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Weyl Quantization for Principal Series

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Abstract. Let G be a connected semisimple non-compact Lie group and π a principal series representation of G. Let $\mathcal{O}$ be the coadjoint orbit of G associated by the Kirillov-Kostant method of orbits to the representation π. By dequantizing π we construct an explicit symplectomorphism between a dense open set of $\mathcal{O}$ and a symplectic product $\mathbb{R}^{2n} \times \mathcal{O}'$ where $\mathcal{O}'$ is a coadjoint orbit of a compact subgroup of G. This allows us to obtain a Weyl correspondence on $\mathcal{O}$ which is adapted to the representation π in the sense of [6].

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1. Introduction

Let G be a connected Lie group, $\mathfrak{g}$ the Lie algebra of G and $\mathfrak{g}^*$ the dual space of $\mathfrak{g}$. Let π be a unitary irreducible representation of G on a Hilbert space H. We suppose that the representation π is associated to a coadjoint orbit $\mathcal{O}$ of G by the Kirillov-Kostant method of orbits [21], [25]. The notion of adapted Weyl correspondence was introduced in [4] (see also [5] and [6]) in order to generalize the usual quantization rules [1], [15].

Definition 1.1. *An adapted Weyl correspondence is an isomorphism W from a vector space $\mathcal{A}$ of complex-valued (or real-valued) smooth functions on the orbit $\mathcal{O}$ (called symbols) to a vector space $\mathcal{B}$ of (not necessarily bounded) linear operators on H satisfying the following properties:*

(i) *the elements of* $\mathcal{B}$ *preserve a fixed dense domain* D *of* H*;*

(ii) *the constant function* 1 *belongs to* $\mathcal{A}$*, the identity operator* I *belongs to* $\mathcal{B}$ *and* $W(1) = I$*;*

(iii) $A \in \mathcal{B}$ *and* $B \in \mathcal{B}$ *implies* $AB \in \mathcal{B}$*;*

(iv) *for each* f *in* $\mathcal{A}$ *the complex conjugate* $\bar{f}$ *of* f *belongs to* $\mathcal{A}$ *and the adjoint of* $W(f)$ *is an extension of* $W(\bar{f})$ *(in the real case: for each* f *in* $\mathcal{A}$ *the operator* $W(f)$ *is symmetric);*

(v) *the elements of* D *are* C^∞*-vectors for the representation* π*, the functions* $\tilde{X}$ $(X \in \mathfrak{g})$ *defined on* $\mathcal{O}$ *by* $\tilde{X}(\xi) =< \xi, X >$ *are in* $\mathcal{A}$ *and* $W(i\tilde{X})\, v = d\pi(X)v$ *for each* $X \in \mathfrak{g}$ *and each* $v \in D$*.*

Let us illustrate this definition by two important examples, the nilpotent case and the compact case. Suppose first that G is a connected simply-connected nilpotent Lie group and $\mathcal{O}$ is an arbitrary coadjoint orbit of G. Let $n = 1/2 \dim \mathcal{O}$. There exists a symplectomorphism from $\mathbb{R}^{2n}$ endowed with its natural symplectic structure onto the orbit $\mathcal{O}$ endowed with its Kostant-Kirillov symplectic 2-form [3], [29]. The representation π associated to $\mathcal{O}$ can be realized on the Hilbert space $L^2(\mathbb{R}^n)$. Then, the usual Weyl correspondence from the space of polynomial functions on $\mathcal{O} \simeq \mathbb{R}^{2n}$ onto the space of polynomial differential operators acting on the Schwartz space $D = S(\mathbb{R}^n)$ is an adapted Weyl correspondence [29]. Suppose now that G is a connected simply-connected semisimple compact Lie group and $\mathcal{O}$ is an integral coadjoint orbit of G. The unitary irreducible representation of G associated to the orbit $\mathcal{O}$ is usually realized on a finite-dimensional complex vector space E whose elements are the holomorphic sections of a Hermitian line bundle on the orbit $\mathcal{O}$. The Berezin calculus is a map which associates to any operator on E a function on $\mathcal{O}$ [1], [10]. Its inverse map is an adapted Weyl correspondence on the orbit $\mathcal{O}$ defined on a finite-dimensional space of functions on $\mathcal{O}$ [4], [5].

The relationship between adapted Weyl correspondences and the notions of prequantization and quantization introduced by Mark Gotay [16] is briefly described in [10]. In fact, our original motivation for constructing adapted Weyl correspondences was to build covariant star-products on coadjoint orbits [5]. A more recent motivation is that adapted Weyl correspondences can be used to study contractions of representations of Lie groups in the setting of the Kirillov-Kostant method of orbits [7], [8], [9], [12].

In [5], adapted Weyl correspondences on the coadjoint orbits associated to the principal series representations of a connected semisimple non-compact Lie group were constructed by combining the Berezin calculus and a symbolic calculus on the cotangent bundle of a nilpotent Lie group (see also Section 5). In [10], we have considered the case when G is the semidirect product $V \rtimes K$ where K is a connected semisimple non-compact Lie group acting linearly on a finite-dimensional real vector space V and $\mathcal{O}$ is a coadjoint orbit of G associated by the method of orbits to a unitary irreducible representation π of G. Under the assumption that the corresponding little group K_0 is a maximal compact subgroup of K, we have shown that the orbit $\mathcal{O}$ is symplectomorphic to the symplectic product $\mathbb{R}^{2n} \times \mathcal{O}'$ where $n = \dim(K) - \dim(K_0)$ and $\mathcal{O}'$ is a coadjoint orbit of K_0. Thus we have obtained an adapted Weyl correspondence on $\mathcal{O}$ by combining the usual Weyl correspondence on $\mathbb{R}^{2n}$ and the Berezin calculus on $\mathcal{O}'$ (see also [6] and [7] for earlier results concerning the Poincaré group).

In the present paper, we revisit the case when G is a connected semisimple non-compact Lie group and $\mathcal{O}$ is a coadjoint orbit of G associated to a principal series representation π of G. We use the dequantization procedure introduced in [10] in order to obtain an adapted Weyl correspondence on $\mathcal{O}$ using only the usual Weyl correspondence and the Berezin calculus. In Section 2, we introduce a principal series representation π and the associated coadjoint orbit $\mathcal{O}$. In particular, we realize the representation π on a Hilbert space of functions on $\mathbb{R}^n$. In Section 3, we give an explicit formula for the derived representation $d\pi$ and we dequantize the representation $d\pi$ by means of the usual Weyl correspondence on $\mathbb{R}^{2n}$ and the Berezin calculus on a coadjoint orbit $\mathcal{O}'$ of a compact subgroup of G. Then we obtain in Section 4 an explicit symplectomorphism from the symplectic product $\mathbb{R}^{2n}\times\mathcal{O}'$ onto a dense open set of $\mathcal{O}$ (Theorem 4.6) and the desired adapted Weyl correspondence on $\mathcal{O}$ (Theorem 4.7). Finally, in Section 5, we compare this adapted Weyl correspondence to the symbolic calculus introduced in [5]. We could hope for further applications of these results to the study in the spirit of [7] and [12] of the contractions of the principal series representations of G to the unitary irreducible representations of its Cartan motion group (see [13]).

