Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 48 (2007), No. 1, 151-173.

# The Possibility of Extending Factorization Results to Infinite Abelian Groups

**Arthur D. Sands    Sándor Szabó**

*Department of Mathematics, University of Dundee*
*Dundee DD1 4HN, Scotland, UK*

*Institute of Mathematics and Informatics, University of Pécs*
*Ifjúság u. 6, 7624 Pécs, Hungary*

**Abstract.** We shall consider three results on factoring finite abelian groups by subsets. These are the Hajós', Rédei's and simulation theorems. As L. Fuchs has done in the case of Hajós' theorem we shall obtain families of infinite abelian groups to which these results cannot be extended. We shall then describe classes of infinite abelian groups for which the extension does hold.

MSC 2000: 20K99 (primary); 52C22 (secondary)
Keywords: factorization of finite and infinite abelian groups, Hajós-Rédei theory

## 1. Introduction

Throughout the paper the word group will be used to mean additive abelian group. Let $A_i$, $i \in I$ be a family of subsets of a group $G$ with $0 \in A_i$ for each $i$. If each element $g \in G$ can be written uniquely as

$$g = \sum a_i, \quad a_i \in A_i$$

and only a finite number of elements $a_i$ being non-zero, then

$$G = \sum A_i$$

is called a *factorization* of $G$. If $A$ is a finite subset of $G$ and $g$ is an element of finite order of $G$, then $|A|$ and $|g|$ denote, respectively the orders of $A$ and of $g$. A subset $A$ of $G$ is said to be *cyclic* if there is an element $a$ of $G$ and an integer $r$ with $2 \leq r \leq |a|$ such that

$$A = \{0, a, 2a, \dots, (r-1)a\}.$$

Clearly $A$ is a subgroup if and only if $r = |a|$. G. Hajós proved a long standing conjecture of H. Minkowski by showing that this geometric conjecture was equivalent to the statement that, in any factorization of a finite group $G$ into cyclic subsets, one of the subsets must be a subgroup and then proving that this is so [6].

L. Rédei proved that in any factorization of a finite group $G$ into factors of prime order one of the factors must be a subgroup [7].

Rédei's theorem is a generalization of Hajós' theorem. If $r = st$, $s \geq 2$, $t \geq 2$, then

$$\{0, a, 2a, \dots, (r-1)a = \{0, a, 2a, \dots, (s-1)a\} + \{0, sa, 2sa, \dots, (t-1)sa\}.$$

The first set is a subgroup if and only if $|a| = r$. This holds if and only if $|sa| = t$ and so if and only if the last set is a subgroup. By continuing in this way each cyclic set is seen to be a sum of cyclic sets of prime order. The original set is a subgroup if and only if one of these sets of prime order is a subgroup.

A subset $A$ of order at least 3 is said to be *simulated* by a subgroup $H$ of a group $G$ if either $A = H$ or there is exactly one element of $A$ not in $H$ and exactly one element of $H$ not in $A$. Thus in this second case

$$H = (A \cap H) \cup \{h\}, \quad A = (A \cap H) \cup \{h + d\}, \quad h \in H \setminus \{0\}, \quad d \notin H.$$

In the finite case an equivalent definition is that $|A| = |H| \leq |A \cap H| + 1$. As usual it is assumed that $0 \in A$. The case $|A| = 2$ is omitted because every subgroup of order 2 would simulate every subset $\{0, a\}$. In addition in the $|A| = 2$ case $A$ is a cyclic subset.

In [3] it is shown that if a finite group $G$ is a direct sum of simulated subsets, then one of these subsets must be a subgroup.

In [5, 85.1] it is shown that each group $G$ may be decomposed into a sum of cyclic subsets of prime order. Thus it makes sense, for each group, to ask if the Hajós' or Rédei's theorem holds true.

A group $G$ will be said to satisfy Hajós' theorem if in every decomposition of $G$ into a direct sum of cyclic subsets one of these subsets must be a subgroup. The group $G$ will be said to satisfy Rédei's theorem if in every decomposition of $G$ into a direct sum of subsets of prime order one of these subsets must be a subgroup. All finite groups belong to both classes. In the infinite case examples will be presented to show that there are groups which satisfy Hajós' theorem but do not satisfy Rédei's theorem.

The standard definitions of abelian group theory as found in Fuchs [5] will be used. The symbol $Z(n)$ will denote the cyclic group of order $n$. If $p$ is a prime, then $Z(p^\infty)$ will denote the quasi-cyclic or Prüferian group belonging to $p$. (See Fuchs [5, §4].) $\langle A\rangle$ will be used to denote the subgroup generated by a subset $A$.

