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A Note on the Existence of $\{k, k\}$-equivelar Polyhedral Maps

Basudeb Datta

Department of Mathematics, Indian Institute of Science, Bangalore 560 012, India
e-mail: dattab@math.iisc.ernet.in

Abstract. A polyhedral map is called $\{p, q\}$-equivelar if each face has p edges and each vertex belongs to q faces. In [12], it was shown that there exist infinitely many geometrically realizable $\{p, q\}$-equivelar polyhedral maps if $q > p = 4$, $p > q = 4$ or $q - 3 > p = 3$. It was shown in [6] that there exist infinitely many $\{4, 4\}$- and $\{3, 6\}$-equivelar polyhedral maps. In [1], it was shown that $\{5, 5\}$- and $\{6, 6\}$-equivelar polyhedral maps exist. In this note, examples are constructed, to show that infinitely many self dual $\{k, k\}$-equivelar polyhedral maps exist for each $k \geq 5$. Also vertex-minimal non-singular $\{p, p\}$-patterns are constructed for all odd primes p.

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1. Introduction and results

A *polyhedral complex* (of dimension 2) is collection of cycles (finite connected 2-regular graphs) together with the edges and the vertices in the cycles such that the intersection of any two cycles is empty, a vertex or an edge. The cycles are called the *faces* of the polyhedral complex. For a polyhedral complex K, $V(K)$ denotes its vertex-set and $\mathrm{EG}(K)$ denotes its edge-graph or 1-skeleton. We say K *finite* if $V(K)$ is finite. If $\mathrm{EG}(K)$ is connected then K is said to be *connected*.

A polyhedral complex is called a *polyhedral* 2*-manifold* (or an *abstract polyhedron*) if for each vertex v the faces containing v are of the form $F_1, \dots, F_m$, where $F_1 \cap F_2, \dots, F_{m-1} \cap$

$F_m, F_m \cap F_1$ are edges for some $m \geq 3$. A connected finite polyhedral 2-manifold is called a *polyhedral map*. A *combinatorial 2-manifold* is a polyhedral 2-manifold whose faces are 3-cycles. A polyhedral map is called *$\{p,q\}$-equivelar* if each face is a p-cycle and each vertex is in q faces. A polyhedral map is called *equivelar* if it is $\{p,q\}$-equivelar for some p, q (cf. [10, 3, 4, 11]).

To each polyhedral complex K, we associate a pure 2-dimensional simplicial complex $B(K)$ (called the *barycentric subdivision* of K) whose 2-faces are of the form ueF, where (u,e,F) is a flag (i.e., e is an edge of the face F and u is a vertex of e) in K. The geometric carrier of $B(K)$ is called the *geometric carrier* of K and is denoted by $|K|$. Clearly, K is a polyhedral 2-manifold if and only if $B(K)$ is a combinatorial 2-manifold (equivalently, $|K|$ is a 2-manifold). A polyhedral 2-manifold K is called *orientable* if $|K|$ is orientable.

An *isomorphism* between two polyhedral complexes K and L is a bijection $\varphi\colon V(K) \to V(L)$ such that $(v_1,\dots,v_m)$ is a face of K if and only if $(\varphi(v_1),\dots,\varphi(v_m))$ is a face of L. Two complexes are called *isomorphic* if there is an isomorphism between them. We identify two isomorphic polyhedral complexes. An isomorphism from K to itself is called an *automorphism* of K. The set $\Gamma(K)$ of automorphisms of K forms a group. A polyhedral 2-manifold K is called *combinatorially regular* if $\Gamma(K)$ is transitive on flags (cf. [10]).

For a polyhedral 2-manifold K, consider the polyhedral complex $\widetilde{K}$ whose vertices are the faces of K and $(F_1,\dots,F_m)$ is a face of $\widetilde{K}$ if $F_1,\dots,F_m$ have a common vertex and $F_1 \cap F_2, \dots, F_{m-1} \cap F_m, F_m \cap F_1$ are edges. Then $\widetilde{K}$ is a polyhedral 2-manifold and called the *dual* of K. If $\widetilde{K}$ is isomorphic to K then K is called *self dual*.

A *pattern* is an ordered pair (M,G), where M is a connected closed surface in some Euclidean space and G is a finite graph on M such that each component of $M \setminus G$ is simply connected. The closure of each component of $M \setminus G$ is called a *face* of (M,G). For a face F, the closed path (in G) consisting of all the edges and the vertices in F is called the *boundary* of F. A pattern (M,G) is said to be *non-singular* if the boundary of each face is a cycle. A non-singular pattern is said to be a *polyhedral pattern* if the intersection of any two faces is empty, a vertex or an edge. A pattern (M,G) is called a *$\{p,q\}$-pattern* if each face contains p edges and the degree of each vertex in G is q (cf. [7]).