2. Preliminaries

Let G be a connected non-compact semisimple real Lie group with finite center. Let $\mathfrak{g}$ be the Lie algebra of G. We identify G-equivariantly $\mathfrak{g}$ to its dual space $\mathfrak{g}^*$ using the Killing form of $\mathfrak{g}$ defined by $<X,Y>=\mathrm{Tr}(\mathrm{ad}\,X\,\mathrm{ad}\,Y)$ for X and Y in $\mathfrak{g}$. Let θ be a Cartan involution of $\mathfrak{g}$ and $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ the corresponding Cartan decomposition of $\mathfrak{g}$. Let K be the connected compact subgroup of G with Lie algebra $\mathfrak{k}$. Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{p}$ and M be the centralizer of $\mathfrak{a}$ in K. Let $\mathfrak{m}$ denote the Lie algebra of M. We can decompose $\mathfrak{g}$ under the adjoint action of $\mathfrak{a}$:

$$\mathfrak{g}=\mathfrak{a}\oplus\mathfrak{m}\oplus\sum_{\lambda\in\Delta}\mathfrak{g}_\lambda$$

where $\mathfrak{g}_\lambda=\{X\in\mathfrak{g}:\quad [H,X]=\lambda(H)X\quad\forall H\in\mathfrak{a}\}$ for $\lambda\in\mathfrak{a}^*$ and $\Delta=\{\lambda\in\mathfrak{a}^*\setminus(0):\mathfrak{g}_\lambda\neq(0)\}$ is the set of restricted roots. We fix a Weyl chamber in $\mathfrak{a}$ and we set $\mathfrak{n}=\sum_{\lambda>o}\mathfrak{g}_\lambda$ and $\bar{\mathfrak{n}}=\sum_{\lambda<o}\mathfrak{g}_\lambda$. Then $\bar{\mathfrak{n}}=\theta(\mathfrak{n})$. Let A, N and $\bar{N}$ denote the analytic subgroups of G with algebras $\mathfrak{a}$, $\mathfrak{n}$, $\bar{\mathfrak{n}}$, respectively. We fix a regular element ξ_1 in $\mathfrak{a}$ (i.e. $\lambda(\xi_1)\neq 0$ for each $\lambda\in\Delta$) and an element ξ_2 in $\mathfrak{m}$. Let $\xi_0=\xi_1+\xi_2$ and denote by $O(\xi_0)$ the orbit of ξ_0 in $\mathfrak{g}^*\simeq\mathfrak{g}$ under the (co)adjoint action of G and by $O(\xi_2)$ the orbit of ξ_2 in $\mathfrak{m}$ under the adjoint action of M.

Let M_0 be the connected component of the identity of M and let σ_0 be a unitary irreducible representation of M_0. We have $M=M_0.Z'$ where Z' is a central finite abelian subgroup of M [24, Lemma 9.13]. Then the unitary irreducible representations σ of M such that $\sigma|_{M_0}=\sigma_0$ constitute (up to unitary equivalence) a finite family (σ_χ) parametrized by the characters χ of Z' satisfying $\chi_{|Z'\cap M_0}=\sigma_{0|Z'\cap M_0}$.

Henceforth we assume that the orbit $O(\xi_2)$ is associated to a unitary irreducible representation σ_0 of M_0 as in [30], Section 4 (see also [26]) and we fix a unitary irreducible representation σ of M realized in a finite dimensional complex vector space E such that $\sigma|_{M_0} = \sigma_0$.

The Berezin calculus associates to each operator A on the finite-dimensional complex vector space E a complex-valued function $s(A)$ on the orbit $\mathcal{O}(\xi_2)$ called the symbol of the operator A (see [2], [11]). The following properties of the Berezin calculus can be found in [2], [4], [5].

Proposition 2.1.

1) *The map $A \to s(A)$ is injective.*
2) *For each operator A on E, we have $s(A^*) = \overline{s(A)}$.*
3) *For $\varphi \in \mathcal{O}(\xi_2)$, $k \in K$ and for an operator A on E, we have*

$$s(A)(\mathrm{Ad}(k)\varphi) = s(\sigma(k)^{-1}A\sigma(k))(\varphi).$$

4) *For $X \in \mathfrak{m}$ and $\varphi \in \mathcal{O}(\xi_2)$, we have $s(d\sigma(X))(\varphi) = i < \varphi, X >$ where $d\sigma$ denotes the derived representation of σ.*
5) *There exists a constant ε (which depends only on the orbit $\mathcal{O}(\xi_2)$) such that, for each operator A on E,*

$$\mathrm{Tr}(A) = \varepsilon \int_{\mathcal{O}(\xi_2)} s(A)(\varphi)\, d\mu_0(\varphi)$$

where $d\mu_0(\varphi)$ is the Liouville measure on $\mathcal{O}(\xi_2)$.

Now we consider the unitarily induced representation

$$\hat{\pi} = \mathrm{Ind}_{MAN}^{G}\ (\sigma \otimes \exp(i\nu) \otimes 1_N)$$

where $\nu =< \xi_1, \cdot >\in \mathfrak{a}^*$. The representation $\hat{\pi}$ belongs to the unitary principal series of G and is usually realized on the space $L^2(\bar{N}, E)$ which is the Hilbert space completion of the space of compactly supported smooth functions $\psi : \bar{N} \to E$ relative to the norm

$$\|\psi\|^2 = \int_{\bar{N}} < \psi\,(y)\,,\, \psi(y) >_E\ dy$$

where dy is the Haar measure on $\bar{N}$ normalized as follows. Let $(E_1, E_2, \ldots, E_n)$ be an orthonormal basis for $\bar{\mathfrak{n}}$ with respect to the scalar product $(\,Y, Z\,) := -\, < Y, \theta(Z) >$. Denote by $(Y_1, Y_2, \ldots, Y_n)$ the coordinates of $Y \in \bar{\mathfrak{n}}$ in this basis and let $dY = dY_1 dY_2 \ldots dY_n$ be the Euclidian measure on $\bar{\mathfrak{n}}$. The exponential map exp is a diffeomorphism from $\bar{\mathfrak{n}}$ onto $\bar{N}$ and we set $dy = \log^*(dY)$ where $\log = \exp^{-1}$. For $g \in G$ the action of the operator $\hat{\pi}(g)$ is given by (see [24], [28])

$$(\hat{\pi}(g)\psi)(y) = e^{-(\rho+i\nu)\log a(g^{-1}y)}\sigma(m(g^{-1}y))^{-1}\ \ \psi(\bar{n}(g^{-1}y)) \tag{2.1}$$

where $\rho(H) := \frac{1}{2}\mathrm{Tr}(\mathrm{ad}\, H|_{\bar{\mathfrak{n}}})$ for $H \in \mathfrak{a}$ and $h = \bar{n}(h)m(h)a(h)n(h)$ is the decomposition of $h \in \bar{N}MAN$, so the functions $\bar{n}$, m, a and n are defined on an open dense subset of G (see [28]). For our purpose, it is more convenient to realize $\hat{\pi}$ on the Hilbert space $L^2(\bar{\mathfrak{n}}, E)$ defined as the completion of the space $C_0(\bar{\mathfrak{n}}, E)$ of compactly supported smooth functions $\phi : \bar{\mathfrak{n}} \to E$ with respect to the norm

$$\|\phi\|^2 = \int_{\bar{\mathfrak{n}}} < \phi(Y), \phi(Y) >_E \, dY.$$

To this end, we use the isometry B from $L^2(\bar{\mathfrak{n}}, E)$ to $L^2(\bar{N}, E)$ defined by $B(\phi)$ $(\exp Y) = \phi(Y)$ and we put $\pi(g) = B^{-1}\hat{\pi}(g)B$ for $g \in G$. We immediately obtain, for $g \in G$

$$(\hat{\pi}(g)\phi)(Y) = e^{-(\rho+i\nu)\log a(g^{-1}\exp Y)}\sigma(m(g^{-1}\exp Y))^{-1} \quad \phi(\log \bar{n}(g^{-1}\exp Y)). \quad (2.2)$$

3. Dequantization of the derived representation

In this section, we first give an explicit formula for the differential $d\pi$ of the representation π of G. Then we dequantize $d\pi$ by means of the Berezin-Weyl calculus on $\bar{\mathfrak{n}} \times \bar{\mathfrak{n}} \times \mathcal{O}(\xi_2)$. For $X \in \bar{\mathfrak{n}}$ we denote by X^+ the right invariant vector field on $\bar{N}$ generated by X, that is, $X^+(y) = \frac{d}{dt}(\exp tX)y|_{t=0}$ for $y \in \bar{N}$.

Lemma 3.1.