L. Fuchs [4, 5] has shown that any group satisfying Hajós' theorem must be of the form

$$G = F + \sum_{i=1}^{s} Z(p_i^\infty) + \sum_{\mu} Z(p),$$

where $F$ is finite $p, p_1, \dots, p_s$ are primes and $\mu$ is any cardinal. It will be shown that if

$$G = F + \sum_{i=1}^{s} Z(p_i^\infty),$$

where $F$ is a finite group and $p_1, \dots, p_s$ are distinct primes not dividing $|F|$, then Rédei's theorem, and hence Hajós' theorem, holds for $G$. Fuchs [4, 5] claims that Hajós' theorem holds for groups $G$ of type

$$F + \sum_{\mu} Z(p),$$

where $F$ is finite and $p$ is any prime and $\mu$ is any cardinal. We believe that the condition $p$ does not divide $|F|$ is needed for his proof to hold and shall present a modification of his proof. Clearly the groups

$$\sum_{\mu} Z(p)$$

satisfy Hajós' theorem. We shall show that, if $p \geq 3$ and $\mu$ is an infinite cardinal, then they do not satisfy Rédei's theorem. We shall show that if

$$G = F + \sum_{\mu} Z(2),$$

where $F$ is a finite group of odd order and $\mu$ is any cardinal, then $G$ satisfy Rédei's theorem.

A group $G$ is said to satisfy the simulation theorem if in every decomposition of $G$ into a direct sum of simulated factors one factor must be a subgroup. In this case the factors may be infinite. It is shown that in any factorization of this type involving only a finite number of factors one of the factors must be a subgroup. It is deduced that all subgroups of groups of the form

$$\sum_{i=1}^{s} Z(p_i^\infty) + \sum_{\mu} Q,$$

where $p_1, \dots, p_s$ are primes and $\mu$ is a finite cardinal, satisfy the simulation theorem. A group which is an infinite direct sum of non-zero subgroups will be shown not to satisfy the simulation theorem.

A subset $A$ of a group $G$ is said to be *periodic* if there exist $g \in G$, $g \neq 0$, such that $g + A = A$. The set of all such periods together with 0 forms a subgroup $H$

of $G$. If $K$ is any subgroup of $H$, then $A$ is a union of cosets of $K$. Equivalently there is a subset $D$ of $G$ such that $A = K + D$, where the sum is direct. Any periodic set which is cyclic or has prime order or is simulated must itself be a subgroup of $G$.

A finite group $G$ is said to have the Hajós $k$-property if in any factorization

$$G = A_1 + \cdots + A_k,$$

at least one of the factors $A_i$ is periodic.

## 2. Preliminary results

Results are presented which will be used later or are of more general interest.

**Lemma 1.** *If $A$ is a direct factor of a group $G$ and $A + d \subseteq A$ for some $d \in G$, $d \neq 0$, then $d$ is a period of $A$.*

*Proof.* Let $B$ be a subset such that $G = A + B$, where the sum is direct. Let $a \in A$. Now $A + d + B = G + d = G$. So there exist $a' \in A$, $b \in B$ with $(a' + d) + b = a + 0$. Now $a' + d \in A$. Since the sum of $A$ and $B$ is direct it follows that $a' + d = a$, $b = 0$. Hence $A + d = A$ and $d$ is a period of $A$.

It should be noted that some condition on $A$ is required. If $G = Z$ and $A$ is the set of positive integers, then $A + d \subseteq A$ for all $d \in A$, but $A$ is not periodic.

**Lemma 2.** *If a finite subset $A$ is a direct factor of a torsion group $G$, then $|A|$ divides $\big|\langle A\rangle\big|$.*

*Proof.* Let $G = A + B$. Since $A \subseteq \langle A\rangle$ it follows that $\langle A\rangle = A + \big(B \cap \langle A\rangle\big)$. Since $G$ is a torsion group and $A$ is finite it follows that $\langle A\rangle$ is finite. Hence $|A|$ divides $\big|\langle A\rangle\big|$.

In particular if $\{0, a, 2a, \ldots, (r-1)a\}$ is a cyclic direct factor of a torsion group $G$, then $r$ divides $|a|$.

A direct factor $A$ of a group $G$ is said to be *replaceable* by a subset $D$ of $G$ if whenever $G = A + B$ is a factorization, then so also is $G = D + B$. In Proposition 3 [10] it is shown that if $A$ is a finite direct factor of $G$ and $k$ is relatively prime to $|A|$, then $A$ may be replaced by $kA = \{ka : a \in A\}$. Two consequences of this result are now presented.

Let $G$ be a torsion group. Then if the set $P$ of primes is expressed as a disjoint union $P_1 \cup P_2$, $G$ is a direct sum of the subgroups

$$H = \sum_{p \in P_1} G_p, \quad K = \sum_{p \in P_2} G_p.$$

Let $A$ be a subset of $G$. Then each $a \in A$ may be written uniquely as

$$a = a_H + a_K, \quad a_H \in H, \quad a_K \in K.$$

This gives rise to subsets

$$A_H = \{a_H : a \in A\}, \quad A_K = \{a_K : a \in A\}.$$

With these conditions satisfied the following replacement results hold.

