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On Minimum Size Blocking Sets of External Lines to a Quadric in PG$(3, q)$

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Abstract. We present the characterization of the minimum size blocking sets with respect to the external lines to a quadric in PG$(3, q)$, $q \geq 9$.

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1. Introduction

A *blocking set* in a projective space $\mathbb{P} = \mathrm{PG}(d, q)$ is a subset of $\mathbb{P}$ which meets every line. Blocking sets have been investigated by several authors from many points of view. The reader is referred to [2, 5, 7, 8] and papers cited there for a survey on this topic.

Now, let $\mathcal{F}$ be a set of lines of $\mathbb{P}$. A point set B of $\mathbb{P}$ is a *blocking set with respect to* $\mathcal{F}$ (or an $\mathcal{F}$*-blocking set*) if every line in $\mathcal{F}$ is incident with at least one point of B. In [1] and [6], all minimum size blocking sets with respect to the set of the external lines to a conic in PG$(2, q)$ have been determined for q odd and q even, respectively.

Let $\mathcal{Q}$ be a non-singular quadric or a cone in PG$(3, q)$, and let $\mathcal{F}$ be the set of the external lines to $\mathcal{Q}$. If π is a plane, then $\pi \backslash \mathcal{Q}$ is an $\mathcal{F}$-blocking set. The minimum size of $\pi \backslash \mathcal{Q}$ is:

(a) $q^2 - q$, if $\mathcal{Q}$ is a hyperbolic quadric (π is a tangent plane);
(b) q^2, if $\mathcal{Q}$ is an elliptic quadric (π is a secant plane);
(c) $q^2 - q$, if $\mathcal{Q}$ is a cone (π is a plane sharing two distinct lines with $\mathcal{Q}$).

The following two questions immediately arise.

1. Are these the correct sizes for the smallest blocking sets with respect to the external lines to a non-singular quadric or a cone?
2. A blocking set of such a size is always of type $\pi \setminus \mathcal{Q}$, for some plane π?

In [4], an affirmative answer is given to both questions when $\mathcal{Q}$ is a hyperbolic quadric and q is even. In this paper, we examine the other cases and again the answer will be affirmative.

We note that if π is a plane and B is a blocking set with respect to the set of the external lines to $\mathcal{Q}$, then $\pi \cap B$ is a blocking set in π with respect to the lines in π external to $\pi \cap \mathcal{Q}$. Moreover, if B is of minimum size, then $B \cap \mathcal{Q} = \emptyset$.

Now, we state the main results of [1] and [6] which will be useful for the sequel of this article.

Theorem 1.1. *Let Γ be an irreducible conic of $PG(2,q)$, $q \geq 9$ odd. If B is a blocking set with respect to the set of the external lines to Γ, then $|B| \geq q-1$ and $|B| = q-1$ if and only if $B = L \setminus \Gamma$, where L is a secant line to Γ.*

Theorem 1.2. *Let Γ be an irreducible conic of $PG(2,q)$, q even, and let n be its nucleus. If B is a blocking set with respect to the set of the external lines to Γ, then $|B| \geq q-1$ and $|B| = q-1$ if and only if one of the following cases occurs:*

(i) *$B = L \setminus \Gamma$, where L is a secant line to Γ;*
(ii) *$B = L \setminus (\Gamma \cup \{n\})$, where L is a tangent line to Γ;*
(iii) *q is a square and $B = \Pi \setminus (\Gamma \cup \{n\})$, where Π is a Baer subplane such that $|\Pi \cap \Gamma| = \sqrt{q}+1$.*

Finally, if $\mathcal{Q}$ is a quadric in $\mathrm{PG}(3,q)$, by an *external* (*tangent* or *secant*) *line* we mean a line external (tangent or secant) to $\mathcal{Q}$. Similarly, a plane tangent or secant to $\mathcal{Q}$ will be sometimes referred to as a *tangent* or *secant plane*.

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2. Hyperbolic quadric, q odd

Let $\mathcal{Q}$ be a hyperbolic quadric of $\mathrm{PG}(3,q)$, $q \geq 9$ odd, and let $\mathcal{F}$ be the set of all external lines to $\mathcal{Q}$. Throughout this section, B denotes an $\mathcal{F}$-blocking set of minimum size. Since, for any tangent plane π to $\mathcal{Q}$, $\pi\backslash\mathcal{Q}$ is an $\mathcal{F}$-blocking set of size q^2-q, necessarily

$$|B| \leq q^2 - q. \tag{2.1}$$

Proposition 2.1. *For an $\mathcal{F}$-blocking set B of minimum size, $|B| = q^2 - q$. Moreover, $|L \cap B| = 1$ for any $L \in \mathcal{F}$.*