1) *For $X \in \bar{\mathfrak{n}}$ and $Y \in \bar{\mathfrak{n}}$, we have*

$$d\log(\exp Y)\left(X^+(\exp Y)\right) = \frac{\operatorname{ad} Y}{e^{\operatorname{ad} Y} - 1}(X).$$

2) *Let $p_{\mathfrak{a}}$, $p_{\mathfrak{m}}$ and $p_{\bar{\mathfrak{n}}}$ be the projections of $\mathfrak{g}$ onto $\mathfrak{a}$, $\mathfrak{m}$ and $\bar{\mathfrak{n}}$ associated with the direct decomposition $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{n} \oplus \bar{\mathfrak{n}}$. For $X \in \mathfrak{g}$ and $y \in \bar{N}$, we have*

$$\begin{aligned}
\frac{d}{dt}a(\exp(tX)y)\Big|_{t=0} &= p_{\mathfrak{a}}(\operatorname{Ad}(y^{-1})X) \\
\frac{d}{dt}m(\exp(tX)y)\Big|_{t=0} &= p_{\mathfrak{m}}(\operatorname{Ad}(y^{-1})X) \\
\frac{d}{dt}\bar{n}(\exp(tX)y)\Big|_{t=0} &= (\operatorname{Ad}(y)\, p_{\bar{\mathfrak{n}}}(\operatorname{Ad}(y^{-1})\, X))^+(y).
\end{aligned}$$

Proof. 1) is an immediate consequence of the well-known expression for the derivative of the exponential map (see [17], for instance).

To prove 2), we consider the diffeomorphism $\mu : \bar{N} \times M \times A \times N \to \bar{N}MAN$ defined by $\mu(y, m, a, n) = yman$. We have, for $y \in \bar{N}$, $Y \in \bar{\mathfrak{n}}$, $U \in \mathfrak{m}$, $H \in \mathfrak{a}$ and $Z \in \mathfrak{n}$:

$$\begin{aligned}
d\mu(y, e, e, e)(Y^+(y), U, H, Z) &= \frac{d}{dt}\exp(tY)y\exp(tU)\exp(tH)\exp(tZ)\Big|_{t=0} \\
&= (Y + \operatorname{Ad} y(U + H + Z))^+(y).
\end{aligned}$$

Now, let $X \in \mathfrak{g}$. We write $\operatorname{Ad} y^{-1}X = Y_0 + U + H + Z$ where $Y_0 \in \bar{\mathfrak{n}}$, $U \in \mathfrak{m}$, $H \in \mathfrak{a}$ and $Z \in \mathfrak{n}$. Then the previous equality implies that $d\bar{n}(y)(X^+(y)) =$

$(\mathrm{Ad}(y)\,Y_0)^+(y)$. This proves the last equality of 2). The other equalities are proved similarly. $\square$

From Lemma 3.1 we deduce immediately:

Proposition 3.2. *For $X \in \mathfrak{g}$ we have*

$$\begin{aligned}(d\pi(X)\phi)(Y) &= (\rho + i\nu)(p_{\mathfrak{a}}(\mathrm{Ad}(y^{-1})X))\phi(y) \\ &+ d\sigma(p_{\mathfrak{m}}(\mathrm{Ad}(y^{-1})X))\phi(Y) \\ &- d\phi(Y)\Big(\frac{\mathrm{ad}\,Y}{e^{\mathrm{ad}\,Y}-1}\,\mathrm{Ad}(y)\,p_{\bar{\mathfrak{n}}}(\mathrm{Ad}(y^{-1})X)\Big)\end{aligned} \tag{3.1}$$

where $\phi \in C_0(\bar{\mathfrak{n}}, E)$, $Y \in \bar{\mathfrak{n}}$ and $y = \exp Y$.

Now we introduce the Berezin-Weyl calculus on $\bar{\mathfrak{n}} \times \bar{\mathfrak{n}} \times \mathcal{O}(\xi_2)$. We say that a complex-valued smooth function $f : (Y, Z, \varphi) \to f(Y, Z, \varphi)$ is a symbol on $\bar{\mathfrak{n}} \times \bar{\mathfrak{n}} \times \mathcal{O}(\xi_2)$ if for each $(Y, Z) \in \bar{\mathfrak{n}} \times \bar{\mathfrak{n}}$ the function $\varphi \to f(Y, Z, \varphi)$ is the symbol in the Berezin calculus on $\mathcal{O}(\xi_2)$ of an operator on E denoted by $\hat{f}(Y, Z)$. A symbol f on $\bar{\mathfrak{n}} \times \bar{\mathfrak{n}} \times \mathcal{O}(\xi_2)$ is called an S-symbol if the function $\hat{f}$ belongs to the Schwartz space of rapidly decreasing smooth functions on $\bar{\mathfrak{n}} \times \bar{\mathfrak{n}}$ with values in $\mathrm{End}(E)$. Here we consider the Weyl calculus for $\mathrm{End}(E)$-valued functions. This is a slight refinement of the usual Weyl calculus for complex-valued functions [15], [19], [20]. For any S-symbol on $\bar{\mathfrak{n}} \times \bar{\mathfrak{n}} \times \mathcal{O}(\xi_2)$ we define an operator $\mathcal{W}(f)$ on the Hilbert space $L^2(\bar{\mathfrak{n}}, E)$ by

$$(\mathcal{W}(f)\phi)(Y) = (2\pi)^{-n} \int_{\bar{\mathfrak{n}}\times\bar{\mathfrak{n}}} e^{i\langle T, Z\rangle} \hat{f}\left(Y + \frac{1}{2}T, Z\right) \phi(Y + T)\, dT\, dZ \tag{3.2}$$

for $\phi \in C_0(\bar{\mathfrak{n}}, E)$.

In fact the Weyl calculus can be extended to much larger classes of symbols (see for instance [20]). Here we only consider a class of polynomial symbols. We say that a symbol f on $\bar{\mathfrak{n}} \times \bar{\mathfrak{n}} \times \mathcal{O}(\xi_2)$ is a P-symbol if the function $\hat{f}(Y, Z)$ is polynomial in Z. Let f be the P-symbol defined by $f(Y, Z, \varphi) = u(Y)Z^\alpha$ where $u \in C^\infty(\bar{\mathfrak{n}}, E)$ and $Z^\alpha := Z^{\alpha_1} Z^{\alpha_2} \cdots Z^{\alpha_n}$ for each multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. Then

$$(\mathcal{W}(f)\phi)(Y) = \left(i\frac{\partial}{\partial Z}\right)^\alpha \left(u(Y + \frac{1}{2}Z)\, \phi(Y + Z)\right)\Big|_{Z=0} \tag{3.3}$$

(see [27]). In particular, if $f(Y, Z, \varphi) = u(Y)$ then

$$(\mathcal{W}(f)\phi)(Y) = u(Y)\, \phi(Y) \tag{3.4}$$

and if $f(Y, Z, \varphi) = u(Y)Z_k$ then

$$(\mathcal{W}(f)\phi)(Y) = i\left(\frac{1}{2}\partial_k u(Y)\, \phi(Y) + u(Y)\partial_k \phi(Y)\right) \tag{3.5}$$

where ∂_k denotes the partial derivative with respect to the variable Y_k.

The correspondence $f \to \mathcal{W}(f)$ is called the Berezin-Weyl calculus on $\bar{\mathfrak{n}} \times \bar{\mathfrak{n}} \times \mathcal{O}(\xi_2)$. In order to dequantize the derived representation $d\pi$, that is, to calculate the Berezin-Weyl symbol of the operators $d\pi(X)$ $(X \in \mathfrak{g})$, we need the following lemma. The trace of an endomorphism u of $\bar{\mathfrak{n}}$ is denoted by $\mathrm{Tr}_{\bar{\mathfrak{n}}}\, u$.