**Theorem 1.** *Let $A$ be a finite direct factor of a torsion group $G$. Let $P$ be the set of primes and let $P_1 = \{p \in P : p \text{ divides } |A|\}$. Then $A$ may be replaced by $A_H$.*

*Proof.* Let the lowest common multiple of the orders of the elements in $A_H$ be $n$ and let the lowest common multiple of the orders of the elements in $A_K$ be $m$. Then $m$ and $n$ are relatively prime. Hence there exists $k$ such that $mk \equiv 1 \pmod{n}$. By Proposition 3 [10] $A$ may be replaced by $mkA$. Let $a = a_H + a_K$, where $a \in A$. Then $mka_K = 0$ and $mka_H = a_H$. Hence $mkA = A_H$, as required.

We should note that Proposition 3 [10] also implies that $|A| = |A_H|$, that is, the elements $a_H$, $a \in A$ are distinct. This implies that $A \cap K = \{0\}$ as $a_H = 0$ implies $a = 0$.

**Theorem 2.** *Let $A$ be a finite direct factor of a torsion group $G$. Let $P$ be the set of primes and let $P_1 = \{p \in P : p \text{ divides } |A|\}$. Then $A$ may be replaced by a set $D$ such that $A_H = D_H$ and each non-zero $d_K$, $d \in D$, has prime order $q$, where $q$ depends only on $A$.*

*Proof.* Let the lowest common multiple of the orders of the elements in $A_H$ be $n$ and let the lowest common multiple of the orders of the elements in $A_K$ be $m$. If $m = 1$, then $A = A_H$ and we may choose $A = D$. Otherwise let $q$ be a prime factor of $m$. Then there exists $l$ such that $(m/q)l \equiv 1 \pmod{n}$. If $q$ divides $l$ we may replace $l$ by $l + n$. Thus we may assume that $l$ has been chosen in such a way that $q$ does not divide $l$. By Proposition 3 [10] we may replace $A$ by $D = (m/q)lA$. Then $A_H = D_H$. Let $a \in A$. If $(m/q)a_K = 0$, then the corresponding element $d_K = 0$, where $d = (m/q)la$. Since $m$ is the lowest common multiple of the orders of the elements in $A_K$ there exists $a \in A$ such that $(m/q)a_K \neq 0$. Thus $a_K$ has order $q$. Since $q$ does not divide $l$ it follows that $d_K = (m/q)la_K$ has order $q$. Thus $D$ has the required property.

Again we should note that $|D| = |A|$. We should note also that in this case $A$ is a subgroup of $G$ if and only if $A_H$ is a subgroup of $G$ and $A = A_H$. So if $A$ is not a subgroup of $G$ it is being replaced by a subset which is also not a subgroup of $G$.

If $G = \sum A_i$ is a factorization of a group $G$ and $A_i$ is replaceable by a subset $B_i$ for each $i$, then clearly any finite direct sum $A_{i(1)} + \cdots + A_{i(k)}$ may be replaced by $B_{i(1)} + \cdots + B_{i(k)}$. Also the sum $\sum B_i$ is direct since any $g$ in $\sum B_i$ belongs to such a finite direct sum. In general we cannot claim that $\sum B_i = G$. For example if $G = Z$, then

$$Z = \{0, 1\} + \{0, -2\} + \{0, 4\} + \cdots .$$

Here $A_i = \{0, (-2)^{i-1}\}$. We may replace each $A_i$ by $3A_i$. Clearly $\sum 3A_i = 3Z \neq Z$.

If $A$ is a subset of a torsion group $G$ and $p$ is a prime we shall use $(A)_p$ rather than $A_{G_p}$ to denote the subset consisting of the $p$-components of the elements of $A$. $(A)_{p'}$ will be used to denote the subset consisting of the complimentary components. If $a$ is an element of a torsion group $G$ and $p$ is a prime we will use $(a)_p$ to denote $a_{G_p}$, that is, the $p$-component of $a$.

**Theorem 3.** *Let $G$ be a torsion group and let*

$$G = \sum_{i \in I} A_i$$

*be a factorization in which each factor has prime power order. If, for some prime $p$, $G_p$ is finite, then*

$$G_p = \sum (A_i)_p,$$

*where the summation is taken over all $i$ such that $|A_i|$ is a power of $p$.*

*Proof.* Let $I_1 = \{i \in I : |A_i| \text{ is a power of } p\}$. Then

$$\sum_{i \in I_1} (A_i)_p$$

is a direct sum contained in $G_p$. Since $G_p$ is finite it follows that $I_1$ is finite. Let $I_2 = I \setminus I_1$. Then

