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A Note on *k*-very Ampleness of a Bundle on a Blown up Plane

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Abstract. In the paper we answer the question how many generic points on a projective plane can be blown up to get the pullback bundle k-very ample in a given 0-dimensional subscheme Z.

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Introduction

The problem of embeddings of blown-up varieties has been recently studied by many authors, cf. eg. [3], [7], [1].

D'Almeida and Hirschowitz in [5] consider the case of points in generic position on $\mathbb{P}^2$, and give a criterion for the pullback bundle to be very ample, i.e. 1-very ample. They prove the following theorem.

Theorem 1. (D'Almeida, Hirschowitz) *Let $P_1, \ldots, P_r$ be generic points of $\mathbb{P}^2$. Let $X := \mathrm{Bl}_{P_1,\ldots,P_r}(\mathbb{P}^2) \xrightarrow{\pi} \mathbb{P}^2$ be the blowing up of $\mathbb{P}^2$ in $P_1, \ldots, P_r$. By $E_i, i = 1, \ldots, r$ denote the exceptional divisors. Let $\tilde{L} := \pi^*\mathcal{O}_{\mathbb{P}^2}(d) - \sum_{i=1}^r E_i$. If $d \geq 5$ and $r \leq \frac{(d+1)(d+2)}{2} - 6$, then $\tilde{L}$ is very ample.*

This paper gives a generalization of the above theorem for 'k-very ample in given subschemes' case. It is easy to see that considering the line bundle $\tilde{L} := \pi^*\mathcal{O}_{\mathbb{P}^2}(d) - \sum_{i=1}^r E_i$, so having $\tilde{L}.E_i = 1$, we cannot consider k-very ampleness of this bundle in every given subscheme (if $k \geq 2$). Still, we can ask when $\tilde{L}$ is k-very ample in a given 0-dimensional subscheme Z, where Z is 'admissible', i.e. $\mathrm{length}(Z \cap E_i) \leq 2$, for all i. The main result is contained in the following theorem:

Theorem 2. *Assume that $W = \{P_1, \dots, P_r\}$, where P_i are points of $\mathbb{P}^2$ and r is a positive integer. Let $M = \mathcal{O}_{\mathbb{P}^2}(d)$ and $X = \mathrm{Bl}_W\mathbb{P}^2 \xrightarrow{\pi} \mathbb{P}^2$, be the blowing up of $\mathbb{P}^2$ in W. By E_i denote the exceptional divisors. Let Z be a 0-dimensional subscheme of X, of length $l(Z) = k+1$, where k is a nonnegative integer. Then, for*

1. *W sufficiently general,*
2. *$d \geq 2k+3$,*
3. *Z admissible, i.e. such that $\forall_{i=1,\dots,r}\ l(Z \cap E_i) \leq 2$,*
4. *$r \leq \frac{(d+1)(d+2)}{2} - 3(k+1)$,*

the pullback bundle $\tilde{L} := \pi^*(\mathcal{O}_{\mathbb{P}^2}(d)) - \sum_{i=1}^r E_i$ is k-very ample in Z.

This results improves the one from [11], and is connected with the papers [9] and [10], where the analogous problem is considered for abelian and for ruled surfaces.

Notation

We work throughout over the field of complex numbers, $\mathbb{C}$. All varieties are assumed to be smooth and projective. By $H^i(X, \mathcal{F}) = H^i(\mathcal{F})$ we denote the cohomology groups of X, and by $h^i(X, \mathcal{F}) = h^i(\mathcal{F})$ their dimensions over $\mathbb{C}$. For line bundles L and divisors D on X we use exchangeably the notation $L + D$, $\mathcal{O}_X(D) \otimes L$ or $\mathcal{O}_X(D+L)$. We will write $\pi^*(I_W \otimes M)$ as well as $\pi^*(M) - \sum_{i=1}^r E_i$. For $\pi^{-1}I_W \cdot \mathcal{O}_X$ we will write $\tilde{I}_W$. A blow up of a surface S in r points in general position will be called a general blow up of S.

Basic definiton and lemmas

Let X be a smooth projective variety, let Z be a 0-dimensional subscheme of X and let I_Z be its ideal sheaf, thus $\mathcal{O}_Z = \mathcal{O}_X/I_Z$. By the length of the subscheme Z we understand $l(Z) := \dim H^0(\mathcal{O}_Z)$.

Let us remind the definition of k-very ampleness in a given subscheme Z and of k-very ampleness, cf. [2].

Definition 3. 1. *Let X be a variety (smooth and projective) and let L be a line bundle on X. We say that L is k-very ample in a given 0-dimensional subscheme Z of length $k+1$ if the mapping*

$$H^0(L) \longrightarrow H^0(L \otimes \mathcal{O}_Z)$$

is surjective.

2. *We say that L is k-very ample if it is k-very ample for every 0-dimensional subscheme Z of length $k+1$.*

Let now $\pi : X \longrightarrow \mathbb{P}^2$ be a general blow up of of $\mathbb{P}^2$ in r points. The following lemma reformulates of the definition of k-very ampleness.