If (M,G) is a polyhedral pattern then clearly the boundaries of the faces of (M,G) form a polyhedral map. Conversely, for a polyhedral map K, let $M = |K|$ and $G = \mathrm{EG}(K)$. Then (M,G) is a polyhedral pattern and the faces of K are the boundaries of the faces of (M,G). This pattern (M,G) is called a *geometric realization* of K. A geometric realization (M,G) (in some $\mathbb{R}^n$) is called *linear* if each face of M is a convex polygon and no two adjacent faces (i.e., faces which share a common edge) lie in the same plane. If a polyhedral map has a linear geometric realization in $\mathbb{R}^3$ then it is called *geometrically realizable*.

If $f_0(K)$, $f_1(K)$ and $f_2(K)$ are the number of vertices, edges and faces respectively of a polyhedral complex K then the number $\chi(K) := f_0(K) - f_1(K) + f_2(K)$ is called the *Euler characteristic* of K. Observe that $\chi(B(K)) = \chi(K)$. If u and v are vertices of a face F and uv is not an edge of F then uv is called a *diagonal*. Clearly, if $d(K)$ is the number of diagonals of a polyhedral complex K then $d(K) + f_1(K) \leq \binom{f_0(K)}{2}$ and in the case of equality each pair of vertices belongs to a face. A polyhedral map K is called a *weakly neighbourly polyhedral map* (in short, *wnp map*) if each pair of vertices belongs to a common face.

We know (cf. [6]) that there exists a unique $\{p,q\}$-equivelar polyhedral map if $(p,q) = (3,3)$,

$(3,4)$ or $(4,3)$ and there are exactly two $\{p,q\}$-equivelar polyhedral maps if $(p,q) = (3,5)$ or $(5,3)$. In [12], McMullen et al. constructed infinitely many geometrically realizable $\{p,q\}$-equivelar polyhedral maps for each $(p,q) \in \{(r,4) : r \geq 5\} \cup \{(4,s) : s \geq 5\} \cup \{(3,k) : k \geq 7\}$. In [6], it was shown that there exist infinitely many $\{4,4\}$- and $\{3,6\}$-equivelar polyhedral maps. It was also shown that there are exactly two neighbourly $\{3,8\}$-equivelar polyhedral maps and there are exactly 14 neighbourly $\{3,9\}$-equivelar polyhedral maps.

In [5], Coxeter constructed a geometrically realizable combinatorially regular infinite polyhedral 2-manifold whose faces are hexagons and each vertex is in six faces (namely, $\{6,6\,|\,3\}$). In [9], Grünbaum constructed another combinatorially regular infinite polyhedral 2-manifold of type $\{6,6\}$ (namely, $\{6,6\}_4$) (cf. [10]). In [8], Gott constructed a geometrically realizable infinite polyhedral 2-manifold whose faces are pentagons and each vertex is in five faces. If K is a $\{p,q\}$-equivelar polyhedral map on n vertices then $d(K) = nq(p-3)/2$ and $f_1(K) = nq/2$. Therefore, if K is an n-vertex $\{p,p\}$-equivelar polyhedral map then $np(p-3)/2 + np/2 \leq n(n-1)/2$ and hence $n \geq (p-1)^2$. Clearly, equality holds if and only if K is a wnp map. Let $\alpha(p)$ denote the smallest n such that there exists an n-vertex $\{p,p\}$-equivelar polyhedral map. Clearly, the 4-vertex 2-sphere (the boundary of a 3-simplex) is the unique $\{3,3\}$-equivelar wnp map. In [1], Brehm proved that there exist exactly three $\{4,4\}$-equivelar wnp maps and constructed the 16-vertex $\{5,5\}$-equivelar polyhedral map $M_{5,16}$ (of Example 1). It was shown in [2] that $M_{5,16}$ is the unique $\{5,5\}$-equivelar polyhedral map on 16 vertices. So, $\alpha(k) = (k-1)^2$ for $k \leq 5$. In [1], Brehm also constructed the 26-vertex $\{6,6\}$-equivelar polyhedral map $M_{6,26}$ (of Example 1). Here we show:

Theorem 1. *For each $m \geq 3$ and $n \geq 0$, there exist a $2(3^{m-1}+2n-1)$-vertex self dual $\{2m-1, 2m-1\}$-equivelar polyhedral map and a (3^m+2n-1)-vertex self dual $\{2m,2m\}$-equivelar polyhedral map.*