$$G = \sum_{i \in I_1} (A_i)_p + \sum_{i \in I_2} A_i.$$

Let $g \in G_p$. Then there exists $a_i \in A_i$ such that

$$g = \sum_{i \in I_1} (a_i)_p + \sum_{i \in I_2} a_i.$$

Since $g \in G_p$ it follows that

$$g = \sum_{i \in I} (a_i)_p$$

and that

$$\sum_{i \in I_2} (a_i)_{p'} = 0.$$

Now for $i \in I_2$ we may replace $A_i$ by $(A_i)_{p'}$ and so

$$\sum_{i \in I_2} (A_i)_{p'}$$

is a direct sum. Hence $(a_i)_{p'} = 0$ for all $i \in I_2$. We also have that $|A_i| = \left|(A_i)_{p'}\right|$. Thus $(a_i)_{p'} = 0$ implies that $a_i = 0$. Therefore

$$g = \sum_{i \in I_1} (a_i)_p.$$

Thus

$$G_p = \sum_{i \in I_1} (A_i)_p,$$

as required.

We note also that

$$|G_p| = \prod_{i \in I_1} |A_i|.$$

Let $G$ be a torsion group and let $G = \sum_{i \in I} A_i$ be a factorization in which each $|A_i|$ is a power of a prime. Let $P_1$ be a set of primes such that, for each $p \in P_1$, $G_p$ is finite. Let $H = \sum_{p \in P_1} G_p$. Then $H = \sum_{p \in P_1} \sum (A_i)_p$, where the inner summation is taken over all $i$ such that $|A_i|$ is equal to a power of $p$.

**Theorem 4.** *Let $G$ be a torsion group such that every finitely generated subgroup is cyclic and let there be a factorization $G = A_1 + \cdots + A_k + B$, where each $A_i$ has order a power of a given prime $p$. Then one of the factors of $G$ is periodic.*

*Proof.* Let $H = \langle A_1 \cup \cdots \cup A_k \rangle$. Then $H$ is a finite cyclic group. Let $C$ be a complete set of coset representatives for $G$ modulo $H$. Then, for each $c \in C$

$$A_1 + \cdots + A_k + \big(B \cap (H + c)\big) = H + c.$$

There is a translate $B_c$, containing 0, of $B \cap (H+c)$ such that $A_1 + \cdots + A_k + B_c = H$. It follows by Theorem 2 [10] that one of these factors is periodic. If no factor $A_i$ is periodic, then $B_c$ is periodic for each $c \in C$.

In order to complete the proof a closer examination of the proof of Theorem 2 [10] is needed. Each subset $A_i$ may be replaced by its $p$-component. If none of these subsets $(A_i)_p$ is periodic, it is shown in [10] that the unique subgroup $P$ of $H$ of order $p$ is a group of periods of $B_c$. Since

$$B = \bigcup_{c \in C} \big(B \cap (H + c)\big)$$

it follows that $P$ is a group of periods of $B$. If one of the subsets, say $(A_1)_p$, is periodic there is defined a subgroup $K$ depending only on $A_1$. If $K = \{0\}$, then it is shown that $A_1$ is periodic. If $K \neq \{0\}$, then it is shown that $K$ is a group of periods of $B_c$. As above it follows that $B$ is periodic.

**Corollary.** *If $p$ is a prime $G = Z(p^\infty)$ and $G = A_1 + \cdots + A_k + B$ is a factorization in which each $A_i$ is finite, then some factor is periodic.*

*Proof.* Let $H = \langle A_1 \cup \cdots \cup A_k \rangle$. Then $H$ is a finite subgroup and so $H = Z(p^n)$ for some $n$. Since $A_1 + \cdots + A_k + (B \cap H) = H$ it follows that each $|A_i|$ is a power of $p$. The result now follows from Theorem 4.

**Theorem 5.** *If $p$, $q$ are distinct primes $G = Z(p^\infty) + Z(q)$ and $G = A_1 + \cdots + A_k + B$ is a factorization in which all the subsets $A_i$ are finite, then one of the factors of $G$ is periodic.*

*Proof.* If all the factors $A_i$ have order equal to some power of $p$, then the result follows by Theorem 4. If this is not so then $q$ divides $|A_i|$ for some value of $i$. Let then $H = \langle A_1 \cup \cdots \cup A_k \rangle$. So $H = Z(p^n) + Z(q)$ for some $n$. Let $C$ be a complete set of coset representatives for $G$ modulo $H$. Then, for each $c \in C$,

$$A_1 + \cdots + A_k + \big(B \cap (H + c)\big) = H + c.$$

As before there exists $B_c$ containing 0 and equal to a translate of $B \cap (H + c)$ such that $A_1 + \cdots + A_k + B_c = H$. Since $Z(p^n) + Z(q)$ has the Hajós $m$-property (Theorem 2 [9]) it follows that one of these subsets is periodic. If no subset $A_i$ is periodic it follows that $B_c$ is periodic for each $c \in C$. Since $q$ divides $|A_i|$ for some $i$, it follows that $B_c$ has order a power of $p$. Hence the unique subgroup $P$ of order $p$ is a group of periods of $B_c$. As before it follows that $P$ is a group of periods of $B$.