Lemma 4. *$\pi^*(I_W \otimes M)$ is k-very ample on X in a given subscheme Z of length $k+1$, if and only if $H^0(X, I_Z \otimes \pi^*(I_W \otimes M))$ has codimension $k+1$ in $H^0(X, \pi^*(I_W \otimes M))$.*

Proof. From the definition of k-very ampleness, $L = \pi^*(I_W \otimes M)$ is k-very ample in the given Z iff the following sequence is exact:

$$0 \longrightarrow H^0(L \otimes I_Z) \longrightarrow H^0(L) \longrightarrow H^0(L \otimes \mathcal{O}_Z) \longrightarrow 0$$

and the exactness of the sequence is equivalent with $H^0(X, I_Z \otimes L)$ having codimension $k+1$ in $H^0(X, L)$. □

The next three lemmas are simple generalizations of lemmas from [5] for the 'k-very ample in a given Z' case. The proofs go analogously to those in [5]. We enclose the full proofs for the convenience of the reader.

Lemma 5. *$\pi^*(I_W \otimes M)$ is k-very ample on X in a given subscheme Z of length $k+1$, if and only if $H^0(\mathbb{P}^2, \pi_*(I_Z \otimes \tilde{I}) \otimes M)$ has codimension $k+1$ in $H^0(\mathbb{P}^2, I_W \otimes M)$.*

Proof. From the Projection Formula (cf. [8]):

$$H^0(X, \pi^*(I_W \otimes M)) \cong H^0(\mathbb{P}^2, I_W \otimes M)$$

and

$$H^0(X, (I_Z \otimes \tilde{I}) \otimes \pi^* M) \cong H^0(\mathbb{P}^2, \pi_*(I_Z \otimes \tilde{I}) \otimes M)$$

so, using Lemma 4 we are done. □

Lemma 6. *For an admissible subscheme Z of length l on X (i.e. $h^0(\mathcal{O}_Z) = l$) we have*

$$dim H^0(I_W / \pi_*(I_Z \otimes \tilde{I})) = l.$$

Proof. Without loss of generality we may assume that $W = \{P\}$. Observe that if $l = 0$ then $Z = \emptyset$ and we are done. Assume then, that $l > 0$ and consider the exact sequence:

$$0 \longrightarrow I_Z(-E) \longrightarrow \mathcal{O}(-E) \longrightarrow \mathcal{O}_Z(-E) \longrightarrow 0.$$

Applying the left exact functor π_* to this sequence, we get:

$$0 \longrightarrow \pi_*(I_Z \otimes \pi^* I_P) \longrightarrow I_P \stackrel{\alpha}{\longrightarrow} \pi_* \mathcal{O}_Z(-E).$$

Observe that $h^0(\pi_*(\mathcal{O}_Z(-E))) = l(\mathcal{O}_Z) = l$. (Indeed, for any sheaf $\mathcal{F}$ and for any morphism $\pi : X \longrightarrow Y$ we have: $h^0(\pi_*(\mathcal{F})) = \dim_{\mathbb{C}}(\pi_*\mathcal{F})(Y) = \dim_{\mathbb{C}}\mathcal{F}(\pi^{-1}Y) = \dim_{\mathbb{C}}\mathcal{F}(X) = h^0(\mathcal{F})$.)

So, to prove our lemma we have to show that α is surjective. To this end, consider the two exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_X(-2E) & \longrightarrow & \mathcal{O}_X(-E) & \longrightarrow & \mathcal{O}_X(-E)|_E & \longrightarrow & 0 \quad (1) \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_{Z'}(-2E) & \longrightarrow & \mathcal{O}_Z(-E) & \longrightarrow & \mathcal{O}_{Z''}(-E) & \longrightarrow & 0 \quad (2) \end{array}$$

where (2) is obtained as the restriction of (1) to Z. Note, that if $Z'' = \emptyset$ then the claim of our lemma is obvious, so we may assume that Z'' is not empty.

Applying π_* to (1) and (2), we get

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_*(\mathcal{O}_X(-2E)) & \longrightarrow & I_P & \stackrel{\beta_1}{\longrightarrow} & \pi_*(\mathcal{O}_X(-E)|_E) & \longrightarrow & 0 \quad (1) \\ & & \downarrow{\scriptstyle \gamma_1} & & \downarrow{\scriptstyle \gamma_2 = \alpha} & & \downarrow{\scriptstyle \gamma_3} & & \\ 0 & \longrightarrow & \pi_*(\mathcal{O}_{Z'}(-2E)) & \longrightarrow & \pi_*(\mathcal{O}_Z(-E)) & \stackrel{\beta_2}{\longrightarrow} & \pi_*(\mathcal{O}_{Z''}(-E)) & \longrightarrow & 0 \quad (2) \end{array}$$

Now observe that:

1. β_1, β_2 are surjective.
2. γ_3 is surjective, as it is the restriction of sections of $\mathcal{O}(-E)|_E = \mathcal{O}_E(1)$ (which is 1-very ample) to Z'', and, as Z is admissible, Z'' has length at most 2.
3. γ_1 is surjective as $Z' \cap E = \emptyset$ and I_P^2 surjects on $\mathcal{O}_{Z'}$.