Thus $(2m-2)^2 \leq \alpha(2m-1) \leq 2(3^{m-1}-1)$ and $(2m-1)^2 \leq \alpha(2m) \leq 3^m - 1$ for all $m \geq 3$. In [13], using a computer, Nilakantan has shown that there does not exist any 25-vertex $\{6,6\}$-equivelar polyhedral map. So, $\alpha(6) = 26$ and hence there does not exist any $\{6,6\}$-equivelar wnp map. We believe the following is true:

Conjecture 1. *There does not exist any $\{k,k\}$-equivelar wnp map for $k \geq 7$.*

For the existence of an n-vertex $\{k,k\}$-pattern n must be at least $k+1$. Here we show:

Theorem 2. *There exists a $(p+1)$-vertex non-singular $\{p,p\}$-pattern for each odd prime p.*

2. Examples and proofs of the results

We first construct infinitely many $\{k,k\}$-equivelar polyhedral maps. We need these to prove our results. We identify a polyhedral complex with the set of faces in it.

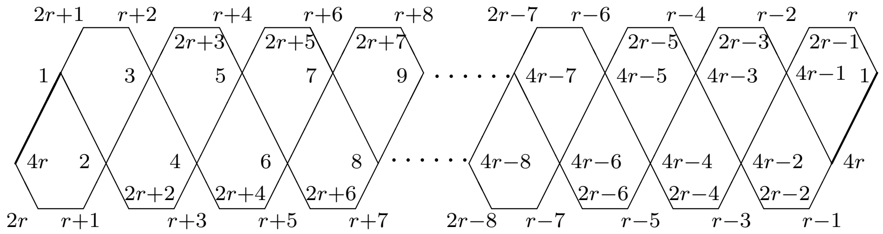
Example 1. For $m \geq 3$ and $n \geq 0$, let

$$\begin{aligned} M_{2m-1,2(3^{m-1}+2n-1)} &= \{F_{i,2m-1} : 1 \leq i \leq 2(3^{m-1}+2n-1)\}, \\ M_{2m,3^m+2n-1} &= \{F_{i,2m} : 1 \leq i \leq 3^m+2n-1\}, \end{aligned}$$

where $b_{2l-1} = 3^{l-1} - 1$, $b_{2l} = 2 \times 3^{l-1} - 1$, for $l \geq 1$ and

$$F_{i,2m-1} = (i+b_1, i+b_2, \dots, i+b_{2m-3}, i+b_{2m-2}+n, i+b_{2m-1}+2n),$$
$$F_{i,2m} = (i+b_1, i+b_2, \dots, i+b_{2m-2}, i+b_{2m-1}, i+b_{2m}+n)$$

are cycles ($(2m-1)$-cycles and $(2m)$-cycles respectively) with vertices from $\mathbb{Z}_{2(3^{m-1}+2n-1)}$ and $\mathbb{Z}_{3^m+2n-1}$ respectively. Clearly, there are $2m$–1 faces through each vertex in $M_{2m-1,2(3^{m-1}+2n-1)}$ and there are $2m$ faces through each vertex in $M_{2m,3^m+2n-1}$. So, $f_1(M_{2m-1,2(3^{m-1}+2n-1)}) = (3^{m-1}+2n-1)(2m-1)$ and $f_1(M_{2m,3^m+2n-1}) = (3^m+2n-1)m$. Thus, $\chi(M_{2m-1,2(3^{m-1}+2n-1)}) = (3^{m-1}+2n-1)(5-2m)$ and $\chi(M_{2m,3^m+2n-1}) = (3^m+2n-1)(2-m)$. By Lemma 2 below, $M_{2m+1,2(3^{m-1}+2n-1)}$ and $M_{2m,3^m+2n-1}$ are polyhedral maps. But, by Lemma 4, none of these polyhedral maps are combinatorially regular.



$M_{5,4r}$

Lemma 1. *For a collection $\mathcal{C}$ of cycles, let $\bar{\mathcal{C}}$ be the 2-dimensional pure simplicial complex whose 2-faces are of the form xyF, where $F \in \mathcal{C}$ and xy is an edge in F. If $B(\mathcal{C})$ is as defined earlier then the following three are equivalent.*

(i) *$B(\mathcal{C})$ is a combinatorial 2-manifold.*

(ii) *$\bar{\mathcal{C}}$ is a combinatorial 2-manifold.*

(iii) *For any vertex v, the cycles containing v are of the form $F_1 = (v, v_{1,1}, \dots, v_{1,n_1}), \dots, F_m = (v, v_{m,1}, \dots, v_{m,n_m})$ such that $v_{1,n_1} = v_{2,1}, \dots, v_{m-1,n_{m-1}} = v_{m,1}$, $v_{m,n_m} = v_{1,1}$ for some $m \geq 2$.*

Proof. Clearly, $B(\mathcal{C})$ is a subdivision of $\bar{\mathcal{C}}$. Therefore, (i) and (ii) are equivalent.