Lemma 3.3. *For $X \in \mathfrak{g}$ let $c_X : \bar{\mathfrak{n}} \to \bar{\mathfrak{n}}$ be the map defined by*

$$c_X(Y) = s(\mathrm{ad}\, Y) p_{\bar{\mathfrak{n}}}(\mathrm{Ad}(y^{-1})\, X)$$

where s is the function defined by $s(z) = \frac{ze^z}{e^z-1}$ for $z \neq 0$ and $s(0) = 1$. Then we have

$$\mathrm{Tr}_{\bar{\mathfrak{n}}}\, dc_X(Y) = -2\rho\left(p_{\mathfrak{a}}(\mathrm{Ad}(y^{-1})\, X)\right). \tag{3.6}$$

Proof. Let $s(z) = \sum_{k\geq 0} a_k z^k$ be the power-series expansion of the function s. Note that for $Y \in \bar{\mathfrak{n}}$ the sum $s(\mathrm{ad}\, Y) = \sum_{k\geq 0} a_k (\mathrm{ad}\, Y)^k$ is finite since $\mathrm{ad}\, Y$ is a nilpotent endomorphism of $\bar{\mathfrak{n}}$. We have, for Y and Z in $\bar{\mathfrak{n}}$

$$\begin{aligned} dc_X(Y) &= \frac{d}{dt} s(\mathrm{ad}(Y+tZ))\Big|_{t=0} p_{\bar{\mathfrak{n}}}(\mathrm{Ad}(y^{-1})X) + \\ &\quad s(\mathrm{ad}\, Y) p_{\bar{\mathfrak{n}}}\left(\frac{d}{dt}\mathrm{Ad}(\exp(-Y-tZ))X\Big|_{t=0}\right). \end{aligned} \tag{3.7}$$

Now

$$\begin{aligned} \frac{d}{dt} s(\mathrm{ad}(Y+tZ))\Big|_{t=0} &= \sum_{k\geq 0} a_k \frac{d}{dt}(\mathrm{ad}\, Y + t\,\mathrm{ad}\, Z)^k\Big|_{t=0} \\ &= \sum_{k\geq 0} a_k \left(\sum_{r=0}^{k-1} (\mathrm{ad}\, Y)^r\, \mathrm{ad}\, Z (\mathrm{ad}\, Y)^{k-r-1}\right). \end{aligned}$$

Then, since the endomorphism of $\bar{\mathfrak{n}}$ defined by

$$\begin{aligned} Z \to \quad & (\mathrm{ad}\, Y)^r\, \mathrm{ad}\, Z (\mathrm{ad}\, Y)^{k-r-1} p_{\bar{\mathfrak{n}}}(\mathrm{Ad}(y^{-1})\, X) \\ &= -(\mathrm{ad}\, Y)^r\, \mathrm{ad}\left((\mathrm{ad}\, Y)^{k-r-1} p_{\bar{\mathfrak{n}}}(\mathrm{Ad}(y^{-1})\, X)\right)(Z) \end{aligned}$$

is clearly nilpotent, the endomorphism of $\bar{\mathfrak{n}}$ given by

$$Z \to \frac{d}{dt} s(\mathrm{ad}(Y+tZ))\Big|_{t=0} p_{\bar{\mathfrak{n}}}(\mathrm{Ad}(y^{-1})\, X)$$

has trace zero. Now we also have

$$\begin{aligned} \frac{d}{dt}\mathrm{Ad}(\exp(-Y-tZ))X\Big|_{t=0} &= \frac{d}{dt}\mathrm{Ad}\left(\exp(-Y)\exp(Y+tZ)\right)^{-1}\mathrm{Ad}(\exp(-Y))X\Big|_{t=0} \\ &= -\,\mathrm{ad}\left(\frac{1-e^{-\mathrm{ad}\, Y}}{\mathrm{ad}\, Y} Z\right)\mathrm{Ad}(\exp Y)^{-1} X \\ &= \mathrm{ad}\left(\mathrm{Ad}(\exp Y)^{-1} X\right)\left(\frac{1-e^{-\mathrm{ad}\, Y}}{\mathrm{ad}\, Y}\right) Z. \end{aligned}$$

Then the trace of the endomorphism of $\bar{\mathfrak{n}}$ defined by

$$Z \to s(\operatorname{ad} Y)p_{\bar{\mathfrak{n}}}\left(\frac{d}{dt}\operatorname{Ad}(\exp(-Y-tZ))\,X\Big|_{t=0}\right)$$

is

$$\begin{aligned}
& \operatorname{Tr}_{\bar{\mathfrak{n}}}\Big(s(\operatorname{ad} Y)p_{\bar{\mathfrak{n}}}\circ \operatorname{ad}(\operatorname{Ad}(\exp Y)^{-1}X)\,\frac{1-e^{-\operatorname{ad} Y}}{\operatorname{ad} Y}\Big)\\
= \; & \operatorname{Tr}_{\bar{\mathfrak{n}}}\Big(\frac{1-e^{-\operatorname{ad} Y}}{\operatorname{ad} Y} s(\operatorname{ad} Y)\,p_{\bar{\mathfrak{n}}}\circ \operatorname{ad}(\operatorname{Ad}(\exp Y)^{-1}X)\Big)\\
= \; & \operatorname{Tr}_{\bar{\mathfrak{n}}}\left(\frac{1-e^{-\operatorname{ad} Y}}{\operatorname{ad} Y}\,\frac{\operatorname{ad} Y}{e^{\operatorname{ad} Y}-1}\exp(\operatorname{ad} Y)\,p_{\bar{\mathfrak{n}}}\circ \operatorname{ad}(\operatorname{Ad}(\exp Y)^{-1}X)\right)\\
= \; & \operatorname{Tr}_{\bar{\mathfrak{n}}}\left(p_{\bar{\mathfrak{n}}}\circ \operatorname{ad}(\operatorname{Ad}(\exp Y)^{-1}X)\right).
\end{aligned}$$

Thus the lemma will be proved if we can show that, for each $X \in \mathfrak{g}$,

$$\operatorname{Tr}_{\bar{\mathfrak{n}}}(p_{\bar{\mathfrak{n}}}\circ \operatorname{ad} X) = -2\rho(p_{\mathfrak{a}}(X)).$$

If $X \in \bar{\mathfrak{n}}$ then $p_{\bar{\mathfrak{n}}}\circ \operatorname{ad} X = \operatorname{ad} X$ is a nilpotent endomorphism of $\bar{\mathfrak{n}}$. Thus $\operatorname{Tr}_{\bar{\mathfrak{n}}}(p_{\bar{\mathfrak{n}}}\circ \operatorname{ad} X) = 0$. If $X \in \mathfrak{n}$ then, noting that $[\mathfrak{n}\,,\,\mathfrak{g}_\lambda] \subset \mathfrak{a} + \sum_{\mu>\lambda}\mathfrak{g}_\mu$ for each $\lambda < 0$, we also obtain that $\operatorname{Tr}_{\bar{\mathfrak{n}}}(p_{\bar{\mathfrak{n}}}\circ \operatorname{ad} X) = 0$. If $X \in \mathfrak{m}$ then $p_{\bar{\mathfrak{n}}}\circ \operatorname{ad} X = \operatorname{ad} X$ is an endomorphism of $\bar{\mathfrak{n}}$ which is skew-symmetric with respect to $(\cdot,\cdot)$ [17]. Thus $\operatorname{Tr}_{\bar{\mathfrak{n}}}(p_{\bar{\mathfrak{n}}}\circ \operatorname{ad} X) = 0$. Finally, if $X \in \mathfrak{a}$ then $\operatorname{Tr}_{\bar{\mathfrak{n}}}(p_{\bar{\mathfrak{n}}}\circ \operatorname{ad} X) = \operatorname{Tr}_{\bar{\mathfrak{n}}}(\operatorname{ad} X) = -2\rho(X)$. This ends the proof of the lemma. □

Proposition 3.4. *For each $X \in \mathfrak{g}$, the Berezin-Weyl symbol of the operator $-id\pi(X)$ is the P-symbol f_X on $\bar{\mathfrak{n}}\times\bar{\mathfrak{n}}\times\mathcal{O}(\xi_2)$ given by*

$$f_X(Y,Z,\varphi) = \nu\left(p_{\mathfrak{a}}(\operatorname{Ad}(y^{-1})\,X)\right) + <\varphi, p_{\mathfrak{m}}(\operatorname{Ad}(y^{-1})\,X)> + (\,c_X(Y), Z\,) \tag{3.8}$$

where $y = \exp Y$.

