.2, we obtain that $D_k \oplus J = A'_k$, and hence $J \leqslant A'_k$, for all $k \in \mathbb{N}$. This implies that $(n+k-1)J \leqslant (n+k-1)A'_k \leqslant (n-1)eR$. As $k$ is arbitrary, we obtain $\aleph_0 J \leqslant (n-1)eR$. By Lemma 3.4, we immediately see that $J = 0$ and so $\bigvee_n A_n = eR$. Now, by applying the replacement Lemma 3.1, we can find a strongly independent sequence of principal ideals $\{E_n\}_{n \geqslant 1}$ and $\{C_n\}_{n \geqslant 1}$ contained in $fR$ satisfying the conditions $E_n \cong B_n$ and $C_n = E_{n+1} \oplus C_{n+1}$ with $\bigwedge_n C_n = 0$. By using the arguments in Lemma 3.4 repeatedly, we can show that the sequence $\{A_n \oplus E_n\}_{n \geqslant 1}$ is also strongly independent. Thus, by Lemma 2.1, we have $eR = \bigvee_n A_n \cong \bigvee_n E_n \leqslant fR$. This result contradicts our assumption. Hence, $eR \leqslant fR$ and the theorem is proved. □

**Corollary 3.7.** *Let $R$ be a Busque regular ring. If $neR \cong nfR$ then $eR \cong fR$.*

## 4. Matrix-isomorphism theorem

In this section, we give a positive answer to the well known matrix-isomorphism problem for Busque regular rings. As hinted by Goodearl [8], the answer to the matrix-isomorphism problem for general regular rings might be negative.

**Theorem 4.1.** (Main theorem) *Let $\Phi : M_n(R) \to M_n(S)$ be an isomorphism between the matrix rings over the rings $R$ and $S$ respectively. If $S$ is a Busque regular ring then $R \cong S$.*

*Proof.* Let $e_i$ be an $n \times n$ matrix whose $(i,i)$ entry is 1 and 0 otherwise. Let $u_i$ be the elementary matrix obtained by interchanging the i-th row with the (i+1)-th row of the $(n \times n)$-identity matrix, $I_n$. Then, it is easy to see that $e_i^2 = e_i$, $u_i^2 = u_i$ for all $i$ and $u_i e_i u_i = e_{i+1}$, where $i = 1, 2, \ldots, n-1$. It is now clear that the identity matrix $I_n$ can be expressed by

$$I_n = e_1 + \sum_{i=1}^{n-1} u_i \cdots u_1 e_1 u_1 \cdots u_i.$$

Since $\Phi$ is an isomorphism, we can let $f_i = \Phi(e_i)$, $v_i = \Phi(u_i)$. As a result, we can derive that

$$I_n = f_1 + \sum_{i=1}^{n-1} v_i \cdots v_1 f_1 v_1 \cdots v_i.$$

Then, by applying the isomorphism $\Phi$ on $M_n(R)$, we obtain the following isomorphic relations of $R$:

$$\begin{aligned} R &\cong End_R(R) \\ &\cong End_R(e_1nR) \\ &\cong e_1M_n(R)e_1 \\ &\cong \Phi(e_1)M_n(S)\Phi(e_1) \\ &\cong f_1M_n(S)f_1 \\ &\cong End_S(f_1nS) \end{aligned}$$

In proving $R \cong S$, our final step is to show that $f_1nS \cong S$. In fact, by Theorem 2.5, we observe that $f_1 \in M_n(S)$ is equivalent to a diagonal matrix with idempotent entries in $S$ since $S$ is a unit-regular ring. This implies that there exist $\alpha, \beta \in GL_n(S)$ such that

$$\alpha f_1 \beta = \begin{pmatrix} c_1 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & c_n \end{pmatrix},$$

where each $c_i$ is an idempotent element. Thus, $I_m(f_1) \cong \oplus_{i \geqslant 1} c_iS$. Furthermore, since $v_k \cdots v_1 f_1 v_1 \cdots v_k = f_k$, we have $I_m(f_1) \cong I_m(f_k)$ for all $k = 1, 2, \ldots, n$. Because $I_n = f_1 + \sum_{i=1}^{n-1} v_i \cdots v_1 f_1 v_1 \cdots v_i$, we obtain

$$nS = I_m(f_1) \oplus I_m(f_2) \oplus \cdots \oplus I_m(f_m) \cong nI_m(f_1) \cong nc_1S \oplus \cdots \oplus nc_nS.$$

Thus, by the cancellation law of the unit-regular ring $S$, we obtain

$$n(1 - c_1)S \cong nc_2S \oplus \cdots \oplus nc_nS,$$

and thereby $nc_2S \leqslant n(1 - c_1)S$. This implies that $c_2S \leqslant (1 - c_1)S$, by Theorem 3.6. Consequently, we obtain $c_1S \oplus c_2S \leqslant S$. Moreover, we also have the following decomposition $(1 - c_1)S \cong c_2S \oplus D_2$, where $D_2$ is a principal right ideal of $R$ such that $nc_2S \oplus nD_2 \cong n(1 - c_1)S \cong nc_2S \oplus \cdots \oplus nc_nS$. Applying Theorem 3.6, we get $nD_2 \cong nc_3S \oplus \cdots \oplus nc_nS$. By applying the same arguments repeatedly, we get $c_3S \leqslant D_2$ and $c_1S \oplus c_2S \oplus c_3S \leqslant S$. Continue the above process, we eventually show that $c_1S \oplus c_2S \oplus \cdots \oplus c_nS \leqslant S$. Now by letting $F$ be a finitely generated projective right $S$-module such that $c_1S \oplus c_2S \oplus \cdots \oplus c_nS \oplus F \cong S$, then we have

$$nc_1S \oplus nc_2S \oplus \cdots \oplus nc_nS \oplus nF \cong nS \oplus nF \cong nS.$$

Applying Theorem 3.6 again, we get $nF = 0$ and thus $F = 0$. In other words, we have $c_1S \oplus c_2S \oplus \cdots \oplus c_nS \cong S$. This shows that $I_m(f_1) \cong S$ and consequently $R \cong S$. Our proof is completed. □

**Remark.** It was noticed by O'Meara that a directly finite simple regular ring $R$ is unit-regular provided that $R$ satisfies the condition given in Theorem 3.6, that is, whenever $e, f \in R$ with $n(eR) \leqslant n(fR)$ then $eR \leqslant fR$ (see [12, Corollary 3]). However, Goodearl gave some counterexamples in [8] to illustrate that the above property no longer holds for simple unit-regular rings. In other words, this result prevents us to use Theorem 3.6 to solve the matrix-isomorphism for simple unit-regular rings and this implies that the answer for directly finite simple regular rings might be negative in general. Thus, the crucial step in solving the problem is to seek some classes of unit-regular rings satisfying the condition of Theorem 3.6.

In closing this paper, we ask the question whether we can extend our Theorem 4.1 to infinite matrix rings?

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