For a 2-dimensional pure simplicial complex X, the link of a vertex v is the graph $\mathrm{lk}_X(v)$ whose vertex-set is $\{u \in V(X) : uv \in X\}$ and edge-set is $\{xy : xyv \in X\}$. Clearly, X is a combinatorial 2-manifold if and only if $\mathrm{lk}_X(v)$ is a cycle for each $v \in V(X)$.

Let v be a vertex of $\bar{\mathcal{C}}$. If $v = F \in \mathcal{C}$ then $\mathrm{lk}_{\bar{\mathcal{C}}}(v)$ is F itself. Let v be a vertex of $\bar{\mathcal{C}}$ which is not a cycle in $\mathcal{C}$. If the cycles containing v are of the form $F_1 = (v, v_{1,1}, \dots, v_{1,n_1}), \dots, F_m = (v, v_{m,1}, \dots, v_{m,n_m})$ such that $v_{1,n_1} = v_{2,1}, \dots, v_{m-1,n_{m-1}} = v_{m,1}$, $v_{m,n_m} = v_{1,1}$ for some $m \geq 2$ then $\mathrm{lk}_{\bar{\mathcal{C}}}(v)$ is the cycle $v_{1,1}F_1v_{2,1}F_2 \cdots v_{m,1}F_m$. Conversely, if $\mathrm{lk}_{\bar{\mathcal{C}}}(v)$ is a cycle then, from the definition of $\bar{\mathcal{C}}$, $\mathrm{lk}_{\bar{\mathcal{C}}}(v)$ must be of the form $v_{1,1}F_1v_{2,1}F_2 \cdots v_{m,1}F_m$, where $F_1 = (v, v_{1,1}, \dots, v_{1,n_1}), \dots, F_m = (v, v_{m,1}, \dots, v_{m,n_m})$ such that $v_{1,n_1} = v_{2,1}, \dots, v_{m-1,n_{m-1}} = v_{m,1}$, $v_{m,n_m} = v_{1,1}$. This proves that (ii) and (iii) are equivalent. □

Lemma 2. *$M_{2m-1,2(3^{m-1}+2n-1)}$ and $M_{2m,3^m+2n-1}$ are polyhedral maps for $m \geq 3$, $n \geq 0$.*

Proof. Since $\{i, i+1\}$ is an edge in $M_{2m-1,2(3^{m-1}+2n-1)}$ for each i, $\mathrm{EG}(M_{2m-1,2(3^{m-1}+2n-1)})$ is connected. Similarly, $\mathrm{EG}(M_{2m,3^m+2n-1})$ is connected.

Observe that the faces in $M_{2m-1,2(3^{m-1}+2n-1)}$ containing i are $F_i, F_{i-b_2}, F_{i-b_3}, F_{i-b_4}, \ldots, F_{i-b_{2m-3}}, F_{i-b_{2m-2}-n}, F_{i-b_{2m-1}-2n}$, where $F_i = F_{i,2m-1} = (i+b_1, i+b_2, \ldots, i+b_{2m-3}, i+b_{2m-2}+n, i+b_{2m-1}+2n)$. Clearly, $F_i \cap F_{i-b_3} = \cdots = F_i \cap F_{i-b_{2m-2}-n} = \cdots = F_{i-b_{2m-1}-2n} \cap F_{i-b_2} = \cdots = F_{i-b_{2m-1}-2n} \cap F_{i-b_{2m-3}} = \{i\}$.

Since $b_{2j+1} = 2b_{2j} - b_{2j-1}$ for all j, $F_{i-b_{2l-1}} \cap F_{i-b_{2l}}$ is the edge $\{i, i+b_{2l}-b_{2l-1}\}$, $F_{i-b_{2l}} \cap F_{i-b_{2l+1}}$ is the edge $\{i+b_{2l}-b_{2l+1}, i\}$ for $1 \le l \le m-2$, $F_{i-b_{2m-3}} \cap F_{i-b_{2m-2}-n}$ is the edge $\{i, i+b_{2m-1}-b_{2m-2}+n\}$ and $F_{i-b_{2m-2}-n} \cap F_{i-b_{2m-1}-2n}$ is the edge $\{i+b_{2m-3}-b_{2m-2}-n, i\}$. Again, since $2b_{2m-1}+4n \equiv 0 \pmod{2(3^{m-1}+2n-1)}$, $F_{i-b_{2m-1}-2n} \cap F_i$ is the edge $\{i, i+b_{2m-1}+2n\}$. Thus, any pair of faces containing i intersects in either at i or on an edge through i and the faces containing i form a single cycle of adjacent faces (sharing a common edge). Therefore, $M_{2m-1,2(3^{m-1}+2n-1)}$ is a polyhedral map.