Proof. Counting in two ways the point-line pairs (x, L), $x \in B \cap L$ and $L \in \mathcal{F}$, yields

$$|B|\frac{q^2-q}{2} \geq \frac{q^2(q-1)^2}{2}. \tag{2.2}$$

From (2.1) and (2.2),

$$\frac{q^2(q-1)^2}{2} = (q^2-q)\frac{q^2-q}{2} \geq |B|\frac{q^2-q}{2} \geq \frac{q^2(q-1)^2}{2}. \tag{2.3}$$

From (2.3), it follows that the equality holds in both (2.1) and (2.2); so, the statement is proved. □

Proposition 2.2. *The following properties are valid:*

(i) *through any point in B, there exist q secant lines whose points not in $\mathcal{Q}$ are in B;*

(ii) *if π is a secant plane, then $|\pi \cap B| = q-1$ or q.*

Proof. (i) Let $p \in B$ and let L be an external line through p. By Proposition 2.1, $L \cap B = \{p\}$. Every plane π through L is a secant plane so, by Theorem 1.1, $|\pi \cap B| \geq q-1$. This implies, by Proposition 2.1, that there exist q planes $\pi_1, \pi_2, \ldots, \pi_q$ through L meeting B in $q-1$ points and just one plane through L intersecting B in q points. By Theorem 1.1, $\pi_i \cap B = L_i\backslash\mathcal{Q}$ $(i = 1, \ldots, q)$, where L_i is a secant line to $\mathcal{Q}$.

(ii) Consider a line L in π external to $\mathcal{Q}$. The arguments in (i) imply the statement. □

Proposition 2.3. *Through any point in B, there exists a tangent plane π such that $|\pi \cap B| \geq (q-1)^2$.*

Proof. Let $p \in B$. By Proposition 2.2 (i), p lies on q secant lines $L_1, L_2, \ldots, L_q$ such that $|L_i \cap B| = q-1$, $i = 1, 2, \ldots, q$. As $q \geq 9$, (ii) of Proposition 2.2 implies that the plane joining L_i and L_j, $i \neq j$, is tangent to $\mathcal{Q}$. Now, consider the two tangent planes π' and π'' through L_1. The lines $L_2, \ldots, L_q$ are in $\pi' \cup \pi''$; so, $q \geq 9$ implies that one of the two planes π' and π'', say π', contains at least two of such lines. Let $L_2, L_3 \subseteq \pi'$. If a line L_i, $i \geq 4$, exists not in π', then through L_i there pass three distinct tangent planes $\langle L_1, L_i\rangle, \langle L_2, L_i\rangle, \langle L_3, L_i\rangle$, a contradiction. Hence, the lines $L_1, L_2, \ldots, L_q$ all lie in π' and the statement is proved. □

Now we can prove the following classification result.

Theorem 2.4. *Let $\mathcal{Q}$ be a hyperbolic quadric in $PG(3, q)$, $q \geq 9$ odd. If B is a minimum size blocking set with respect to the set of the external lines to $\mathcal{Q}$, then $|B| = q^2 - q$ and $B = \pi\backslash\mathcal{Q}$ for some plane π tangent to $\mathcal{Q}$.*

Proof. By Proposition 2.1, $|B| = q^2 - q$. Now, let $p \in B$. By Proposition 2.3, a tangent plane π through p exists sharing with B at least $(q-1)^2$ points. Assume that a point p' exists in $B \backslash \pi$. Again by Proposition 2.3, p' lies in a tangent plane π' such that $|\pi' \cap B| \geq (q-1)^2$. Since $|(\pi \cap \pi') \cap B| \leq q-1$, then $|(\pi \cup \pi') \cap B| \geq 2q^2 - 5q + 3$, a contradiction to $|B| = q^2 - q$ as $q \geq 9$. Hence, $B \subseteq \pi$; so, $|B| = q^2 - q$ implies that $B = \pi \backslash \mathcal{Q}$ and the statement is completely proved. □

3. Elliptic quadric

Let $\mathcal{Q}$ be an elliptic quadric of $\mathrm{PG}(3, q)$, $q \geq 9$, and let $\mathcal{F}$ be the set of all external lines to $\mathcal{Q}$. In this section, B denotes an $\mathcal{F}$-blocking set of minimum size. Since, for any secant plane π, $\pi \backslash \mathcal{Q}$ is an $\mathcal{F}$-blocking set of size q^2, necessarily

$$|B| \leq q^2. \tag{3.1}$$

Proposition 3.1. *For any tangent plane π, $|\pi \cap B| \geq q$.*

If a tangent plane π meets B in exactly q points, then $B \cap \pi = L \backslash \mathcal{Q}$ for some line L in π tangent to $\mathcal{Q}$.

