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The Geometry of Pseudo Harmonic Morphisms

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Abstract. We study a class of maps, called Pseudo Horizontally Weakly Conformal (PHWC), which includes horizontally weakly conformal mappings. We give geometrical conditions ensuring the harmonicity of a (PHWC) map, making it a pseudo harmonic morphism, a generalisation of harmonic morphism, for which we broaden the Baird-Eells Theorem.
Finally, considering pseudo horizontally homothetic maps, we extend a theorem of Aprodu, Aprodu and Brinzanescu to pseudo harmonic morphisms, and show that the dual stress-energy of such maps is horizontally covariant constant.

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1. Introduction

Pseudo horizontally weakly conformal (PHWC) maps from compact Riemannian manifolds into Kähler manifolds (cf. Definition 5) were first considered in [4], though the name itself only appeared later on, in a study of stable harmonic maps into irreducible Hermitian symmetric space of compact type. The denomination is due to their property of generalising horizontally weakly conformal maps, which in turn embrace Riemannian submersions. This theme was picked up again in [11] where to a (PHWC) map is associated an f-structure (see Definition 1),

and conditions on this f-structure that force the harmonicity of the map are given. This particular line of investigation is completed here in Section 4.
Independently, Aprodu, Aprodu and Brinzanescu define in [1] a special class of (PHWC) map, called pseudo horizontally homothetic, with which they construct minimal submanifolds. We extend their result in Proposition 7.
Finally, the dual stress-energy tensor (compare with the stress-energy tensor of Baird-Eells in [2]), which vanishes for horizontally weakly conformal maps, is defined and shown to be horizontally covariant constant for pseudo horizontally homothetic maps.
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2. Definitions and essential properties of f-structures

Definition 1. *An f-structure on a Riemannian manifold (M^m, g) is a (smooth) skew-symmetric section F of $\mathrm{End}(TM)$ such that:*

$$F^3 + F = 0. \tag{1}$$

This concept was introduced by Yano in [16] (see [9] as well).

Condition (1) implies that on $T^{\mathbb{C}}M$, the complexification of TM, F admits three distinct eigenvalues: $+i$, $-i$ and 0, whose eigenspaces T^+M, T^-M and T^0M provide an orthogonal decomposition

$$T^{\mathbb{C}}M = T^+M \oplus T^-M \oplus T^0M, \tag{2}$$

with respect to the Hermitian metric $h(X,Y) = g(X,\bar{Y})$.

In 1977, Stong showed in [14] that the rank k of an f-structure is even and constant.

The notion of f-structure includes almost complex structures ($k = m$) and almost contact structures ($k = m-1$).

As is pointed out in [16], the existence of an f-structure is equivalent to a reduction of the structure group of the tangent bundle from $\mathrm{O}(m)$ to $\mathrm{U}(\frac{k}{2}) \times \mathrm{O}(m-k)$.

The eigenspaces T^+M and T^-M are g-isotropic, and in fact, there exists a bijection between g-isotropic subbundles of $T^{\mathbb{C}}M$ and f-structures (cf. [13]).

As for almost complex structures, the integrability of f-structures, i.e. the existence of local coordinates respecting the decomposition (2), boils down to the vanishing of the Nijenhuis tensor (cf. [7]):

$$N(X,Y) = [FX,FY] - F[FX,Y] - F[X,FY] + F^2[X,Y], \quad \forall X,Y \in TM. \tag{3}$$

Definition 2. *A map $\phi : (M,g,F) \to (N,h,\tilde{F})$ satisfying*

$$d\phi \circ F = \pm \tilde{F} \circ d\phi$$

is called $\pm f$-holomorphic.

3. (1,2)-symplectic and cosymplectic f-structures

Note that, when restricted to $H^{\mathbb{C}} = T^{+}M \oplus T^{-}M = \ker\left[F^2 + I\right]^{\mathbb{C}}$, an f-structure is an almost complex structure. This justifies the introduction of the following definitions:

Definition 3. *An f-structure F is called $(1,2)$-symplectic if $F\big|_{H^{\mathbb{C}}}$ is a $(1,2)$-symplectic almost complex structure, that is if:*

$$F\left[(\nabla_X F)(Y) + (\nabla_{FX} F)(FY)\right] = 0, \tag{4}$$

where X and $Y \in \ker\left[F^2 + I\right]$.

This generalises condition (A) of Rawnsley [13]. Similarly,

Definition 4. *An f-structure F is called cosymplectic if $F\big|_{H^{\mathbb{C}}}$ is a cosymplectic almost complex structure, that is, if:*

$$\sum_{j=1}^{k/2} F\left[\left(\nabla_{e_j} F\right)(e_j) + \left(\nabla_{Fe_j} F\right)(Fe_j)\right] = 0, \tag{5}$$

where $\{e_j, Fe_j\}_{j=1,\dots,\frac{k}{2}}$ an orthonormal basis of $\ker\left[F^2 + I\right]$.

From these definitions, it readily follows that a (1,2)-symplectic f-structure is always cosymplectic.

Let ω be the fundamental 2-form associated to the almost complex structure $F\big|_{H^{\mathbb{C}}}$. From our knowledge of almost complex structures, we infer that F is a (1,2)-symplectic f-structure if and only if $(d\omega)^{(1,2)} = 0$ and a cosymplectic f-structure if and only if $d^{*}\omega = 0$.

Let $\Omega(X,Y) = g(X, FY)$ be the fundamental 2-form associated to F, then it is clear that $(d\Omega)^{(1,2)} = 0$ implies $(d\omega)^{(1,2)} = 0$ but $d^{*}\Omega = 0$ implies $d^{*}\omega = 0$ only when the distribution $\ker F$ is minimal, since, for $X \in H$, $(d^{*}\Omega - d^{*}\omega)(X) = (m-k)g(F\eta, X)$ (η being the mean curvature of the distribution $\ker F$), or F is an almost complex structure.