The faces in $M_{2m,3^m+2n-1}$ containing i are $C_i, C_{i-b_2}, C_{i-b_3}, \ldots, C_{i-b_{2m-1}}, C_{i-b_{2m}-n}$, where $C_i = F_{i,2m} = (i+b_1, i+b_2, \ldots, i+b_{2m-1}, i+b_{2m}+n)$ and $C_i \cap C_{i-b_3} = \cdots = C_i \cap C_{i-b_{2m-1}} = \cdots = C_{i-b_{2m}-n} \cap C_{i-b_2} = \cdots = C_{i-b_{2m}-n} \cap C_{i-b_{2m-2}} = \{i\}$. Also, since $2b_{2m} - b_{2m-1} + 2n \equiv 0 \pmod{3^m+2n-1}$, $C_{i-b_{2l-1}} \cap C_{i-b_{2l}}$ is the edge $\{i, i+b_{2l}-b_{2l-1}\}$, $C_{i-b_{2l}} \cap C_{i-b_{2l+1}}$ is the edge $\{i+b_{2l}-b_{2l+1}, i\}$ for $1 \le l \le m-1$, $C_{i-b_{2m-1}} \cap C_{i-b_{2m}-n}$ is the edge $\{i, i-b_{2m}-n\}$ and $C_{i-b_{2m}-n} \cap C_i$ is the edge $\{i+b_{2m}+n, i\}$. Thus, any pair of faces containing i intersects in either at i or on an edge through i and the faces containing i form a single cycle of adjacent faces. Therefore, $M_{2m,3^m+2n-1}$ is a polyhedral map. □

From the uniqueness of 16-vertex $\{5,5\}$-equivelar polyhedral map it follows that $M_{5,16}$ is self dual. Here we prove.

Lemma 3. *$M_{2m-1,2(3^{m-1}+2n-1)}$ and $M_{2m,3^m+2n-1}$ are self dual for $m \ge 3$ and $n \ge 0$.*

Proof. Let $\varphi\colon M_{2m-1,2(3^{m-1}+2n-1)} \to \widetilde{M}_{2m-1,2(3^{m-1}+2n-1)}$ be the mapping given by $\varphi(i) = F_i := F_{-i,2m-1}$. Clearly φ is a bijection. Consider the face $F_i = (i+b_1, \ldots, i+b_{2m-3}, i+b_{2m-2}+n, i+b_{2m-1}+2n)$. Now, $(\varphi(i+b_1), \ldots, \varphi(i+b_{2m-3}), \varphi(i+b_{2m-2}+n), \varphi(i+b_{2m-1}+2n)) = (F_{-i-b_1}, \ldots, F_{-i-b_{2m-3}}, F_{-i-b_{2m-2}-n}, F_{-i-b_{2m-1}-2n}) = \widetilde{F}_{-i}$ (say). From the proof of Lemma 2, $\widetilde{F}_{-i}$ is a cycle of adjacent faces (sharing a common edge) containing the common vertex $-i$. Therefore, by the definition, $\widetilde{F}_{-i}$ is a face of $\widetilde{M}_{2m-1,2(3^{m-1}+2n-1)}$. This implies that $\widetilde{M}_{2m-1,2(3^{m-1}+2n-1)}$ is isomorphic to $M_{2m-1,2(3^{m-1}+2n-1)}$. Similarly, $\psi\colon M_{2m,3^m+2n-1} \to \widetilde{M}_{2m,3^m+2n-1}$, given by $\psi(i) = F_{-i,2m}$ defines an isomorphism. □

Clearly, $\Gamma(M_{2m-1,2(3^{m-1}+2n-1)})$ and $\Gamma(M_{2m,3^m+2n-1})$ are transitive on the vertices and the faces. Here we prove.