Proof. Counting in two ways the point-line pairs (x, L), $x \in \pi \cap B \cap L$ and L a line in π external to $\mathcal{Q}$, gives $|\pi \cap B| q \geq q^2$. The first statement follows.

For the proof of the second statement, let $\pi \cap \mathcal{Q} = \{p_0\}$. The set $(B \cap \pi) \cup \{p_0\}$ is a blocking set in π and $|(B \cap \pi) \cup \{p_0\}| = q + 1$; so, $(B \cap \pi) \cup \{p_0\}$ is a line. The statement follows. □

Proposition 3.2. *There exists a tangent line skew to B.*

Proof. Assume that any tangent line shares at least one point with B. Counting in two ways the point-line pairs (x, L), $x \in B \cap L$ and L a tangent line, yields $|B|(q+1) \geq (q+1)(q^2+1)$, a contradiction to (3.1); so, the statement is proved. □

Proposition 3.3. *If L is a tangent line skew to B, then the tangent plane through L meets B in exactly q points and $|B \cap \pi| = q - 1$ for any secant plane through L. Moreover, $|B| = q^2$.*

Proof. Counting points of B on planes through L, we obtain, by Proposition 3.1 and Theorems 1.1 and 1.2,

$$|B| \geq q + q(q-1) = q^2;$$

so, the statement follows from (3.1). □

Proposition 3.4. *There exist $q + 1$ lines $L_1, \ldots, L_{q+1}$ tangent to $\mathcal{Q}$ at distinct points such that $L_i \backslash \mathcal{Q} \subseteq B$, $i = 1, \ldots, q + 1$, and such that the tangent plane through L_i only intersects B in $L_i \setminus \mathcal{Q}$.*

Proof. By Proposition 3.1, every tangent plane shares at least q points with B. Denote by t the number of tangent planes meeting B in exactly q points. Counting in two ways the point-plane pairs (x, π), $x \in B \cap \pi$ and π a tangent plane, gives

$$|B|(q+1) \geq tq + (q^2 + 1 - t)(q+1)$$

from which, by Proposition 3.3, $t \geq q+1$. The statement follows from Proposition 3.1. □

Now we can prove the following classification result.

Theorem 3.5. *Let $\mathcal{Q}$ be an elliptic quadric in $PG(3, q)$, $q \geq 9$. If B is a minimum size blocking set with respect to the set of the external lines to $\mathcal{Q}$, then $|B| = q^2$ and $B = \pi \backslash \mathcal{Q}$ for some secant plane π to $\mathcal{Q}$.*

Proof. By Proposition 3.3, $|B| = q^2$.
Case 1 q odd. By Proposition 3.2, there exists a tangent line L skew to B. Set $\{p\} = L \cap \mathcal{Q}$. Moreover, denote by α_0 the tangent plane to the point p and by $\alpha_i, i = 1, \ldots, q$, the secant planes through L. By Propositions 3.3 and 3.1 and Theorem 1.1, $\alpha_0 \cap B = T \backslash \{p\}$, where T is a line in α_0 tangent to $\mathcal{Q}$, and $\alpha_i \cap B = L_i \backslash \mathcal{Q}$, where L_i is a secant line on p.

Let π_i be the plane through L_1 and L_i, $i = 2, \ldots, q$. Since π_i is a secant plane through p and $|\pi_i \cap B| \geq 2q - 2$, then Proposition 3.3 implies that $\pi_i \cap \alpha_0 = T$. Hence, the planes π_i, $i = 2, \ldots, q$, all coincide with the secant plane π joining L_1 and T. Since $|\pi \cap B| \geq q^2$, necessarily $B = \pi \backslash \mathcal{Q}$.

Case 2 q even. By Proposition 3.4, there exist $q + 1$ lines $L_1, \ldots, L_{q+1}$ tangent to $\mathcal{Q}$ at distinct points such that $L_i \backslash \mathcal{Q} \subseteq B$, and such that the tangent plane π_i through L_i meets B in $L_i \backslash \mathcal{Q}$. So, L_i meets π_j, $i \neq j$, at a point of L_j. Therefore, the lines L_i pairwise intersect in a point of B.