4. The construction of F^{ϕ}

Definition 5. *Let (M,g) be a Riemannian manifold and (N,h,J) an almost Hermitian manifold. A map $\phi : (M,g) \to (N,h,J)$ is called pseudo horizontally weakly conformal (PHWC) if the map*

$$d\phi \circ (d\phi)^{*} : T^{\mathbb{C}}N \to T^{\mathbb{C}}N,$$

with the identifications $T^{\mathbb{C}}M \simeq T^{\mathbb{C}^{*}}M$ and $T^{\mathbb{C}}N \simeq T^{\mathbb{C}^{*}}N$, commutes with J, i.e.:

$$\left[d\phi \circ (d\phi)^{*}, J\right] = 0. \tag{PHWC}$$

This condition was first studied in [4].

It is easy to see that condition (PHWC) is equivalent to $(d\phi)^{*}\left(T^{(1,0)}N\right)$ being isotropic, and therefore to a (PHWC) map there exists an associated f-structure F^{ϕ} on M (cf. [11]). Note that $\ker F^{\phi} = \ker d\phi$.

Considering the complex structure J on N as an f-structure, one can verify that a (PHWC) map $\phi : (M, g, F^\phi) \to (N, h, J)$ is f-holomorphic, but an f-holomorphic map is not necessarily (PHWC).

Our first aim will be to find a link between F^ϕ and the tension field of ϕ.

Theorem 1. *Let $\phi : (M, g) \to (N, h, J)$ be a (PHWC) map from a Riemannian to a Kähler manifold, then the tension field of ϕ is given by:*

$$\tau(\phi) = -d\phi(F^\phi \delta F^\phi), \tag{6}$$

where F^ϕ is the f-structure associated to ϕ and $\delta F^\phi = \operatorname{trace} \nabla F^\phi$ is the divergence of F^ϕ.

Remark 1. As will be clear in the proof, equation (6) is, in fact, valid for any $\pm f$-holomorphic map from (M, g, F) to (N, h, J).
Besides, we could easily drop the integrability hypothesis on the target, and replace the "Kähler" condition by "(1,2)-symplectic".
Such an extended version could be seen as a generalisation of [6, Lemma 4.1].

Proof. Based on [6].
We work at a regular point, i.e. a point at which $d\phi$ does not vanish.
Consider an adapted frame $\{e_i, F^\phi e_i, e_\alpha\}$ (i.e. an orthonormal frame such that $e_\alpha \in \ker F^\phi$) and let $Z_j = \frac{1}{\sqrt{2}}\left(e_j - iF^\phi e_j\right)$. One can easily verify that:

$$\begin{aligned}
&\sum_{j=1}^{k/2} F^\phi \left[\left(\nabla_{e_j} F^\phi\right)(e_j) + \left(\nabla_{F^\phi e_j} F^\phi\right)(F^\phi e_j)\right] = \\
&\sum_{j=1}^{k/2} -F^\phi (i - F^\phi)\nabla_{\bar{Z}_j} Z_j - F^\phi (i + F^\phi)\nabla_{Z_j} \bar{Z}_j = \\
&\sum_{j=1}^{k/2} 2\left(\nabla_{\bar{Z}_j} Z_j\right)^- + 2\left(\nabla_{Z_j} \bar{Z}_j\right)^+,
\end{aligned}$$

where $X^\pm = -\frac{1}{2}F^\phi\left(F^\phi \pm i\right)X \in T^\pm M$ for any vector $X \in T^{\mathbb{C}}M$.

We require the following result, the proof of which is delayed:

Lemma 1.

$$F^\phi \delta_H F^\phi = \sum_{j=1}^{k/2} F^\phi \left[\left(\nabla_{e_j} F^\phi\right)(e_j) + \left(\nabla_{F^\phi e_j} F^\phi\right)(F^\phi e_j)\right] = F^\phi \delta F^\phi - (m-k)\eta,$$

where η is the mean curvature of the fibres and δ_H denotes the horizontal divergence (*see* [15])*:*

$$\delta_H F^\phi = \Sigma(\nabla_{e_A} F^\phi)(e_A)$$

where $\{e_A\}$ is an orthonormal basis of $\ker[(F^\phi)^2 + I]$.

On the other hand

$$\begin{aligned}\tau(\phi) &= \operatorname{trace}_g \nabla d\phi \\ &= \sum_{j=1}^{k/2} \left[(\nabla d\phi)(e_j, e_j) + (\nabla d\phi)(F^\phi e_j, F^\phi e_j)\right] + \sum_{\alpha=k+1}^{m} (\nabla d\phi)(e_\alpha, e_\alpha) \\ &= 2\sum_{j=1}^{k/2} \left(\nabla^{\phi^{-1}TN}_{\bar{Z}_j} d\phi(Z_j)\right) - (m-k)d\phi(\eta) - 2\sum_{j=1}^{k/2} d\phi\left(\nabla_{\bar{Z}_j} Z_j\right).\end{aligned}$$

Now, since (N, h, J) is Kähler, $\left[\nabla^{\phi^{-1}TN}_{\bar{Z}_j} d\phi(Z_j)\right]^{(0,1)} = 0$.
Therefore

$$\tau(\phi)^{(0,1)} = d\phi\left[(m-k)\eta - 2\left(\nabla_{\bar{Z}_j} Z_j\right)^{(0,1)}\right] = -\left[d\phi\left(F^\phi \delta F^\phi\right)\right]^{(0,1)}. \tag{7}$$

Since all the ingredients of equation (7) are real, we obtain:

$$\tau(\phi) = -d\phi\left(F^\phi \delta F^\phi\right). \tag{8}$$

In the case of a critical point p, either $d\phi$ vanishes in a neighbourhood p, or there exists a sequence of regular points converging towards p for which equation (8) holds. □

Proof of Lemma 1.