Lemma 4. *$M_{2m-1,2(3^{m-1}+2n-1)}$ and $M_{2m,3^m+2n-1}$ are not combinatorially regular for all $m \ge 3$ and $n \ge 0$.*

Proof. Let $\mu = 2(3^{m-1}+2n-1)$. If $m > 3$ then consider the flags $\mathcal{F}_1 = (0, \{0, b_m - b_{m+1}\}, F_{-b_{m+1}})$ and $\mathcal{F}_2 = (0, \{0, b_{m+2} - b_{m+1}\}, F_{-b_{m+1}})$ in $M_{2m-1,\mu}$. If possible let there exist $\varphi \in \Gamma(M_{2m-1,\mu})$ such that $\varphi(\mathcal{F}_1) = \mathcal{F}_2$. Then $\varphi(0) = 0$, $\varphi(F_{-b_{m+1}}) = F_{-b_{m+1}}$ and hence

$\varphi(1-b_{m+1}) = -b_{m+1}$ and $\varphi(1) = 1$. If $m > 5$ then, by considering the faces containing 1, $\varphi(F_{1-b_{m+2}}) = F_{1-b_{m+2}}$, $\varphi(F_{1-b_{m+1}}) = F_{1-b_{m+3}}$. These imply $1 + b_4 - b_{m+3} = \varphi(1-b_{m+1}) = -b_{m+1}$ in $\mathbb{Z}_\mu$, a contradiction. If $m = 5$ then $\varphi(F_{1-b_6}) = F_{1-b_8-n}$ and hence $1 + b_4 - b_8 - n = \varphi(1-b_6) = -b_6$ in $\mathbb{Z}_\mu$. This is not possible. If $m = 4$ then $\varphi(F_{1-b_5}) = F_{1-b_7-2n}$ and hence $1 + b_4 - b_7 - 2n = \varphi(1 - b_{-5}) = -b_5$ in $\mathbb{Z}_\mu$, a contradiction.

For $m = 3$, if $\psi \in \Gamma(M_{5,\mu})$ such that $\psi((0, \{0, 3+n\}, F_{-b_4-n})) = (0, \{0, 13+3n\}, F_{-b_4-n})$ then $\psi(12+3n) = 11+3n$ and $\psi(F_{1-b_4-n}) = F_1$ and hence $3 = \psi(12+3n) = 11+3n$ in $\mathbb{Z}_\mu$. This is also not possible.

Thus, $M_{2m-1,2(3^{m-1}+2n-1)}$ always has a pair of flags $\mathcal{F}_1$ and $\mathcal{F}_2$ such that $\varphi(\mathcal{F}_1) \neq \mathcal{F}_2$ for all $\varphi \in \Gamma(M_{2m-1,2(3^{m-1}+2n-1)})$. So, $M_{2m-1,2(3^{m-1}+2n-1)}$ is not combinatorially regular.

Let $\eta = 3^m + 2n - 1$ and $C_i = F_{i,2m}$. Consider the flags $\mathcal{C}_1 = (0, \{0, (-1)^m(b_{m+2} - b_{m+1})\}, C_{-b_{m+1}})$ and $\mathcal{C}_2 = (0, \{0, (-1)^m(b_{m+2} - b_{m+1})\}, C_{-b_{m+2}})$ in $M_{2m,\eta}$. If $\varphi \in \Gamma(M_{2m,\eta})$ such that $\varphi(\mathcal{C}_1) = \mathcal{C}_2$, then $\varphi(\mathcal{C}_2) = \mathcal{C}_1$, $\varphi(1-b_{m+2}) = -b_{m+1}$ and $\varphi(1) = 1$. If $m > 3$ then $\varphi(C_{1-b_{m+2}}) = C_{1-b_{m+3}}$ and hence $1+b_2-b_{m+3} = \varphi(1-b_{m+2}) = -b_{m+1}$ in $\mathbb{Z}_\eta$, a contradiction. If $m = 3$ then $\varphi(C_{1-b_5}) = C_{1-b_6-n}$ and hence $1 + b_2 - b_6 + n = \varphi(1-b_5) = -b_4$ in $\mathbb{Z}_\eta$. This is not possible. Therefore, by similar argument as before, $M_{2m,3^m+2n-1}$ is not combinatorially regular. □

Example 2. Let $\mathcal{C}_4$ be the collection of 4-cycles of the complete graph K_5 on the vertex set $\mathbb{Z}_4 \cup \{u\}$ given by $\mathcal{C}_4 = \{(0,1,2,3), (u, i, i+1, i+3) \ : i \in \mathbb{Z}_4\}$. Then $|\bar{\mathcal{C}}_4|$ is the torus and hence $(|\bar{\mathcal{C}}_4|, K_5)$ is a non-singular $\{4,4\}$-pattern.