Consider the plane π through L_1 and L_2, and let $\{n\} = L_1 \cap L_2$. Obviously, n is the nucleus of the conic $\pi \cap \mathcal{Q}$. Since any line L_i, $i = 3, \ldots, q+1$, meets L_1 and L_2, then $L_i \subseteq \pi$ and $n \in L_i$ since L_i cannot intersect L_1 and L_2 in two distinct points. Otherwise, it would be skew or secant to $\pi \cap \mathcal{Q}$. It therefore passes through n. Since the only tangents to $\mathcal{Q}$ through n lie in π, the lines $L_3, \ldots, L_{q+1}$ all are contained in π. Hence, the lines $L_1, \ldots, L_{q+1}$ all are contained in the plane π; so, $|\pi \cap B| \geq q^2$. It follows that $B = \pi \backslash \mathcal{Q}$ and the statement is completely proved. □

4. Cone

Let $\mathcal{C}$ be a cone of $\mathrm{PG}(3, q)$, $q \geq 9$, and let $\mathcal{F}$ be the set of all external lines to $\mathcal{Q}$. In this section, B denotes an $\mathcal{F}$-blocking set of minimum size. Since for any plane π sharing two distinct lines with $\mathcal{C}$, $\pi \backslash \mathcal{C}$ is an $\mathcal{F}$-blocking set of size $q^2 - q$, necessarily

$$|B| \leq q^2 - q. \tag{4.1}$$

Proposition 4.1. $|B| = q^2 - q$ *and, for any secant plane π, $|B \cap \pi| = q - 1$.*

Proof. By Theorems 1.1 and 1.2, $|\pi \cap B| \geq q-1$. Therefore, counting in two ways the point-plane pairs (x, π), $x \in B \cap \pi$ and π a secant plane, gives

$$|B|q^2 \geq q^3(q-1). \tag{4.2}$$

The statement follows from (4.1) and (4.2). □

Proposition 4.2. *For any secant plane π, $B \cap \pi = L \backslash \mathcal{C}$, where L is a line in π secant to $\mathcal{C}$.*

Proof. Let $B' = B \cap \pi$ and let $\Gamma = \mathcal{C} \cap \pi$. By Proposition 4.1, $|\pi \cap B| = q-1$.

If q is odd, then the statement follows from Theorem 1.1.

If q is even, then one of the three cases in Theorem 1.2 must occur. Assume that $B' = T \backslash (\Gamma \cup \{n\})$, where T is a line in π tangent to Γ and where n is the nucleus of Γ. If π' is a secant plane through T other than π, then $T \backslash (\Gamma \cup \{n\}) \subseteq B \cap \pi'$; so, by Proposition 4.1, $T \backslash (\Gamma \cup \{n\}) = B \cap \pi'$, a contradiction to Theorem 1.2 since the nucleus of the conic $\mathcal{C} \cap \pi'$ is a point of $T \backslash \Gamma$ distinct from n.

Now, assume that q is a square and $B' = \Pi \backslash (\Gamma \cup \{n\})$, with Π a Baer subplane of π sharing exactly $\sqrt{q}+1$ points with Γ. Consider a line T in π tangent to Γ at a point of Π and a secant plane π' through T, distinct from π, such that the nucleus n' of $\mathcal{C} \cap \pi'$ is in Π. By Proposition 4.1, $|B \cap \pi'| = q-1$; so, Theorem 1.2 implies that $n' \notin B \cap \pi'$, a contradiction.

Hence, only the case (i) of Theorem 1.2 can occur.

The statement is completely proved. □

Theorem 4.3. *Let $\mathcal{C}$ be a cone in $PG(3,q)$, $q \geq 9$. If B is a minimum size blocking set with respect to the set of the external lines to $\mathcal{C}$, then $|B| = q^2 - q$ and $B = \pi \backslash \mathcal{C}$ for some plane π sharing two distinct lines with $\mathcal{C}$.*

Proof. Let v be the vertex of $\mathcal{C}$ and let $p \in \mathcal{C} \backslash \{v\}$. By Proposition 4.1, $|B| = q^2 - q$; so, through p, a secant line L skew to B exists. Denote by r the point in $\mathcal{C} \cap L$ different from p, and by $\pi_1, \ldots, \pi_q$ the q secant planes through L.

By Proposition 4.2, $\pi_i \cap B = L_i \backslash \mathcal{C}$, L_i a secant for any $i = 1, \ldots, q$. Obviously, L_i passes through p or r, $i = 1, \ldots, q$. Assume that $p \in L_i$ and $r \in L_j$, $i \neq j$. Since $q \geq 9$, there exists a secant plane through L_i and a point in $L_j \cap B$; a contradiction to Proposition 4.2. Hence, the q lines L_i, $i = 1, \ldots, q$, all pass through one of the two points p and r, say p. By Proposition 4.2, the plane through L_1 and L_j, $j = 2, \ldots, q$, is not a secant plane, so it is the plane π joining L_1 and v. Therefore, $|\pi \cap B| \geq q(q-1)$; so $B = \pi \backslash \mathcal{C}$ and the statement is proved. □

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