$$\begin{aligned}F^\phi \delta F^\phi &= \sum_{j=1}^{k/2} F^\phi\left[\left(\nabla_{e_j} F^\phi\right)(e_j) + \left(\nabla_{F^\phi e_j} F^\phi\right)(F^\phi e_j)\right] + \sum_{\alpha=k+1}^{m} F^\phi\left[\left(\nabla_{e_\alpha} F^\phi\right)(e_\alpha)\right] \\ &= \sum_{j=1}^{k/2} F^\phi\left[\left(\nabla_{e_j} F^\phi\right)(e_j) + \left(\nabla_{F^\phi e_j} F^\phi\right)(F^\phi e_j)\right] \\ &\quad + \sum_{\alpha=k+1}^{m} F^\phi\left[\left(\nabla_{e_\alpha} F^\phi(e_\alpha)\right) - F^\phi\left(\nabla_{e_\alpha} e_\alpha\right)\right] \\ &= \sum_{j=1}^{k/2} F^\phi\left[\left(\nabla_{e_j} F^\phi\right)(e_j) + \left(\nabla_{F^\phi e_j} F^\phi\right)(F^\phi e_j)\right] - (m-k)(F^\phi)^2(\eta) \\ &= \sum_{j=1}^{k/2} F^\phi\left[\left(\nabla_{e_j} F^\phi\right)(e_j) + \left(\nabla_{F^\phi e_j} F^\phi\right)(F^\phi e_j)\right] + (m-k)\eta,\end{aligned}$$

since $\eta \in \ker\left[(F^\phi)^2 + I\right]$. □

From Lemma 1, we deduce:

Proposition 1. *An f-structure F is cosymplectic if and only if $F\delta F = (m-k)\eta$, or equally, $F\delta_H F = 0$.*

A fairly direct implication of Theorem 1 is:

Proposition 2. *Let $\phi : (M, g) \to (N, h, J)$ be a (PHWC) map from a Riemannian to a Kähler manifold. Then ϕ is harmonic if and only if $F^\phi \delta F^\phi = 0$.*

Proof. From Theorem 1, it is clear that $F^\phi \delta F^\phi = 0$ implies the harmonicity of ϕ.
Conversely, $F^\phi \delta F^\phi$ is in $\ker [(F^\phi)^2 + I]$, since $(F^\phi)^3 + F^\phi = 0$ and, if $d\phi(F^\phi \delta F^\phi) = 0$, then $F^\phi \delta F^\phi \in \ker d\phi = \ker F^\phi$ (by construction of F^ϕ).
Therefore $F^\phi \delta F^\phi \in \ker [(F^\phi)^2 + I] \cap \ker F^\phi = \{0\}$. □

Remark 2. Note that the case of ±holomorphic maps between Hermitian manifolds is dissimilar to the situation at hand, since an almost complex structure has a trivial kernel while holomorphic maps need not (if the dimensions are different).

Definition 6. *A (PHWC) map is called a pseudo harmonic morphism if it is harmonic.*

We generalise a theorem of Baird-Eells [2] to pseudo harmonic morphisms (cf. Example 2 and Proposition 6).

Theorem 2. *Let $\phi : (M, g) \to (N, h, J)$ be a non-constant (PHWC) submersion, then:*

1. *If $\dim N = 2$ then ϕ is a pseudo harmonic morphism if and only if its fibres are minimal.*
2. *If $\dim N > 2$ then any two of the following conditions imply the third*
   (a) *ϕ is a pseudo harmonic morphism.*
   (b) *ϕ has minimal fibres.*
   (c) *F^ϕ is cosymplectic.*

Proof. 1) If $\dim N = 2$, the notions of (PHWC) and horizontally weakly conformal (see Definition 9) coincide and we are exactly in the situation of [2].
2) If $\dim N > 2$, the statement is a direct consequence of Proposition 1 and Proposition 2. □

5. Homotopy invariant

Definition 7. *Let $\phi : (M, g, F) \to (N, h, \tilde{F})$ be a smooth map between Riemannian manifolds equipped with f-structures, the domain being compact.*
Consider a component of the differential $d\phi$:

$$(d\phi)^+ : T^{\mathbb{C}}M \to T^+N$$

As for holomorphic maps, we can define the partial energy densities:

$$e^+(\phi) = \frac{1}{2}\sum_{j=1}^{k/2} h\left((d\phi)^+(Z_j), \overline{(d\phi)^+(Z_j)}\right)$$

and

$$e^-(\phi) = \frac{1}{2}\sum_{j=1}^{k/2} h\left((d\phi)^+(\overline{Z_j}), \overline{(d\phi)^+(\overline{Z_j})}\right).$$

Let

$$E^+(\phi) = \int_M e^+(\phi)\, v_g \qquad \textit{and} \qquad E^-(\phi) = \int_M e^-(\phi)\, v_g$$

and put:

$$K(\phi) = E^+(\phi) - E^-(\phi).$$

The next proposition is a slight extension of results due to Rawnsley [13] and Burstall [3], the initial ideas are to be found in Lichnerowicz [10]:

Proposition 3. *Let (M, g, F) be a compact Riemannian manifold equipped with an f-structure F such that $d^*\Omega^M = 0$ and $(N, h, \tilde{F})$ a Riemannian manifold with an f-structure satisfying $d\Omega^N = 0$. Then $K(\phi) = E^+(\phi) - E^-(\phi)$ is a homotopy invariant.*

We omit the proof as it is only an adaptation of the methods in [13, 3].