Lemma 5. *Suppose* $\mathcal{C}(\pi_p) = \{(0, 1, \ldots, p-1), (u, i+\pi_p(1), \ldots, i+\pi_p(p-1)) \ : \ i \in \mathbb{Z}_p\}$ *is a collection of cycles of the complete graph* K_{p+1} *on the vertex set* $\mathbb{Z}_p \cup \{u\}$*, where* p *is an odd prime and* π_p *is a permutation of* $\mathbb{Z}_p \setminus \{0\} = \{1, \ldots, p-1\}$*. If*

(pp1) $\pi_p(i) + \pi_p(p-i) = p$ *for* $1 \leq i \leq p-1$*,*

(pp2) $\pi_p(\frac{p-1}{2}) = \frac{p-1}{2}$ *and*

(pp3) *exactly one of* j*,* $-j$ *is in* $\{\pi_p(2) - \pi_p(1), \pi_p(3) - \pi_p(2), \ldots, \pi_p(\frac{p+1}{2}) - \pi_p(\frac{p-1}{2})\}$

then $\bar{\mathcal{C}}(\pi_p)$ *is a connected combinatorial* 2*-manifold.*

Proof. Since edges of cycles of $\mathcal{C}(\pi_p)$ form a connected graph, $\mathrm{EG}(\bar{\mathcal{C}}(\pi_p))$ is connected.

Let $a_i = \pi_p(i+1) - \pi_p(i)$ for $1 \leq i \leq p-2$. Then, by (pp1), $a_i = a_{p-1-i}$. Let $r = \frac{p-3}{2}$. Then, by (pp1), (pp2), $a_{r+1} = 1$ and, by (pp3), $\{a_1, \ldots, a_{r+1}, -a_1, \ldots, -a_{r+1}\} = \mathbb{Z}_p \setminus \{0\}$.

If r is even then the cycles containing i are $(i, u, \ldots, i+a_1)$, $(i, i+a_1, \ldots, i-a_2)$, $(i, i-a_2, \ldots, i+a_3), \ldots, (i, i+a_{r-1}, \ldots, i-a_r)$, $(i, i-a_r, \ldots, i+1)$, $(i, i+1, i+2, \ldots, i+p-1)$, $(i, i+p-1, \ldots, i+a_{r+2}), \ldots, (i, i+a_{2r}, \ldots, i-a_{2r+1})$, $(i, i-a_{2r+1}, \ldots, u)$.

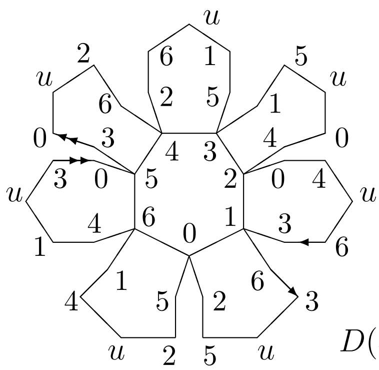
If r is odd then the cycles containing i are $(i, u, \ldots, i+a_1)$, $(i, i+a_1, \ldots, i-a_2)$, $(i, i-a_2, \ldots, i+a_3), \ldots, (i, i-a_{r-1}, \ldots, i+a_r)$, $(i, i+a_r, \ldots, i+p-1)$, $(i, i+p-1, \ldots, i+2, i+1)$, $(i, i+1, \ldots, i-a_{r+2}), \ldots, (i, i+a_{2r}, \ldots, i-a_{2r+1})$, $(i, i-a_{2r+1}, \ldots, u)$.

The cycles containing u are $(u, \pi_p(1), \ldots, \pi_p(p-1))$, $(u, 1+\pi_p(1), \ldots, 1+\pi_p(p-1)), \ldots$, $(u, p-1+\pi_p(1), \ldots, p-1+\pi_p(p-1))$. Since $\{\pi_p(p-1), 1+\pi_p(p-1), \ldots, p-1+\pi_p(p-1)\} = \mathbb{Z}_p$, the cycles containing u can be arranged as $(u, \pi_p(i_1), \ldots, \pi_p(j_1)), \ldots, (u, \pi_p(i_p), \ldots, \pi_p(j_p))$, where $j_1 = i_2, \ldots, j_{p-1} = i_p, j_p = i_1$. The lemma now follows by Lemma 1. □

Clearly, π_3 is the identity permutation and $\mathcal{C}(\pi_3)$ is the 4-vertex 2-sphere S^2_4. Also, $\chi(\bar{\mathcal{C}}(\pi_p)) = 2(p+1) - \left(\binom{p+1}{2} + p(p+1)\right) + (p+1)p = (p+1)(4-p)/2$. So, if $p = 4k+1$ for some $k \geq 1$ then $\chi(\bar{\mathcal{C}}(\pi_p))$ is odd and hence $\bar{\mathcal{C}}(\pi_p)$ is non-orientable. Here we prove.