Thus, $\gamma_2 = \alpha$ must be surjective and we are done. □

Lemma 7. *For an admissble Z of length $k+1$ on X, if*

$$h^1(\pi_*(I_Z \otimes \tilde{I}) \otimes M) = h^1(I_W \otimes M)$$

then $\pi^*(I_W \otimes M)$ is k-very ample in Z on X.

Proof. Consider the exact sequence:

$$0 \longrightarrow \pi_*(I_Z \otimes \tilde{I}) \otimes M \longrightarrow I_W \otimes M \longrightarrow I_W/\pi_*(I_Z \otimes \tilde{I}) \otimes M \longrightarrow 0,$$

from which we have the long exact sequence:

$$0 \longrightarrow H^0(\pi_*(I_Z \otimes \tilde{I}) \otimes M) \longrightarrow H^0(I_W \otimes M) \longrightarrow H^0(I_W/\pi_*(I_Z \otimes \tilde{I}) \otimes M) \longrightarrow$$

$$\longrightarrow H^1(\pi_*(I_Z \otimes \tilde{I}) \otimes M) \overset{\text{assumption}}{\cong} H^1(I_W \otimes M) \longrightarrow H^1(I_W/\pi_*(I_Z \otimes \tilde{I}) \otimes M) \longrightarrow 0$$

From Lemma 5, $h^0(I_W/\pi_*(I_Z \otimes \tilde{I}) \otimes M) = k+1$ and, as $h^1(I_W/\pi_*(I_Z \otimes \tilde{I}) \otimes M) = 0$ we have

$$h^0(I_W \otimes M) = h^0(\pi_*(I_Z \otimes \tilde{I}) \otimes M) + (k+1)$$

and from Lemma 4 we are done. □

Following [4], we introduce the following notation: H^N denotes $\mathrm{Hilb}^N_{\mathbb{P}^2}$, the Hilbert scheme of 0-dimensional subschemes of $\mathbb{P}^2$ of length N; $\chi(N,d) = \frac{(d+1)(d+2)}{2} - N$, with χ^+ the positive part of χ. Let

$$W_N^i[d] := \{Z \in H^N | h^0(I_Z(d)) \geq \chi^+ + i + 1\}.$$

Remark 8. Let $\chi \geq 0$.
1. According to [4], Proposition 9.1

$$\mathrm{codim} W_N^0[d] \geq \min(\chi[N,d] + 1, d). \qquad (*)$$

2. As for a 0-dimensional subscheme Z on $\mathbb{P}^2$, $l(Z) = h^0(\mathcal{O}_{\mathbb{P}^2}(d)) - h^0(I_Z(d)) + h^1(I_Z(d))$, for $Z \in H^N$ we have $h^1(I_Z(d)) \neq 0$ if and only if $I_Z(d) \in W_N^0[d]$.

For the proof of our main result we will need the following theorem of Fogarty [6].

Theorem 9. *If F is a nonsingular surface, then* Hilb^N_F *is a nonsingular variety of dimension* $2N$.

Now we are able to prove Theorem 2.

Proof. Take $S \subset H^r \times H^{r+k+1}$, such that:

i. $S \subset \{(W, W') \in H^r \times H^{r+k+1} | W \subset W'\}$,

ii. W is reduced and $h^1(I_W(d)) = 0$,

iii. $I_{W'} = \pi_*(I_Z \otimes \tilde{I})$.

We claim that for $(W, W') \in S$ $\tilde{M}$ is k-very ample in Z. To check this consider the two projections from S to H^N and to H^{r+k+1}. Observe:

1. $p_2^{-1}(W')$ is finite. (W is reduced and $W \subset W'$.)
2. Lemma 7 implies that it is enough to check that $h^1(I_{W'})(d) = 0$ for a sufficiently general W.
3. From Remark 8, 2., we have: $h^1(I_{W'})(d) > 0$ if and only if $W' \in W^0_{r+k+1}[d]$.
4. According to Remark 8, 1., the codimension of $W^0_{r+k+1}[d]$ in H^{r+k+1} satisfies:

$$\mathrm{codim} W^0_{r+k+1}[d] \geq \min(\chi[r+k+1, d] + 1, d),$$

thus from our assumptions:

$$\mathrm{codim} W^0_{r+k+1}[d] \geq 2k + 3.$$

5. From Theorem 9 it follows that $p_1^{-1}(W)$, for $W \in H^N$ is of dimension $2k + 2$.
6. Thus, for W sufficiently general, $p_2^{-1}(W^0_{r+k+1}[d]) \cap p_1^{-1}(W) = \emptyset$, finishing the proof.

□

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