Consequently, in the spirit of [3, Lemma 3.3], we have:

Corollary 1. *Let ϕ be a map from (M, g, F), a compact Riemannian manifold equipped with an f-structure F such that $d^*\Omega^M = 0$, into $(N, h, \tilde{F})$, a Riemannian manifold with an f-structure satisfying $d\Omega^N = 0$, then:*

1. *If ϕ is $\pm f$-holomorphic then ϕ is harmonic and minimises the energy in its homotopy class.*
2. *If ϕ minimises the energy in its homotopy class and is homotopic to a $\pm f$-holomorphic map then ϕ is $\pm f$-holomorphic.*
3. *Let ϕ be a $\pm f$-holomorphic map and ϕ_t a smooth variation of ϕ through harmonic maps, then each ϕ_t is $\pm f$-holomorphic.*
4. *Let K vanish on some homotopy class H. Then any $\pm f$-holomorphic map in H is constant. In particular, any homotopically trivial $\pm f$-holomorphic map is constant.*
5. *Let ϕ_1 be f-holomorphic and ϕ_2 be f-antiholomorphic, then if ϕ_1 and ϕ_2 are homotopic, they are both constant.*

Proposition 4. *Let (M, g) be a closed homology sphere, of dimension at least three, and (N, h, J) a Kähler manifold. Let $\phi : (M, g) \to (N, h, J)$ be a (PHWC) map. If the fundamental 2-form Ω of F^ϕ is co-closed then ϕ is constant.*

Proof. For a homology sphere $H^2(M, \mathbb{R}) = 0$. For any map $\phi : (M,\, g) \to (N,\, h,\, J)$ into a Kähler manifold, let σ be the fundamental form of N. Then $\bar{\sigma} = \sigma$ and

$$d(\phi^*\sigma) = \phi^* d\sigma = 0$$

since $(N,\, h,\, J)$ is Kähler. It follows that

$$[\phi^*\sigma] \in H^2(M, \mathbb{R}) = 0$$

i.e. $\phi^*\sigma$ is exact. Hence we have

$$\begin{aligned} K(\phi) &= \int_M < \Omega,\, \phi^*\sigma > \, v_g \\ &= \int_M < \Omega,\, d\alpha > \, v_g \\ &= \int_M < d^*\Omega,\, \alpha > \, v_g \\ &= 0. \end{aligned}$$

□

Example 1. Consider the Hopf map

$$\begin{aligned} H : \mathbb{S}^3 \subset \mathbb{C}^2 &\to \mathbb{S}^2 \subset \mathbb{C} \times \mathbb{R} \\ (z, w) &\mapsto (2zw, |z|^2 - |w|^2). \end{aligned}$$

It is well-known to be a harmonic morphism with dilation 2. In particular it is horizontally conformal and therefore pseudo horizontally weakly conformal.
However by Proposition 4, the fundamental 2-form of the associated f-structure cannot be co-closed. We can check this fact directly by the following computation: Let $e_1, e_{\bar{1}}$ be a frame of $T^+M \oplus T^-M$ and e_0 a vector spanning the distribution T^0M. They can be chosen to be orthonormal and so that the associated f-structure F acts on these vectors in the following manner

$$Fe_1 = ie_1\,, \quad Fe_{\bar{1}} = -ie_{\bar{1}}\,, \quad Fe_0 = 0.$$

Then

$$\begin{aligned} d^*\Omega(e_1) &= -(\nabla_{e_1}\Omega)(e_{\bar{1}}, e_1) - (\nabla_{e_{\bar{1}}}\Omega)(e_1, e_1) - (\nabla_{e_0}\Omega)(e_0, e_1) \\ &= g(\nabla_{e_1}e_{\bar{1}}, Fe_1) + g(e_{\bar{1}}, F\nabla_{e_1}e_1) + g(\nabla_{e_{\bar{1}}}e_1, Fe_1) + g(e_1, F\nabla_{e_{\bar{1}}}e_1) \\ &+ g(\nabla_{e_0}e_0, Fe_1) + g(e_0, F\nabla_{e_0}e_1) \\ &= i\left(g(e_1, \nabla_{e_1}e_{\bar{1}}) + g(e_{\bar{1}}, \nabla_{e_1}e_1) + g(\nabla_{e_{\bar{1}}}e_1, e_1) - g(e_1, \nabla_{e_{\bar{1}}}e_1) + g(\nabla_{e_0}e_0, e_1)\right) \\ &= ig(\nabla_{e_0}e_0, e_1) \\ &= 0, \end{aligned}$$

since the fibres are totally geodesic. We obtain a similar expression for $d^*\Omega(e_{\bar{1}})$. The last term to compute is:

$$\begin{aligned} d^*\Omega(e_0) &= -(\nabla_{e_1}\Omega)(e_{\bar{1}}, e_0) - (\nabla_{e_{\bar{1}}}\Omega)(e_1, e_0) - (\nabla_{e_0}\Omega)(e_0, e_0) \\ &= -\left(g(\nabla_{e_1}e_0, Fe_{\bar{1}}) + g(Fe_1, \nabla_{e_{\bar{1}}}e_0)\right) \\ &= -i\left(g(e_0, \nabla_{e_1}e_{\bar{1}}) - g(e_0, \nabla_{e_{\bar{1}}}e_1)\right) \\ &= -ig(e_0, [e_1, e_{\bar{1}}]) \\ &= -2iA_{e_1}e_{\bar{1}}, \end{aligned}$$

where A is the O'Neill tensor (cf. [12]), which vanishes if and only if the horizontal distribution is integrable, which is not the case for the Hopf map. Therefore $d^*\Omega(e_0) \neq 0$ and the f-structure associated to the Hopf map is not co-closed.
On the other hand, it can be verified that the associated f-structure satisfies $d\Omega = 0$. There is in fact only one term to compute:

$$\begin{aligned} d\Omega(e_1, e_{\bar{1}}, e_0) &= \nabla_{e_1}(\Omega(e_{\bar{1}}, e_0)) - \nabla_{e_{\bar{1}}}(\Omega(e_1, e_0)) + \nabla_{e_0}(\Omega(e_1, e_{\bar{1}})) \\ &\quad - \Omega([e_1, e_{\bar{1}}], e_0) + \Omega([e_1, e_0], e_{\bar{1}}) - \Omega([e_{\bar{1}}, e_0], e_1) \\ &= g(\nabla_{e_1} e_0 - \nabla_{e_0} e_1, -ie_{\bar{1}}) - g(\nabla_{e_{\bar{1}}} e_0 - \nabla_{e_0} e_{\bar{1}}, ie_1) \\ &= ig(e_0, \nabla_{e_1} e_{\bar{1}} + \nabla_{e_{\bar{1}}} e_1) \\ &= ig(e_0, A_{e_1} e_{\bar{1}} + A_{e_{\bar{1}}} e_1) \\ &= 0, \end{aligned}$$

since the O'Neill tensor A is anti-symmetric.
It is rather surprising that the associated f-structure is closed but not co-closed, especially for such a low dimension 3. This situation is very different from the case of almost complex structures on almost Hermitian manifolds, for which "closed" always implies "co-closed" (cf. [5]). As to the target, the almost Hermitian structure being Kähler, we have $d^*\omega = d\omega = 0$.

6. Pseudo horizontally homothetic map

Definition 8. [1] *A (PHWC) map $\phi : (M, g) \to (N, h, J)$ is called pseudo horizontally homothetic (PHH) if*

$$d\phi\,(\nabla_X (d\phi)^*(JY)) = J d\phi\,(\nabla_X (d\phi)^*(Y))\,, \tag{9}$$

for $X \in H = \ker\,[(F^\phi)^2 + I] \subset TM$ and $Y \in TN$.

Remark 3. Maps which satisfy equation (9) for all $X \in TM$ are called *strongly pseudo horizontally homothetic.*
When (M, g) is Kähler, $\pm$holomorphic maps are strongly (PHH).

A (PHH) map, as a (PHWC) map, admits an associated f-structure F^ϕ. (PHH) maps are used in [1], in particular, to construct minimal submanifolds as the inverse image of complex submanifolds of Kähler manifolds by (PHH) harmonic submersions. We will extend this result to (PHWC) maps later on.

Proposition 5. *Let $\phi : (M, g) \to (N, h, J)$ be a pseudo horizontally homothetic map, then its associated f-structure F^ϕ is $(1, 2)$-symplectic.*

Proof. Let $X \in H$ and $Y \in TN$, then:

$$\begin{aligned} \nabla F^\phi(X, (d\phi)^*(Y)) &= \nabla_X F^\phi((d\phi)^*(Y)) - F^\phi(\nabla_X (d\phi)^*(Y)) \\ &= \nabla_X((d\phi)^*(JY)) - F^\phi(\nabla_X (d\phi)^*(Y)). \end{aligned}$$

Therefore

$$\begin{aligned} d\phi\left(\nabla F^{\phi}(X,(d\phi)^{*}(Y))\right) &= d\phi\left(\nabla_X((d\phi)^{*}(JY))\right) - d\phi\left(F^{\phi}(\nabla_X(d\phi)^{*}(Y))\right) \\ &= Jd\phi\left(\nabla_X((d\phi)^{*}(Y))\right) - d\phi\left(F^{\phi}(\nabla_X(d\phi)^{*}(Y))\right) \\ &= 0, \end{aligned}$$

as ϕ is f-holomorphic. ☐

Remark 4. A by-product of Proposition 5 is that $(\nabla_X F^{\phi})^H = 0$ for all $X \in H$, which, as was noted in [1], is a partial Kähler condition on the horizontal bundle.

Definition 9. *A map $\phi : (M,g) \to (N,h)$ between Riemannian manifolds is called horizontally weakly conformal if, at a regular point x, i.e. at which $d\phi_x \neq 0$,*

$$d\phi_x : (\ker d\phi_x)^{\perp} \to T_{\phi(x)}N$$

is surjective and conformal with some conformal factor $\lambda(x)$. When ϕ is also submersive, we drop the adverb "weakly". If the function $\nabla\lambda^2$ is vertical, i.e. $d\phi(\nabla\lambda^2) = 0$, we call ϕ horizontally homothetic.

Proposition 6. *Let $\phi : (M^m, g) \to (N^n, h, J) \quad (n \geq 3)$ be a horizontally conformal map. Then ϕ is horizontally homothetic if and only if its associated f-structure F^{ϕ} is cosymplectic.*

Proof. Recall that for a (PHWC) map:

$$\tau(\phi) = -d\phi(F^{\phi}\delta F^{\phi}).$$

On the other hand [8]:

$$\tau(\phi) = d\phi\left((1-\frac{n}{2})\nabla\ln\lambda^2 - (m-n)\eta\right).$$

Hence, at a regular point:

$$F^{\phi}\delta F^{\phi} + (1-\frac{n}{2})(\nabla\ln\lambda^2)^H = (m-n)\eta,$$

or equivalently

$$F^{\phi}\delta_H F^{\phi} = -(1-\frac{n}{2})(\nabla\ln\lambda^2)^H,$$

where $(\nabla\ln\lambda^2)^H$ denotes the horizontal part of $\nabla\ln\lambda^2$, remarking that $F^{\phi}\delta F^{\phi} \in H = \ker\left[(F^{\phi})^2 + I\right]$ and $\eta \in H$ (by definition).
Thus, if $n > 2$,

$$\nabla\ln\lambda^2 \in V$$

if and only if

$$F^{\phi}\delta F^{\phi} - (m-n)\eta = 0,$$

i.e. F^{ϕ} is cosymplectic. ☐

We can now generalise [1, Theorem 4.1] and construct minimal submanifolds via pseudo harmonic morphisms.