Lemma 6. *$\bar{\mathcal{C}}(\pi_p)$ is non-orientable for $p > 3$.*

Proof. Let $F = (0, 1, \ldots, p-1)$ and $F_i = (u, i + \pi_p(1), \ldots, i + \pi_p(p-1))$ for $1 \leq i \leq p-1$. We can choose a p-gonal disc (not necessarily convex) in the plane for each cycle in $\bar{\mathcal{C}}(\pi_p)$ so that the disc corresponding to F_i is attached with that for F along the common edge $\{i + \pi_p(\frac{p-1}{2}), i + \pi_p(\frac{p+1}{2})\}$ for each i and there are no other intersections. This gives us a $p(p-1)$-gonal disc $D(\pi_p)$. Then there are two edges in $D(\pi_p)$ corresponding to an edge jk ($j, k \in \mathbb{Z}_p$, $-1 \neq j-k \neq 1$) in some cycle F_i and they appear in the same direction (clockwise or anti-clockwise). Since $|\bar{\mathcal{C}}(\pi_p)|$ is homeomorphic to the space obtained by identifying such pairs of edges (and some more) of $D(\pi_p)$, $|\bar{\mathcal{C}}(\pi_p)|$ is non-orientable. □



$D(\sigma_7)$

Lemma 7. *Let $p > 3$ be a prime.*

(a) *If $p = 4k+3$ for some $k \geq 1$ then the permutation $\sigma_p = (2, 4k+1)(4, 4k-1)\cdots(2k, 2k+3)$ of $\mathbb{Z}_p \setminus \{0\}$ satisfies* (pp1), (pp2) *and* (pp3) *of Lemma* 5.

(b) *If $p = 4l+1$ for some $l \geq 1$ then the permutation $\rho_p = (1, 4l)(3, 4l-2)\cdots(2l-1, 2l+2)$ of $\mathbb{Z}_p \setminus \{0\}$ satisfies* (pp1), (pp2) *and* (pp3) *of Lemma* 5.

Proof. Clearly, σ_p and ρ_p satisfy hypothesis (pp1) and (pp2).

Now, $\{\sigma_p(2) - \sigma_p(1), \ldots, \sigma_p(\frac{p+1}{2}) - \sigma_p(\frac{p-1}{2})\} = \{4k, -(4k-2), 4k-4, \ldots, 4, -2, 1\} = \{-2, 4, -6, \ldots, -(4k-2), 4k, -(4k+2)\}$. Thus σ_p satisfies (pp3).

Again, $\{\rho_p(2) - \rho_p(1), \ldots, \rho_p(\frac{p+1}{2}) - \rho_p(\frac{p-1}{2})\} = \{-(4l-2), 4l-4, -(4l-6), \ldots, 4, -2, 1\} = \{-2, 4, -6, \ldots, (4l-4), -(4l-2), -4l\}$. Thus ρ_p satisfies (pp3). □

Proof of Theorem 1. Let $m \geq 3$ and $n \geq 0$. By Lemma 2, $M_{2m-1, 2(3^{m-1}+2n-1)}$ is a $2(3^{m-1} + 2n - 1)$-vertex polyhedral map and hence a $\{2m-1, 2m-1\}$-equivelar polyhedral map. Again, by Lemma 2, $M_{2m, 3^m+2n-1}$ is a $(3^m + 2n - 1)$-vertex polyhedral map and hence a $\{2m, 2m\}$-equivelar polyhedral map. The theorem now follows from Lemma 3. □

Proof of Theorem 2. Let $p > 3$ be a prime and K_{p+1} be the complete graph on the vertex set $\mathbb{Z}_p \cup \{u\}$. By Lemma 7, there exists a permutation π_p of $\mathbb{Z}_p \setminus \{0\}$ which satisfies (pp1), (pp2) and (pp3) of Lemma 5. Let $\mathcal{C}(\pi_p)$ be as in Lemma 5. Then, by Lemma 5, $\bar{\mathcal{C}}(\pi_p)$ is a connected combinatorial 2-manifold. So, if $N_p := |\bar{\mathcal{C}}(\pi_p)|$ then (N_p, K_{p+1}) is a non-singular $\{p,p\}$-pattern and the cycles in $\mathcal{C}(\pi_p)$ are the boundaries of the faces of (N_p, K_{p+1}). Finally, the 4-vertex 2-sphere S^2_4 gives a $\{3,3\}$-pattern. This completes the proof. □

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