Proof. If we put $c_X^k(Y) = (\,c_X(Y), E_k\,)$ for $k = 1,\ldots,n$ then we can write (3.1) as

$$\begin{aligned}
(-id\pi(X)\phi)(Y) \;=\; & (-i\rho+\nu)(p_{\mathfrak{a}}(\operatorname{Ad}(y^{-1})X))\phi(y)\\
& -id\sigma(p_{\mathfrak{m}}(\operatorname{Ad}(y^{-1})X))\phi(Y)\\
& +i\sum_{k=1}^{n}(\partial_k\phi)(Y)c_X^k(Y).
\end{aligned} \tag{3.9}$$

Using Proposition 2.1 4) and the properties (3.3) and (3.4) of the Weyl calculus, we see that the symbol of $-id\pi(X)$ is

$$\begin{aligned}
f_X(Y,Z,\varphi) \;=\; & (-i\rho+\nu)\left(p_{\mathfrak{a}}(\operatorname{Ad}(y^{-1})X)\right) + <\varphi, p_{\mathfrak{m}}(\operatorname{Ad}(y^{-1})X)>\\
& + \sum_{k=1}^{n} c_X^k(Y)Z_k - \frac{i}{2}\sum_{k=1}^{n}\partial_k c_X^k(Y).
\end{aligned}$$

By Lemma 3.3

$$-\frac{i}{2}\sum_{k=1}^{n}\partial_k c_X^k(Y) = -\frac{i}{2}\,\mathrm{Tr}_{\bar{\mathfrak{n}}}\,(dc_X(Y)) = i\rho(p_{\mathfrak{a}}(\mathrm{Ad}(y^{-1})X)).$$

The result follows. □

4. Adapted Weyl correspondence

In this section, we show that the dequantization procedure introduced in Section 3 allows us to obtain an explicit diffeomorphism from $\bar{\mathfrak{n}}^2\times\mathcal{O}(\xi_2)$ onto the dense open set $\tilde{\mathcal{O}}(\xi_0) := \mathrm{Ad}(\bar{N}MAN)\xi_0$ of $\mathcal{O}(\xi_0)$. This diffeomorphism is a symplectomorphism for the natural symplectic structures. Using this symplectomorphism, we then construct an adapted Weyl correspondence on $\mathcal{O}(\xi_0)$. We retain the notation of the previous sections.

Recall that f_X denotes the Berezin-Weyl symbol of the operator $-id\pi(X)$ for $X\in\mathfrak{g}$. Since the map $X\to f_X(T,S,\varphi)$ is linear there exists a map Ψ from $\bar{\mathfrak{n}}^2\times\mathcal{O}(\xi_2)$ to $\mathfrak{g}^*$ such that

$$f_X(Y,Z,\varphi) =< \Psi(Y,Z,\varphi)\,,\,X >$$

for each $X\in\mathfrak{g}$ and each $(Y,Z,\varphi)\in\bar{\mathfrak{n}}\times\bar{\mathfrak{n}}\times\mathcal{O}(\xi_2)$. From Proposition 3.4 we can deduce a precise expression for Ψ.

Proposition 4.1. *For $(Y,Z,\varphi)\in\bar{\mathfrak{n}}\times\bar{\mathfrak{n}}\times\mathcal{O}(\xi_2)$ we have*

$$\Psi(Y,Z,\varphi) = \mathrm{Ad}(\exp Y)\left(\xi_1+\varphi+p_{\mathfrak{n}}\Big(\frac{\mathrm{ad}\,Y}{e^{\mathrm{ad}\,Y}-1}\,\theta(Z)\Big)\right). \tag{4.1}$$

Proof. Recall that

$$f_X(Y,Z,\varphi) = \nu\left(p_{\bar{\mathfrak{a}}}(\mathrm{Ad}(y^{-1})\,X)\right) + <\varphi, p_{\mathfrak{m}}(\mathrm{Ad}(y^{-1})\,X)> + < c_X(Y),\theta(Z)>$$

where $y=\exp Y$. We have

$$\begin{aligned}
< c_X(Y),\theta(Z) >= &\ < \tfrac{\mathrm{ad}\,Y}{e^{\mathrm{ad}\,Y}-1}\,\mathrm{Ad}(y)\ p_{\bar{\mathfrak{n}}}(\mathrm{Ad}(y^{-1})\,X)\,,\,\theta(Z)>\\
= &\ < p_{\bar{\mathfrak{n}}}(\mathrm{Ad}(y^{-1})\,X)\,,\,\tfrac{\mathrm{ad}\,Y}{e^{\mathrm{ad}\,Y}-1}\,\theta(Z)>\\
= &\ < p_{\bar{\mathfrak{n}}}(\mathrm{Ad}(y^{-1})\,X)\,,\,p_{\mathfrak{n}}\Big(\tfrac{\mathrm{ad}\,Y}{e^{\mathrm{ad}\,Y}-1}\,\theta(Z)\Big)>\\
= &\ < \mathrm{Ad}(y^{-1})\,X\,,\,p_{\mathfrak{n}}\Big(\tfrac{\mathrm{ad}\,Y}{e^{\mathrm{ad}\,Y}-1}\,\theta(Z)\Big)>\\
= &\ < X\,,\,\mathrm{Ad}(y)\,p_{\mathfrak{n}}\Big(\tfrac{\mathrm{ad}\,Y}{e^{\mathrm{ad}\,Y}-1}\,\theta(Z)\Big)>.
\end{aligned}$$

Here we have used that $<\mathfrak{a}+\mathfrak{m},\mathfrak{n}+\bar{\mathfrak{n}}>=(0)$, $<\mathfrak{n},\mathfrak{n}>=(0)$ and $<\bar{\mathfrak{n}},\bar{\mathfrak{n}}>=(0)$. Similarly, we have

$$\begin{aligned}
\nu\left(p_{\mathfrak{a}}(\mathrm{Ad}(y^{-1})\,X)\right) &= <\xi_1, p_{\mathfrak{a}}(\mathrm{Ad}(y^{-1})\,X)>\\
&= <\xi_1,\mathrm{Ad}(y^{-1})\,X>=<\mathrm{Ad}(y)\,\xi_1, X>
\end{aligned}$$

and

$$< \varphi, p_{\mathfrak{m}}(\mathrm{Ad}(y^{-1})X) >=< \varphi, \mathrm{Ad}(y^{-1})\, X >=< \mathrm{Ad}(y)\, \varphi, X > .$$

The result then follows. □

An element $V \in \mathfrak{a} + \mathfrak{m}$ is called *regular* if $\mathrm{Det}(\mathrm{ad}\, V|_{\mathfrak{n}}) \neq 0$. If $V \in \mathfrak{a}$ then the condition that V is regular is just that $\lambda(V) \neq 0$ for each $\lambda > 0$.

Lemma 4.2.

1) *If V_1 is a regular element of $\mathfrak{a}$ and $V_2 \in \mathfrak{m}$ then $V = V_1 + V_2$ is regular.*
2) *Let $G(\xi_0)$ be the stabilizer of ξ_0 in G. Then $G(\xi_0) = A\, M(\xi_2)$ where $M(\xi_2)$ is the stabilizer of ξ_2 in M.*

Proof. 1) For each $\lambda > 0$, we have $(\mathrm{ad}\, V)(\mathfrak{g}_\lambda) \subset \mathfrak{g}_\lambda$. Set $v_\lambda = (\mathrm{ad}\, V)|_{\mathfrak{g}_\lambda}$. Then, for each $X \in \mathfrak{g}_\lambda$, we have $v_\lambda(X) = \lambda(V_1)X + (\mathrm{ad}\, V_2)(X)$. Thus, since $\mathrm{ad}\, V_2$ is skew-symmetric with respect to $(\, X, Y\,) =< X, \theta(Y) >$, the eigenvalues of v_λ are of the form $\lambda(V_1) + ir$ where $\lambda(V_1) \in \mathbb{R} \setminus (0)$ and $r \in \mathbb{R}$. Then $\mathrm{Det}(v_\lambda) \neq 0$ for each $\lambda > 0$. Hence $\mathrm{Det}(\mathrm{ad}\, V|_{\mathfrak{n}}) \neq 0$.