Proposition 7. *Let $\phi : (M, g) \to (N, h, J)$ be a non-constant submersive pseudo harmonic morphism from a Riemannian to a Kähler manifold with a $(1, 2)$-symplectic associated f-structure. Then, if P is a complex submanifold of N, $\phi^{-1}(P)$ is a minimal submanifold of M.*

Proof. We follow [1]. Let $K = \phi^{-1}(P)$ and H_1, H_2 such that:

$$TK = H_1 \oplus V, \quad H = H_1 \oplus H_2,$$

this orthogonal decomposition is possible, as K is made up of fibres. Remark that, at a point $x \in K$:

$$H_{1,x} = \left\{ v \in H_x \,|\, d\phi_x(v) \in T_{\phi(x)}P \right\}.$$

One can easily verify that H_1 is an F^ϕ-invariant subbundle, due to the f-holomorphicity of ϕ and the complexity of P. Thus we can choose an orthogonal frame field

$$\left\{ e_1, \ldots, e_p, F^\phi e_1, \ldots, F^\phi e_p, e_{n+1}, \ldots, e_m \right\}$$

for TK, adapted to the decomposition $TK = H_1 + V$. In this frame, the submanifold K is minimal if

$$\sum_{\alpha=n+1}^{m} \nabla_{e_\alpha} e_\alpha + \sum_{i=1}^{p} \left(\nabla_{e_i} e_i + \nabla_{F^\phi e_i} F^\phi e_i \right) \in TK.$$

From Theorem 2, we know the fibres of ϕ to be minimal, i.e.,

$$\sum_{\alpha=n+1}^{m} \nabla_{e_\alpha} e_\alpha \in V.$$

On the other hand

$$\left(\nabla_{e_i} F^\phi \right)(e_i) + \left(\nabla_{F^\phi e_i} F^\phi \right)(F^\phi e_i) = [e_i, F^\phi e_i] - F^\phi \left(\nabla_{e_i} e_i + \nabla_{F^\phi e_i} F^\phi e_i \right)$$

and, as F^ϕ is (1,2)-symplectic,

$$F^\phi \left[\left(\nabla_{e_j} F^\phi \right)(e_j) + \left(\nabla_{F^\phi e_j} F^\phi \right)(F^\phi e_j) \right] = 0.$$

Finally

$$\begin{aligned}
&\left[\nabla_{e_i} e_i + \nabla_{F^\phi e_i} F^\phi e_i \right]^H \\
&= \frac{-1}{2} F^\phi (F^\phi + i) \left[\nabla_{e_i} e_i + \nabla_{F^\phi e_i} F^\phi e_i \right] - \frac{1}{2} F^\phi (F^\phi - i) \left[\nabla_{e_i} e_i + \nabla_{F^\phi e_i} F^\phi e_i \right] \\
&= -(F^\phi)^2 \left[\nabla_{e_i} e_i + \nabla_{F^\phi e_i} F^\phi e_i \right] \\
&= F^\phi \left[F^\phi e_i, e_i \right],
\end{aligned}$$

since H_1 is F^ϕ-invariant and, as P is a complex submanifold, $\left[F^\phi e_i, e_i \right]^H \in H_1$, so that:

$$\left[\nabla_{e_i} e_i + \nabla_{F^\phi e_i} F^\phi e_i \right]^H = F^\phi \left[F^\phi e_i, e_i \right] = \left(F^\phi \left[F^\phi e_i, e_i \right] \right)^H \subset H_1,$$

and

$$\left[\nabla_{e_i} e_i + \nabla_{F^\phi e_i} F^\phi e_i\right]^H = \left[\nabla_{e_i} e_i + \nabla_{F^\phi e_i} F^\phi e_i\right]^{H_1},$$

and the submanifold K is minimal. □

As is clear in the proof, the "(1,2)-symplectic" condition could be replaced by the weaker condition:

$$F^\phi[(\nabla_X F^\phi)(X) + (\nabla_{F^\phi X} F^\phi)(F^\phi X)] = 0, \tag{10}$$

for $\forall X \in \ker[(F^\phi)^2 + I]$ and $\|X\| = 1$. In fact it is easy to see that F^ϕ is (1,2)-symplectic implies equation (10), which in turn implies that F^ϕ is cosymplectic. For a horizontally weakly conformal (PHWC) map, all these conditions are equivalent to horizontal homothety.

We close this section with a method of constructing (PHWC) maps (or pseudo-harmonic morphisms) which satisfy equation (10) but are not horizontally homothetic.

Proposition 8. *Let $\phi : (M,\, g) \to (\mathbb{C},\, can)$ be a non-constant horizontally weakly conformal map. Then $\Phi = (\phi, \cdots, \phi) : (M,\, g) \to \mathbb{C}^r$ $(r > 1)$ is a (PHWC) map satisfying* (10) *which is not horizontally homothetic.*

Proof. It is easy to verify that

$$\sum_{i,j=1}^m g^{ij} \frac{\partial \Phi^\alpha}{\partial x^i} \frac{\partial \Phi^\beta}{\partial x^j} = \sum_{i,j=1}^m g^{ij} \frac{\partial \phi}{\partial x^i} \frac{\partial \phi}{\partial x^j} = 0$$

i.e. Φ is a (PHWC) map. By Lemma 1 and the proof of Proposition 6, at a regular point,

$$F^\phi \delta_H F^\phi = F^\phi \delta F^\phi - (m-2)\eta = (1 - \frac{2}{2})(\nabla \ln \lambda^2)^H = 0.$$

On the other hand, it is easy to show that

$$H^\Phi = H^\phi; \quad F^\Phi = F^\phi.$$

Hence

$$F^\Phi \delta_H F^\Phi = F^\phi \delta_H F^\phi = 0.$$

Notice that $dim H^\Phi = 2$. We see that Φ satisfies equation (10), since for $X \in \ker[(F^\Phi)^2 + I]$ with $\|X\| = 1$, $\left\{X,\, F^\Phi X\right\}$ form an orthonormal basis and:

$$F^\Phi[(\nabla_X F^\Phi)(X) + (\nabla_{F^\Phi X} F^\Phi)(F^\Phi X)] = F^\Phi \delta_H F^\Phi = 0.$$