**Theorem 6.** *If $p$, $q$ are distinct primes $G = Z(p^\infty) + Z(q)$ and*

$$G = \sum_{i \in I} A_i$$

*is a factorization in which all the subsets $A_i$ are finite, then one of these factors is periodic.*

*Proof.* Let $H$ be the subgroup of $G$ of order $pq$. Then each element of $H$ is contained in a finite sum of factors $A_i$. Hence there exists a finite subset $J$ of $I$ such that

$$H \subseteq \sum_{i \in J} A_i.$$

Let

$$B = \sum_{i \in I \setminus J} A_i.$$

Then

$$G = B + \sum_{i \in J} A_i.$$

By Theorem 5 one of these factors is periodic. If $B$ is periodic it has either a period of order $p$ or a period of order $q$. Now these elements belong to $H$ and $H \subseteq \sum_{i \in J} A_i$ implies that $H + B$ is a direct sum. Thus it is not possible for $B$ to have a group of periods of order $p$ or $q$. Hence one of the subsets $A_i$, $i \in J$ is periodic.

**Theorem 7.** *If $p$, $q$ are distinct primes $G = Z(p^\infty) + Z(q)$ and*

$$G = B + \sum_{i \in I} A_i$$

*is a factorization in which all the factors $A_i$ are finite, then one of the factors of $G$ is periodic.*

*Proof.* If $I$ is finite, then the result follows by Theorem 5. Since $G$ is countable, $I$ is countable and we may assume that $I = \{1, 2, 3, \ldots\}$. For each $k \in I$ let

$$H_k = \langle A_1 \cup \cdots \cup A_k \rangle, \quad B_{k+1} = B + \sum_{i>k} A_i.$$

Let $C_k$ be a complete set of coset representatives for $G$ modulo $H_k$. Thus, for each $c \in C_k$,

$$A_1 + \cdots + A_k + \big(B_{k+1} \cap (H_k + c)\big) = H_k + c.$$

As before either one of the factors from $A_1, \ldots, A_k$ is periodic or there is a period of $B_{k+1} \cap (H_k + c)$ which depends only on $A_1, \ldots, A_k$ and not on $c$. Thus $B_{k+1}$ is periodic with an element $a$ of order $p$ or an element $b$ of order $q$ as period. It follows that either $a$ is a period of $B_{k+1}$ for infinitely many $k$ or that $b$ is so.

Suppose that this holds for $d$, say. Let $g \in B$. Then $g \in B_{k+1}$ and so $g + d \in B_{k+1}$ for infinitely many $k$. Let $k = r$ be such a value. Then

$$g + d = a_{r+1} + \cdots + a_{r+s} + b',$$

where $a_i \in A_i$, $b' \in B$. Let $k = t > r + s$ be another such value. Then

$$g + d = a_{t+1} + \cdots + a_{t+v} + b'',$$

where $a_i \in A_i$, $b'' \in B$. Now the sum $B + \sum A_i$ is direct. It follows that

$$a_{r+1} = \cdots = a_{r+s} = a_{t+1} = \cdots = a_{t+v} = 0, \quad b' = b''.$$

Hence $g + d \in B$. Therefore $B + d \subseteq B$. Since $B$ is a direct factor of $G$ it follows by Lemma 1 that $d$ is a period of $B$.

It is possible to prove the results in Theorems 5, 6, 7 for $G = Z(p^\infty)$ by similar methods. They also may be proved as follows. Let $a$ be a non-zero element in $Z(p^\infty)$ and let $b$ have order $q$ in $Z(q)$. Let

$$D = \{0, b, 2b, \ldots, (q-2)b, (q-1)b + a\}.$$

Then $D$ is finite and is not periodic. Since $Z(p^\infty) + D = Z(p^\infty) + Z(q)$ any counterexample to these results for $Z(p^\infty)$ can be extended to a counterexample for $Z(p^\infty) + Z(q)$.

These results are in a certain sense best possible. $Z(p^\infty)$ is generated by elements $a_1, a_2, \ldots, a_r, \ldots$ satisfying $pa_1 = 0$, $pa_{r+1} = a_r$, $r \geq 1$. Let

$$A_r = \{0, a_r, 2a_r, \ldots, (p-1)a_r\}.$$

Then $Z(p^\infty) = \sum A_i$. Plainly $A_1$ is a subgroup. Hence the other factors, and sums of the other factors, cannot be periodic. In [2] results are given which show that the groups

$$Z(p^3) + Z(q^2), \quad Z(p^3) + Z(q) + Z(r)$$

admit factorizations in which no factor is periodic. Hence the same holds for

$$Z(p^\infty) + Z(q^2), \quad Z(p^\infty) + Z(q) + Z(r)$$

by adding on the factors $\sum_{i\geq 4} A_i$. These can be regarded as a single infinite factor or a family of finite factors or as an infinite factor plus a family of finite factors. So Theorems $5, 6, 7$ do not extend to these groups.