2) We argue as in [24], p. 126, Remark (2). Let $\hat{M}$ be the normalizer of $\mathfrak{a}$ in K. Let $g \in G(\xi_0)$. We can write $g = k \exp X$ with $k \in K$, $X \in \mathfrak{p}$. Applying θ, we see that $k \exp(-X) \in \hat{M}$. Then $\exp(2X) \in \hat{M}$ i.e. $\mathrm{Ad}(\exp(2X))(\mathfrak{a}) \subset \mathfrak{a}$. Since $\mathrm{Ad}(\exp(2X))$ is positive definite on $\mathfrak{g}$, its Hermitian logarithm $\mathrm{ad}(2X)$ is a polynomial in $\mathrm{Ad}(\exp(2X))$. Thus $\mathrm{ad}(2X)(\mathfrak{a}) \subset \mathfrak{a}$ and we easily verify that $X \in \mathfrak{a}$. Then $\mathrm{Ad}(k)(\xi_0) = \xi_0$ or, equivalently, $\mathrm{Ad}(k)(\xi_1 + \xi_2) = \xi_1 + \xi_2$. Since $\mathrm{Ad}(k)\xi_1 \in \mathfrak{k}$ and $\mathrm{Ad}(k)\xi_2 \in \mathfrak{p}$, this gives $\mathrm{Ad}(k)\xi_1 = \xi_1$ and $\mathrm{Ad}(k)\xi_2 = \xi_2$. But $\mathrm{Ad}(k)\xi_1 = \xi_1$ implies that $\mathrm{Ad}(k)\mathfrak{a} \subset \mathfrak{a}$. In other words, k lies in the normalizer $\hat{M}$ of $\mathfrak{a}$ in K. Let $C(\xi_1)$ the Weyl chamber of $\mathfrak{a}$ containing ξ_1. We have $\mathrm{Ad}(k)C(\xi_1) = C(\xi_1)$. Since the Weyl group $\hat{M}/M$ acts simply transitively on the set of the Weyl chambers of $\mathfrak{a}$, we obtain $k \in M$. Hence $k \in M(\xi_2)$. □

The following proposition can be found in [14] (see also [4]). Here we present an elementary proof.

Proposition 4.3. *The map Ψ_0 defined by $\Psi_0(y, Z, \varphi) = \mathrm{Ad}(y)\, (\xi_1 + \varphi + \theta(Z))$ is a diffeomorphism from $\bar{N} \times \bar{\mathfrak{n}} \times \mathcal{O}(\xi_2)$ onto $\tilde{\mathcal{O}}(\xi_0)$.*

Proof. Recall that if $V \in \mathfrak{a}+\mathfrak{m}$ is regular then the map $Y \to \mathrm{Ad}(\exp Y)\, V - V$ is a diffeomorphism of $\mathfrak{n}$ [28]. Let $(y, Z, \varphi) \in \bar{N} \times \bar{\mathfrak{n}} \times \mathcal{O}(\xi_2)$. Write $\varphi = \mathrm{Ad}(m)\xi_2$ with $m \in M$. Since the element $\xi_1 + \varphi = \mathrm{Ad}(m)\xi_0$ is regular, there exists an element $z \in N$ such that $\xi_1 + \varphi + \theta(Z) = \mathrm{Ad}(zm)\xi_0$. Then $\Psi_0(y, Z, \varphi) = \mathrm{Ad}(yzm)(\xi_0)$. Thus Ψ_0 takes values in $\tilde{\mathcal{O}}(\xi_0)$. By the same arguments, we show that Ψ_0 is onto. Now, suppose that $\Psi_0(y, Z, \varphi) = \Psi_0(y', Z', \varphi')$. As before, we write $\Psi_0(y, Z, \varphi) = \mathrm{Ad}(yzm)(\xi_0)$ and $\Psi_0(y', Z', \varphi') = \mathrm{Ad}(y'z'm')(\xi_0)$ with $m \in M$, $m' \in M$, $z \in N$ and $z' \in N$. Then $(z'm')^{-1}(y'^{-1}y)zm$ lies in $G(\xi_0)$. By Lemma 4.2., there exists $m_0 \in M(\xi_2)$ and $a \in A$ such that $(z'm')^{-1}(y'^{-1}y)zm = m_0 a$. By unicity in the $\bar{N}MAN$-decomposition, we have $y'^{-1}y = e$ and $m = m'm_0$. Then $y = y'$, $\varphi = \varphi'$ and $Z = Z'$. This shows that Ψ_0 is injective.

Now we show that Ψ_0 is regular. Let $Y \in \bar{\mathfrak{n}}$, $T \in \bar{\mathfrak{n}}$ and $\phi \in \mathfrak{m}$ such that

$$d\Psi_0(y, Z, \varphi)(Y^+(y), T, \phi^+(\varphi)) = 0.$$

Since

$$\begin{aligned} d\Psi_0(y, Z, \varphi)(Y^+(y), T, \phi^+(\varphi)) &= \frac{d}{dt}\Big(\mathrm{Ad}(\exp(tY)y)(\xi_1 + \exp(t\phi)\varphi + \theta(Z + tT))\Big)\Big|_{t=0} \\ &= \mathrm{Ad}(y)([\phi, \varphi] + \theta(T) + [Y, \Psi_0(y, Z, \varphi)]) \end{aligned}$$

we have

$$[\phi, \varphi] + \theta(T) + [\mathrm{Ad}(y^{-1})Y, \xi_1 + \varphi + \theta(Z)] = 0. \tag{4.2}$$

Since $[\bar{\mathfrak{n}}, \mathfrak{a} + \mathfrak{m} + \mathfrak{n}] \subset \mathfrak{a} + \mathfrak{n} + \bar{\mathfrak{n}}$, we get $\phi^+(\varphi) = [\phi, \varphi] = 0$. Write $\varphi = \mathrm{Ad}(m)\xi_2$ with $m \in M$ and set $T' = \mathrm{Ad}(m^{-1})T \in \mathfrak{m}$, $Y' = \mathrm{Ad}(m^{-1})\,\mathrm{Ad}(y^{-1})Y \in \bar{\mathfrak{n}}$ and $Z' = \mathrm{Ad}(m^{-1})Z \in \bar{\mathfrak{n}}$. Then (4.2) can be rewritten as

$$\theta(T') + [Y', \xi_0] + [Y', \theta(Z')] = 0. \tag{4.3}$$

If $Y' \neq 0$ then we can write $Y' = Y'_\lambda + \sum_{\mu > \lambda} Y'_\mu$ where $Y'_\lambda \in \mathfrak{g}_\lambda \setminus (0)$ and $Y'_\mu \in \mathfrak{g}_\mu$ for each $\mu > \lambda$. But the first member of (4.3) lies in $\mathfrak{a} + \mathfrak{g}_\lambda + \sum_{\mu > \lambda} \mathfrak{g}_\mu$ (direct sum) and its component on $\mathfrak{g}_\lambda$ is $[Y'_\lambda, \xi_0] = -\lambda(\xi_0)Y'_\lambda$. This is a contradiction because ξ_0 is regular. Then $Y = 0$. Hence $T = 0$. □

Lemma 4.4. *Let $Y \in \bar{\mathfrak{n}}$. Then the map*

$$\eta: \quad Z \to p_{\mathfrak{n}}\Big(\frac{\mathrm{ad}\, Y}{e^{\mathrm{ad}\, Y} - 1}\,(Z)\Big)$$

is a diffeomorphism of $\mathfrak{n}$.