However when $d\Phi_x \neq 0$

$$d\Phi_x : H^\Phi = (Kerd\Phi_x)^\perp \to T_{\Phi(x)}\mathbb{C}^r$$

is not surjective. Hence Φ is not a horizontally weakly conformal map, in particular, it is not horizontally homothetic. □

Example 2. Let $\psi : \mathbb{R}^m \to \mathbb{C}$ be defined by

$$\psi(x_1, \ldots, x_m) = \begin{cases} \sum_{j=1}^{r} x_j + i \sum_{j=r+1}^{2r} x_j & \text{if} \quad m = 2r \\ \sum_{j=1}^{r} x_j + i\sqrt{\frac{r}{r-1}} \sum_{j=r+1}^{2r-1} x_j & \text{if} \quad m = 2r-1. \end{cases}$$

It is easy to verify that

$$\sum_{i=1}^{m} \frac{\partial^2 \psi}{\partial x_i^2} = 0,$$

and

$$\sum_{i=1}^{m} \left(\frac{\partial \psi}{\partial x_i} \right)^2 = 0.$$

So ψ is a harmonic morphism.
Let $\mathbb{H}^{m+1} = (\mathbb{R}^m \times \mathbb{R}^{\perp},\, (1/x_{m+1}^2) <,>_{\mathbb{R}^{m+1}})$ and $\pi : \mathbb{H}^{m+1} \to \mathbb{R}^m$ be the projection onto $\mathbb{R}^m$ followed by a homothety, given by $\pi(p,\, x) \to \alpha \cdot p$, where $\alpha \in \mathbb{R} - \{0\}$. Then π is a horizontally homothetic harmonic morphism with totally geodesic fibres. Hence $\psi \circ \pi : \mathbb{H}^{m+1} \to \mathbb{C}$ is a harmonic morphism. Using Proposition 8, we have that, for arbitrary $m \geq 1$ and $n > 1$, there exist pseudo-harmonic morphisms $\phi_1 : \mathbb{R}^m \to \mathbb{C}^n$ and $\phi_2 : \mathbb{H}^{m+1} \to \mathbb{C}^n$ satisfying (10) which are not horizontally homothetic.

7. Dual stress-energy tensor

7.1. Description of the problem

Let (M, g) be a Riemannian manifold, (N, J, h) a Kähler manifold and $\phi : (M, g) \to (N, J, h)$ a pseudo horizontally weakly conformal map.
Consider the section of the pull-back bundle (by the dual of $d\phi$) of $T^*N \otimes T^*N$

$$S_* = \frac{2e_\phi}{n} h^* - (d\phi^*) g^*,$$

called the *dual stress-energy tensor*, where g^* and h^* are the metrics dual to g and h, and $(d\phi^*)g^*$ is the pull-back of g^* by the dual of $d\phi$.
In local coordinates $(x^i)_{i=1,\ldots,m}$ on M and $(z^\alpha)_{\alpha=1,\ldots,n}$ on N, S_* takes the form:

$$S_*^{AB} = \frac{2e_\phi}{n} h^{AB} - \frac{\partial \phi^A}{\partial x^i} \frac{\partial \phi^B}{\partial x^j} g^{ij} \quad \forall A, B = 1, \overline{1}, \ldots, n, \overline{n}.$$

As the metric h is Hermitian, i.e. $h^{\alpha\beta} = 0 \quad \forall \alpha, \beta = 1, \ldots, n$, and ϕ is pseudo horizontally weakly conformal, the (2,0)-part of S_*,

$$S_*^{\alpha\beta} = \frac{2e_\phi}{n} h^{\alpha\beta} - \frac{\partial \phi^\alpha}{\partial x^i} \frac{\partial \phi^\beta}{\partial x^j} g^{ij},$$

vanishes for all $\alpha, \beta = 1, \ldots, n$.
On the other hand, it is easy to see that ϕ is horizontally weakly conformal if and only if the (1,1)-part of S_* is zero, that is:

$$S_*^{\alpha\overline{\beta}} = \frac{2e_\phi}{n} h^{\alpha\overline{\beta}} - \frac{\partial\phi^\alpha}{\partial x^i}\frac{\partial\phi^{\overline{\beta}}}{\partial x^j} g^{ij} = 0 \quad \forall\alpha, \beta = 1, \ldots, n.$$

Observe that

$$e_\phi = \frac{1}{2} g^{ij} \frac{\partial\phi^A}{\partial x^i}\frac{\partial\phi^B}{\partial x^j} h_{AB} = g^{ij}\frac{\partial\phi^\alpha}{\partial x^i}\frac{\partial\phi^{\overline{\beta}}}{\partial x^j} h_{\alpha\overline{\beta}},$$

therefore $\operatorname{trace}_h S_* = 0$.

If ϕ is a pseudo horizontally weakly conformal map, then the (1,1)-part of S_* measures how far ϕ is from being horizontally weakly conformal.

Our aim is to find conditions such that this "obstruction" is a covariant constant 2-tensor. This is equivalent to:

$$S_*^{\alpha\overline{\beta}} = C^{\alpha\overline{\beta}},$$

with the $C^{\alpha\overline{\beta}}$'s covariant constant, and the condition we will study is:

$$\nabla_{e_k}(S_*^{\alpha\overline{\beta}}) = 0.$$

We choose a frame (e_k) adapted to the f-structure F^ϕ on TM and normal complex coordinates $(z^\alpha)_{\alpha=1,\ldots,n}$ on N (which we can use as (N, J, h) is Kähler).