Also in [2, 8] results are given which show that the groups

$$Z(p^2) + Z(p), \quad p \geq 5,$$

$$Z(3^3) + Z(3), \quad Z(2^3) + Z(2^2), \quad Z(2^4) + Z(2) + Z(2)$$

admit factorizations in which no factor is periodic. As above this implies that Theorems $5, 6, 7$ do not hold for

$$Z(p^\infty) + Z(p), \quad p \geq 3,$$

$$Z(2^\infty) + Z(2^2), \quad Z(2^\infty) + Z(2) + Z(2).$$

It is shown in [1] that $Z(2^n) + Z(2)$ has the Hajós $m$-property. Thus counterexamples as above cannot be constructed for $G = Z(2^\infty) + Z(2)$. We now show that the result analogous to Theorem 5 does hold for this group. We use the methods involving group characters which are used in [1]. The necessary results used are also described there.

**Theorem 8.** *Let $G = Z(2^\infty) + Z(2)$. If $G = A_1 + \cdots + A_k + B$ is a factorization in which each factor $A_i$ is finite, then some factor is periodic.*

*Proof.* Let $a$ in $Z(2^\infty)$ have order 2 and let $b$ be another element of order 2. Let $H = \langle A_1 \cup \cdots \cup A_k \rangle$. Let $C$ be a complete set of coset representatives for $G$ modulo $H$. Then for each $c \in C$,

$$A_1 + \cdots + A_k + \big(B \cap (H + c)\big) = H + c.$$

For some translate $B_c$ of $B \cap (H + c)$ with $0 \in B_c$ we have that

$$A_1 + \cdots + A_k + B_c = H.$$

Since $H$ is a finite subgroup of $G$, either $H = Z(2^n)$ or $H = Z(2^n) + Z(2)$ for some $n$.

If some $A_i$ is periodic, then there is nothing to prove. Assume that no factor $A_i$ is periodic and try to establish that each $B_c$ has a common period independently of $c$. This will give that $B$ is periodic. If $H = Z(2^n)$, then $a$ is a period of $B_c$ for each $c$.

If $H = Z(2^n) + Z(2)$ then, we will use the method of the proof of Theorem 10 [1]. Let $\rho$ be a primitive $(2^n)$th root of unity and let $d$ in $H$ be such that $2^{n-1}d = a$. Let characters of $H$ be defined by

$$\chi_1(d) = \rho, \quad \chi_1(b) = 1 \quad \text{and} \quad \chi_2(d) = \rho, \quad \chi_2(b) = -1.$$

Clearly, $\chi_1$ is not the principal character of $H$ and so

$$0 = \chi_1(H) = \chi_1(A_1) \cdots \chi_1(A_k)\chi_1(B_c).$$

It follows that $\chi_1(D) = 0$ for some factor $D$ of the factorization $H = A_1 + \cdots + A_k + B_c$. Similarly $\chi_2(D) = 0$ for some factor. If for some factor $\chi_1(D) = \chi_2(D) = 0$, then by Theorem 1 [11], $a$ is a period of $D$. We may assume that $\chi_1(D) = 0$, $\chi_2(D) = 0$ does not hold for any $D$.

If $\chi_1(A_i) = 0$ and $\chi_2(A_j) = 0$, then by Theorem 2 [11] there are subsets $P$, $Q$, $R$, $S$ of $H$ such that

$$A_i = \big(\{0, 2^{n-1}d\} + P\big) \cup \big(\{0, 2^{n-1}d + b\} + Q\big), A_j = \big(\{0, 2^{n-1}d\} + R\big) \cup \big(\{0, b\} + S\big),$$

where the sums are direct and the unions are disjoint. Since $A_i$ is not periodic, $P$ and $Q$ are non-empty sets. Since $A_j$ is not periodic, $R$ and $S$ are non-empty sets. Choose $a \in P$, $b \in R$ and consider the factorization

$$H = H - a - b = A_1 + \cdots + (A_i - a) + \cdots + (A_j - b) + \cdots + A_k + B_c.$$

By the factorization the sum $A_i + A_j$ is direct and so $(A_i - a) \cap (A_j - b) = \{0\}$. On the other hand $2^{n-1}d \in (A_i - a) \cap (A_j - b)$. This contradiction shows that either $A_i$ or $A_j$ is periodic.