Proof. By using the decomposition $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{n} \oplus \bar{\mathfrak{n}}$, we easily show that the inverse map of η is $Z \to p_{\mathfrak{n}}\Big(\frac{e^{\mathrm{ad}\, Y} - 1}{\mathrm{ad}\, Y}\,(Z)\Big)$. □

Let ω_0 and ω_2 be the Kirillov 2-forms on $\mathcal{O}(\xi_0)$ and $\mathcal{O}(\xi_2)$, respectively. Denote by $\{\cdot, \cdot\}_0$ and $\{\cdot, \cdot\}_2$ the Poisson brackets associated with ω_0 and ω_2. We equip $\bar{\mathfrak{n}}^2$ with the symplectic form $dY \wedge dZ := \sum_{k=1}^n dY_k \wedge dZ_k$. The corresponding Poisson bracket on $C^\infty(\bar{\mathfrak{n}}^2)$ is

$$\{f\,,\, g\} = \sum_{k=1}^{n} \left(\frac{\partial f}{\partial Y_k}\frac{\partial g}{\partial Z_k} - \frac{\partial f}{\partial Z_k}\frac{\partial g}{\partial Y_k} \right). \tag{4.4}$$

We form the symplectic product $\bar{\mathfrak{n}}^2 \times \mathcal{O}(\xi_2)$. We denote by $\{\cdot, \cdot\}_1$ the Poisson bracket associated with the symplectic form $\omega_1 := (dT \wedge dS) \otimes \omega_2$. Let u, $v \in C^\infty(\bar{\mathfrak{n}}^2)$ and a, $b \in C^\infty(\mathcal{O}(\xi_2))$. Observe that for $f(Y, Z, \varphi) = u(Y, Z)a(\varphi)$ and $g(Y, Z, \varphi) = v(Y, Z)b(\varphi)$ we have

$$\{f, g\}_1 = u(Y, Z)v(Y, Z)\{a,\, b\}_2 + a(\varphi)b(\varphi)\{u, v\}. \tag{4.5}$$

Lemma 4.5. *Suppose that f and g are two P-symbols on $\bar{\mathfrak{n}}^2 \times \mathcal{O}(\xi_2)$ of the form*

$$u(Y) + < v(Y), \varphi > + \sum_{k=1}^{n} w_k(Y) Z_k$$

where $u \in C^\infty(\bar{\mathfrak{n}})$, $v \in C^\infty(\bar{\mathfrak{n}}, \mathfrak{m})$ and $w_k \in C^\infty(\bar{\mathfrak{n}})$ for $k = 1, 2, \ldots, n$. Then

$$[\mathcal{W}(f) \, , \, \mathcal{W}(g)] = -i \, \mathcal{W}(\{f \, , \, g\}_1). \tag{4.6}$$

Proof. Direct computation using (3.4), (3.5) and the fact that if $f =< v(Y), \varphi >$ then $(W(f)\phi)(Y) = -i \, d\rho(v(Y))\phi(Y)$. □

Theorem 4.6. *The map Ψ introduced in Proposition 4.1 is a symplectomorphism from $(\bar{\mathfrak{n}}^2 \times \mathcal{O}(\xi_2) \, , \, \omega_1)$ onto $(\tilde{\mathcal{O}}(\xi_0) \, , \, \omega_0|_{\tilde{\mathcal{O}}(\xi_0)})$.*

Proof. By combining Proposition 4.3 and Lemma 4.4, we see that Ψ is a diffeomorphism from $\bar{\mathfrak{n}}^2 \times \mathcal{O}(\xi_2)$ onto $\tilde{\mathcal{O}}(\xi_0)$. Recall that for $X \in \mathfrak{g}$, $\tilde{X}$ denotes the function on $\mathcal{O}(\xi_0)$ defined by $\tilde{X}(\xi) =< \xi, X >$. Observe that $f_X = \tilde{X} \circ \Psi$.

Let X and Y in $\mathfrak{g}$. Then by Proposition 3.4 and Lemma 4.5 we have

$$[\mathcal{W}(f_X) \, , \, \mathcal{W}(f_Y)] = -i\mathcal{W}(\{f_X \, , \, f_Y\}_1).$$

On the other hand

$$[\mathcal{W}(f_X) \, , \, \mathcal{W}(f_Y)] = [-id\pi(X), -id\pi(Y)] = -d\pi([X \, , \, Y]) = -i\mathcal{W}(f_{[X,Y]}).$$

Hence we obtain $f_{[X,Y]} = \{f_X \, , \, f_Y\}_1$. Since $\widetilde{[X,Y]} = \{\tilde{X}, \tilde{Y}\}_0$ we have finally

$$\{\tilde{X}, \tilde{Y}\}_0 \circ \Psi = \{\tilde{X} \circ \Psi \, , \, \tilde{Y} \circ \Psi\}_1.$$

This implies that Ψ is a symplectomorphism. □

In order to construct an adapted Weyl transform on $\mathcal{O}(\xi_0)$ we transfer to $\mathcal{O}(\xi_0)$ the Berezin-Weyl calculus on $\bar{\mathfrak{n}}^2 \times \mathcal{O}(\xi_2)$. We say that a smooth function f on $\mathcal{O}(\xi_0)$ is a symbol on $\mathcal{O}(\xi_0)$ if $f \circ \Psi$ is a symbol for the Berezin-Weyl calculus on $\bar{\mathfrak{n}}^2 \times \mathcal{O}(\xi_2)$. We say that f is a P-symbol (or an S-symbol) on $\mathcal{O}(\xi_0)$ if $f \circ \Psi$ is a P-symbol (or an S-symbol) on $\bar{\mathfrak{n}}^2 \times \mathcal{O}(\xi_2)$.

Theorem 4.7. *Let $\mathcal{A}$ be the space of P-symbols on $\mathcal{O}(\xi_0)$ and let $\mathcal{B}$ be the space of differential operators on $\bar{\mathfrak{n}}$ with coefficients in $C^\infty(\bar{\mathfrak{n}}, E)$. Then the map $W : \mathcal{A} \to \mathcal{B}$ that assigns to each $f \in \mathcal{A}$ the operator $\mathcal{W}(f \circ \Psi)$ on $L^2(\bar{\mathfrak{n}}, E)$ is an adapted Weyl correspondence in the sense of Definition 1.1.*

Proof. Properties (i), (ii) and (iii) of the definition of an adapted Weyl correspondence are clearly satisfied with $D = C_0(\bar{\mathfrak{n}}, E)$.

Property (iv) follows from Proposition 2.1 2) and from the similar result for the usual Weyl calculus.

Finally, property (v) is just a reformulation of Proposition 3.4. □

5. Comparison with the symbolic calculus of [4].

In this section, we first describe the construction of the adapted Weyl transform on $\mathcal{O}(\xi_0)$ introduced in [4]. We retain the notation of the previous sections. In [4], a symbolic calculus on $\bar{N}\times\bar{\mathfrak{n}}\times\mathcal{O}(\xi_2)$ is defined as follows. A smooth function $f\,:\,(y,Z,\varphi)\rightarrow f(y,Z,\phi)$ is said to be a symbol on $\bar{N}\times\bar{\mathfrak{n}}\times\mathcal{O}(\xi_2)$ if for each $(y,Z)\in\bar{N}\times\bar{\mathfrak{n}}$ the function $\phi\rightarrow f(y,Z,\varphi)$ is the symbol in the Berezin calculus on $\mathcal{O}(\xi_2)$ of an operator on E denoted by $\hat{f}(y,Z)$. As in Section 3, we define the notions of S-symbol and P-symbol on $\bar{N}\times\bar{\mathfrak{n}}\times\mathcal{O}(\xi_2)$. We associate with any P-symbol f on $\bar{N}\times\bar{\mathfrak{n}}\times\mathcal{O}(\xi_2)$ the operator $\mathcal{W}_0(f)$ on $C_0(\bar{N},E)$ defined by the integral formula

$$\mathcal{W}_0(f)(\psi)(y)=(2\pi)^{-n}\int\int_{\bar{\mathfrak{n}}\times\bar{\mathfrak{n}}}e^{i(T,Z)}\hat{f}(y\exp(T/2),Z)\psi(y\exp T)dTdZ. \tag{5.1}$$

As for the Weyl calculus, one can extend $\mathcal{W}_0$ to P-symbols. More precisely, let f be the P-symbol on $\bar{N}\times\bar{\mathfrak{n}}\times\mathcal{O}(\xi_2)$ defined by $f(y,Z,\varphi)=u(y)Z^{\alpha}$ with $u\in C^{\infty}(\bar{N},E)$. Then we have

$$\mathcal{W}_0(f)(\psi)(y)=\Big(i\frac{\partial}{\partial Z}\Big)^{\alpha}(u(y\exp{(T/2)})\,\psi(y\exp T))\Big|_{Z=0}. \tag{5.2}$$