7.2. Computation of $\nabla_{e_k}(S_*^{\alpha\overline{\beta}})$

All computations are taken at a chosen point $p \in M$ and, in particular, all Christoffel symbols of the target manifold N vanish at the point $\phi(p)$.

$$\begin{aligned}
\nabla_{e_k}(S_*^{\alpha\overline{\beta}}) &= \nabla_{e_k}(S_*)(dz^\alpha, dz^{\overline{\beta}}) + S_*(\nabla_{e_k} dz^\alpha, dz^{\overline{\beta}}) + S_*(dz^\alpha, \nabla_{e_k} dz^{\overline{\beta}}) \\
&= \nabla_{e_k}(S_*)(dz^\alpha, dz^{\overline{\beta}}) \\
&= \frac{2}{n}\nabla_{e_k}(e_\phi) h^{\alpha\overline{\beta}} + \frac{e_\phi}{n}\nabla_{e_k}(h^{\alpha\overline{\beta}}) - (\nabla_{e_k}(d\phi^*)g)\,(dz^\alpha, dz^{\overline{\beta}}) \\
&= \frac{2}{n}\nabla_{e_k}(e_\phi) h^{\alpha\overline{\beta}} - (\nabla_{e_k}(d\phi^*)g)\,(dz^\alpha, dz^{\overline{\beta}})
\end{aligned}$$

The energy density e_ϕ can be seen as the trace, with respect to the metric g of ϕ^*h, or, by duality, as the trace, w.r.t. the metric h, of $(d\phi^*)g$, i.e. $e_\phi = \frac{1}{2}\operatorname{trace}_h (d\phi^*)g$. Therefore

$$\begin{aligned}
\nabla_{e_k}(e_\phi) &= \operatorname{trace}_h \nabla_{e_k}(d\phi^*)g \\
&= h_{\delta\overline{\gamma}}\,(\nabla_{e_k}(d\phi^*)g)\,(dz^\delta, dz^{\overline{\gamma}}).
\end{aligned}$$

Since it appears twice, we compute $(\nabla_{e_k}(d\phi^*)g)\,(dz^\alpha, dz^{\overline{\beta}})$ separately:

$$
\begin{aligned}
&(\nabla_{e_k}(d\phi^*)g)\,(dz^\alpha, dz^{\overline{\beta}}) = \\
&(\nabla_{e_k} g)\left(d\phi^*(dz^\alpha), d\phi^*(dz^{\overline{\beta}})\right) + g\left((\nabla_{e_k} d\phi^*)(dz^\alpha), d\phi^*(dz^{\overline{\beta}})\right) + g\left(d\phi^*(dz^\alpha), (\nabla_{e_k} d\phi^*)(dz^{\overline{\beta}})\right) \\
&= g\left((\nabla_{e_k} d\phi^*)(dz^\alpha), d\phi^*(dz^{\overline{\beta}})\right) + g\left(d\phi^*(dz^\alpha), (\nabla_{e_k} d\phi^*)(dz^{\overline{\beta}})\right) \\
&= g\left(\nabla_{e_k}(d\phi^*(dz^\alpha)) - d\phi^*\left(\nabla_{d\phi(e_k)} dz^\alpha\right), d\phi^*(dz^{\overline{\beta}})\right) \\
&+ g\left(d\phi^*(dz^\alpha), \nabla_{e_k}(d\phi^*(dz^{\overline{\beta}})) - d\phi^*\left(\nabla_{d\phi(e_k)} dz^{\overline{\beta}}\right)\right) \\
&= g\left(\nabla_{e_k}(d\phi^*(dz^\alpha)), d\phi^*(dz^{\overline{\beta}})\right) + g\left(d\phi^*(dz^\alpha), \nabla_{e_k}(d\phi^*(dz^{\overline{\beta}}))\right),
\end{aligned}
$$

since $\nabla_{d\phi(e_k)} dz^\alpha = -\phi_k^B \Gamma_{BC}^\alpha dz^C = 0$ at the point $\phi(p)$. Similarly $\nabla_{d\phi(e_k)} dz^{\overline{\beta}} = 0$. We conclude that:

$$
\begin{aligned}
\nabla_{e_k}(S_*^{\alpha\overline{\beta}}) = \frac{1}{n} h_{\delta\overline{\gamma}} &\left[g\left(\nabla_{e_k}(d\phi^*(dz^\delta)), d\phi^*(dz^{\overline{\gamma}})\right) + g\left(d\phi^*(dz^\delta), \nabla_{e_k}(d\phi^*(dz^{\overline{\gamma}}))\right)\right] h^{\alpha\overline{\beta}} \\
&- g\left(\nabla_{e_k}(d\phi^*(dz^\alpha)), d\phi^*(dz^{\overline{\beta}})\right) - g\left(d\phi^*(dz^\alpha), \nabla_{e_k}(d\phi^*(dz^{\overline{\beta}}))\right).
\end{aligned}
$$

7.3. Conditions such that $\nabla_{e_k}(S_*^{\alpha\overline{\beta}}) = 0$

The (PHH) condition implies that:

$$
d\phi(\nabla_X F^\phi Z) = d\phi(F^\phi \nabla_X Z) = J d\phi(\nabla_X Z), \quad \forall X \in H, Z \in T^+M,
$$

therefore, if $X \in H$, ∇_X maps sections of T^+M onto sections of $T^+M \oplus T^0M$ so:

$$
g(\nabla_{e_k}(d\phi^*(dz^\alpha)),\, d\phi^*(dz^{\bar{\beta}})) = 0.
$$

Similarly, one can deduce that $\nabla_{e_k} S_*^{\overline{\alpha}\beta} = 0$. This shows that the dual stress-energy tensor of a pseudo horizontally homothetic map is horizontally covariant constant, i.e. it is close to being a horizontally weakly conformal map.

Proposition 9. *Let (M, g) be a Riemannian manifold and (N, J, h) a Kähler manifold. If $\phi : (M, g) \to (N, J, h)$ a pseudo horizontally homothetic map then*

$$
\nabla_{e_k} S_* = 0 \quad \forall e_k \in H = \ker\left[(F^\phi)^2 + I\right],
$$

i.e. the dual stress-energy tensor is horizontally covariant constant. If ϕ is strongly pseudo horizontally homothetic then S_* is covariant constant.

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