If $\chi_1(A_i) = 0$ and $\chi_2(B_c) = 0$, then by Theorem 2 [11] there exist subsets $P$, $Q$, $R$, $S$ of $H$ such that

$$A_i = \big(\{0, 2^{n-1}d\} + P\big) \cup \big(\{0, 2^{n-1}d + b\} + Q\big), B_c = \big(\{0, 2^{n-1}d\} + R\big) \cup \big(\{0, b\} + S\big),$$

where the sums are direct and the unions are disjoint. Since $A_i$ is not periodic, $P$ and $Q$ are non-empty sets. Since $A_i + B_c$ is direct, as above, it follows that $R$ is empty. Hence $b$ is a period of $B_c$ for all $c$.

Similarly, if $\chi_2(A_j) = 0$ and $\chi_1(B_c) = 0$, then it follows that $2^{n-1}d + b$ is a period of $B_c$ for all $c$.

The results analogous to Theorems 6, 7 can now be deduced from Theorem 8 in a similar way to the deductions of Theorems 6, 7 from Theorem 5.

## 3. The simulation theorem

It has been shown in [3] that if a finite group $G$ is a direct sum of simulated subsets, then one of these subsets must be a subgroup. An infinite group $G$ will be said to satisfy the simulation theorem if the result holds for it. The following replacement result has been proved for finite groups by using group characters and sums of roots of unity in [3]. In fact this method is not needed and the result holds also for infinite groups.

**Theorem 9.** *If a subset $A$ is simulated by a subgroup $H$ of a group $G$ and $G = A + B$ is a factorization, then so also is $G = H + B$.*

*Proof.* If $A = H$, then there is nothing to prove. So we may assume that

$$H = (A \cap H) \cup \{h\}, \quad A = (A \cap H) \cup \{h + d\}, \quad h \neq 0, \quad d \neq 0.$$

Let $b \in B$. Then $h + b = a + b_1$, for some $a \in A$, $b_1 \in B$.

If $a = 0$, then $h + b = b_1$. Let $h_1 \in (A \cap H)$, $h_1 \neq 0$. Then $h + h_1 \in (A \cap H)$ and $(h + h_1 + b) = h_1 + b_1$. Since $A + B$ is a direct sum it follows that $h + h_1 = h_1$. This is false and so $a \neq 0$.

If $a \in (A \cap H)$, then $h - a \in (A \cap H)$. Thus $b = (h - a) + b_1$. Since $A + B$ is a direct sum it follows that $h - a = 0$, which is false.

So the remaining case in which $a = h + d$ must occur. Thus $h + b = h + d + b_1$. Hence $(-d) + b \in B$. Therefore $(-d) + B \subseteq B$. By Lemma 1 it follows that $(-d) + B = B$ and so that $B = d + B$. Then $h + B = h + d + B$ and so $H + B = A + B$. The desired result now follows.

**Theorem 10.** *If a group $G$ is a direct sum of a finite number of simulated subsets, then one of these subsets is equal to its simulating subgroup.*

*Proof.* Let $G = A_1 + \cdots + A_k$ be a factorization in which each subset $A_i$ is simulated by a subgroup $H_i$. If $k = 1$, then the result is trivial and we may proceed by induction on $k$. It may be supposed that $A_i \neq H_i$ and so that

$$H_i = (A_i \cap H_i) \cup \{h_i\}, \quad A_i = (A_i \cap H_i) \cup \{h_i + d_i\}, \quad h_i \neq 0, \quad d_i \neq 0.$$

By Theorem 9, the factor $A_i$ may be replaced by the subgroup $H_i$. This leads to the factorization

$$G/H_i = \sum_{r \neq i} (A_r + H_i)/H_i.$$

By the inductive assumption some subset here is equal to its simulating subgroup. So there exists $f(i) \neq i$ such that $A_{f(i)} + H_i = H_{f(i)} + H_i$. Then $h_{f(i)} + d_{f(i)} = h'_{f(i)} + h'_i$ for some $h'_{f(i)} \in H_{f(i)}$, $h'_i \in H_i$. Since, by Theorem 9, the sum $A_{f(i)} + H_i$ is direct it follows that $h'_{f(i)} \notin A_{f(i)}$ and so that $h'_{f(i)} = h_{f(i)}$. Therefore $d_{f(i)} \in H_i$.