In particular, if $f(y,Z,\phi)=u(y)$ then $\mathcal{W}_0(f)(\psi)(y)=u(y)\psi(y)$ and if $f(y,Z,\varphi)=(\,v(y),Z\,)$ with $v\in C^{\infty}(\bar{N},E)$ then

$$\mathcal{W}_0(f)(\psi)(y)=i\Big(\frac{d}{dt}\sum_{k=1}^{n}\Big(E_k,v(y\ \exp\ \frac{t}{2}E_k)\Big)\Big|_{t=0}\psi(y)+\frac{d}{dt}\psi(y\exp(tv(y))\Big|_{t=0}\Big). \tag{5.3}$$

Now we say that a smooth function f on $\mathcal{O}(\xi_0)$ is a P-symbol (resp. an S-symbol) on $\mathcal{O}(\xi_0)$ if $f\circ\Psi_0$ is a P-symbol (resp. an S-symbol) on $\bar{N}\times\bar{\mathfrak{n}}\times\mathcal{O}(\xi_2)$. For each P-symbol (or S-symbol) f on $\mathcal{O}(\xi_0)$ we set $W_0(f)=\mathcal{W}_0(f\circ\Psi_0)$. We have shown in [4] that the map W_0 is then a Weyl correspondence on $\mathcal{O}(\xi_0)$ adapted to the realization $\hat{\pi}$ of the principal series representation (see (2.1)). In particular, we have $d\hat{\pi}(X)\psi=W_0(i\tilde{X})\psi$ for $X\in\mathfrak{g}$ and $\psi\in C_0(\bar{N},E)$ ([4], Proposition 6).

Note that if f is a function on $\mathcal{O}(\xi_0)$ then

$$(f\circ\Psi)(Y,Z,\varphi)=(f\circ\Psi_0)(\exp Y,\theta\Big(p_{\mathfrak{n}}\Big(\frac{\operatorname{ad}Y}{e^{\operatorname{ad}Y}-1}\,\theta(Z)\Big)\Big),\varphi). \tag{5.4}$$

Then we see that f is a P-symbol (resp. an S-symbol) for W_0 if and only f is a P-symbol (resp. an S-symbol) for W. Recall that the unitary operator B from $L^2(\bar{\mathfrak{n}},E)$ to $L^2(\bar{N},E)$ defined by $B(\phi)(\exp Y)=\phi(Y)$ is an intertwining operator between the representations π and $\hat{\pi}$, that is, $B\pi(g)=\hat{\pi}(g)B$ for each $g\in G$. Then, noting that B maps $C_0(\bar{\mathfrak{n}},E)$ onto $C_0(\bar{N},E)$, we have $Bd\pi(X)=d\hat{\pi}(X)B$

on $C_0(\bar{\mathfrak{n}},E)$ for each $X\in\mathfrak{g}$. Equivalently, we have $B\,W(\tilde{X})=W_0(\tilde{X})B$ on $C_0(\bar{\mathfrak{n}},E)$ for each $X\in\mathfrak{g}$. Now, we say that f is a P-symbol of degree ≤ 1 for W_0 (resp. W) if $f\circ\Psi_0$ (resp. $f\circ\Psi$) is a polynomial in the variable Z of degree ≤ 1. It is immediate that f is a P-symbol of degree ≤ 1 for W_0 if and only if f is a P-symbol of degree ≤ 1 for W. In particular, for each $X\in\mathfrak{g}$, the function $\tilde{X}$ is P-symbol of degree ≤ 1 for W_0 and W. We will prove that if f is a P-symbol of degree ≤ 1 then the relation $B\,W(f)=W_0(f)B$ holds.

For $Z\in\bar{\mathfrak{n}}$, we denote by Z^- the vector field on $\bar{N}$ defined by $Z^-(y)=\frac{d}{dt}(y\exp(tZ))|_{t=0}$.

Lemma 5.1. *Let $U\in C^\infty(\bar{\mathfrak{n}},\bar{\mathfrak{n}})$. Define $u\in C^\infty(\bar{\mathfrak{n}},\bar{N})$ by $u(\exp Y)=\frac{1-e^{-\operatorname{ad}Y}}{\operatorname{ad}Y}U(Y)$. For $Y\in\bar{\mathfrak{n}}$, let γ_Y be the endomorphism of $\bar{\mathfrak{n}}$ defined by $\gamma_Y(Z)=du(\exp Y)(Z^-(\exp Y))$. Then we have*

$$Tr_{\bar{\mathfrak{n}}}(\gamma_Y)=\mathrm{Tr}_{\bar{\mathfrak{n}}}(dU(Y))$$

for each $Y\in\bar{\mathfrak{n}}$.

Proof. By differentiating the relation $U(Y)=\frac{\operatorname{ad}Y}{1-e^{-\operatorname{ad}Y}}u(\exp Y)$, we obtain, for Y and Z in $\bar{\mathfrak{n}}$

$$\begin{aligned}dU(Y)(Z)=\ &\sum_{m\geq 0}b_m\sum_{r=0}^{m-1}(\operatorname{ad}Y)^r\operatorname{ad}Z(\operatorname{ad}Y)^{m-r-1}u(\exp Y)\\ &+\ \frac{\operatorname{ad}Y}{1-e^{-\operatorname{ad}Y}}\gamma_Y\left(\frac{1-e^{-\operatorname{ad}Y}}{\operatorname{ad}Y}\right)Z\end{aligned}$$

where $\frac{z}{1-e^{-z}}=\sum_{m\geq 0}b_mz^m$.
Now, we note first that the endomorphism of $\bar{\mathfrak{n}}$ defined by

$$Z\to(\operatorname{ad}Y)^r\operatorname{ad}Z(\operatorname{ad}Y)^{m-r-1}u(\exp Y)=-\operatorname{ad}Y\operatorname{ad}((\operatorname{ad}Y)^{m-r-1}u(\exp Y))\,Z$$

is nilpotent then its trace is zero. Secondly, we have

$$Tr_{\bar{\mathfrak{n}}}\left(\frac{\operatorname{ad}Y}{1-e^{-\operatorname{ad}Y}}\circ\gamma_Y\circ\frac{1-e^{-\operatorname{ad}Y}}{\operatorname{ad}Y}\right)=Tr_{\bar{\mathfrak{n}}}(\gamma_Y).$$

The lemma then follows. □

Proposition 5.2. *Let f be a P-symbol of degree ≤ 1 for W and W_0. Then we have $B\,W(f)=W_0(f)B$ on $C_0(\bar{\mathfrak{n}},E)$.*

Proof. Using the relations (3.5) and (5.3), we see that the result is a consequence of Lemma 5.1. □

Proposition 5.3. *Assume that $[\bar{\mathfrak{n}},[\bar{\mathfrak{n}},\bar{\mathfrak{n}}]]=(0)$. Let f be an S-symbol or a P-symbol for W. Then we have $B\,W(f)=W_0(f)B$ on $C_0(\bar{\mathfrak{n}},E)$.*

Proof. Suppose first that f is an S-symbol for W and W_0. We express $W(f)(\psi \circ \exp)(Y)$ as an integral (see (3.2)) which we transform by means of the change of variables $(T, Z) \to (T', Z')$ where $T' = \log(\exp(-Y)\exp(Y+T))$ and

$$Z' = \theta\Big(p_{\mathfrak{n}}\Big(\frac{\mathrm{ad}(Y+\frac{T}{2})}{e^{\mathrm{ad}(Y+\frac{T}{2})}-1}\,\theta(Z)\Big)\Big).$$

Since $[\bar{\mathfrak{n}}, [\bar{\mathfrak{n}}, \bar{\mathfrak{n}}]] = (0)$, we have $T' = T - \frac{1}{2}[Y, T]$ and we easily verify that $W(f)(\psi \circ \exp)(Y) = W_0(f)(\psi)(\exp Y)$. Then we obtain the result for S-symbols and thus, following [27], the result for P-symbols. □

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