Such a mapping $f$ must give rise to a cycle among the indices $1, 2, \ldots, k$. By reordering the subsets it may be assumed that $1, 2, \ldots, r$ is such a cycle. Thus it follows that

$$d_2 \in H_1, \ldots, d_r \in H_{r-1},\ d_1 \in H_r.$$

Consider the element

$$g = (h_1 + d_1) + (h_2 + d_2) + \cdots + (h_r + d_r),$$

which is in $A_1 + A_2 + \cdots + A_r$. Since $d_{i+1} \in H_i$, $d_{i+1} \neq 0$, it follows that $h_i + d_{i+1} \in A_i \cap H_i$, $1 \leq i \leq r - 1$, and similarly that $d_1 + h_r \in A_r \cap H_r$. Then

$$g = (h_1 + d_2) + (h_2 + d_3) + \cdots + (h_{r-1} + d_r) + (h_r + d_1).$$

Since the sum $\sum A_i$ is direct it follows that

$$h_1 + d_1 = h_1 + d_2, \ldots, h_r + d_r = h_r + d_1.$$

Thus $d_1 = d_2 \in H_1 \cap H_r$. By using two applications of Theorem 9 it may be seen that the sum $H_1 + H_r$ is direct. Therefore $H_1 \cap H_r = \{0\}$ and so $d_1 = 0$. This is false.

It follows that $A_j = H_j$ for some $j$. The result now follows by induction.

**Theorem 11.** *If a group $G$ satisfies the simulation theorem and $G$ is the direct sum of subgroups $H$ and $K$, then these direct summands of $G$ also satisfy the simulation theorem.*

*Proof.* Let $H = \sum A_i$, where the subsets are simulated by subgroups $H_i$.

Suppose first that $K$ contains at least three elements. Choose $h \in H$, $f \in K$ with $h \neq 0$, $f \neq 0$ and form $B$ from $K$ by replacing $f$ by $f + h$, that is set

$$B = \big(K \setminus \{f\}\big) \cup \{f + h\}.$$

Then $B$ is simulated by $K$ and $B \neq K$. Now

$$G = K + H = B + H = B + \sum A_i.$$

Since $G$ satisfies the simulation theorem it follows that $A_i = H_i$ for some $i$.

Now suppose that $K = \{0, f\}$ has only two elements. Let $B = A_1 \cup (H_1 + f)$. Then $B$ is simulated by $H_1 + K$. Also

$$\begin{aligned}
B + \sum_{i \neq 1} A_i &= \big(A_1 \cup (H_1 + f)\big) + \sum_{i \neq 1} A_i \\
&= \Big(A_1 + \sum_{i \neq 1} A_i\Big) \cup \Big(f + H_1 + \sum_{i \neq 1} A_i\Big) \\
&= \Big(\sum A_i\Big) \cup \Big(f + A_1 + \sum_{i \neq 1} A_i\Big) \quad \text{(by Theorem 9)} \\
&= H \cup (f + H) \\
&= \{0, f\} + H \\
&= K + H \\
&= G.
\end{aligned}$$

Since $B \neq H_1 + K$ it follows that $A_i = H_i$ for some $i$.

Therefore $H$ satisfies the simulation theorem.

**Lemma 3.** *Let $G$ be a direct sum of countable many subgroups that are not of exponent two. Then $G$ does not satisfy the simulation theorem.*

*Proof.* Let

$$G = \sum_{i=1}^{\infty} H_i,$$

where $H_i$ are subgroups that are not of exponent two. For each $i$ choose $f_i \in H_i$ such that $2f_i \neq 0$. Form the subset $A_i$ from $H_i$ by replacing $f_i$ by $f_i + f_{i+1}$. Then $A_i$ is simulated by $H_i$, $A_i \neq H_i$,

$$A_i = (A_i \cap H_i) \cup \{f_i + f_{i+1}\}.$$

Let $g \in G$, $g \neq 0$. Suppose that $r$ is the least value and $s$ is the greatest value of $i$ such that the $H_i$-component of $g$ is non-zero. We define the length of $g$ by $n(g) = s - r$. We shall show that $g \in \sum_{i \geq r} A_i$ by using induction on length.

If $n(g) = 0$, then $g \in H_r$. If $g \in A_r$, then the desired result holds. If $g \notin A_r$, then

$$g = f_r = (f_r + f_{r+1}) + (-f_{r+1}).$$

Now $-f_{r+1} \neq f_{r+1}$ and so $-f_{r+1} \in A_{r+1}$. Hence $g \in A_r + A_{r+1}$. So the result holds for $n(g) = 0$.

Let $n(g) = k$ and assume that the result holds for all elements of length less than $k$. Then $g = h_r + \cdots + h_{r+k}$, where $h_r \neq 0$, $h_{r+k} \neq 0$. If $h_r \in A_r$, then $g = a_r + h$, where $n(h) < k$. By the inductive assumption $h \in \sum_{i>r} A_i$. Therefore $g \in \sum_{i \geq r} A_i$. If $h_r \notin A_r$, then $h_r = f_r$. Thus

$$g = f_r = (f_r + f_{r+1}) + (-f_{r+1}) + h_{r+1} + \cdots + h_{r+k} = f_r + f_{r+1} + h,$$

where $n(h) < k$. As before it follows that $g \in \sum_{i \geq r} A_i$.

It follows that $\sum A_i = G$.

We now show that this sum is direct. Suppose that

$$g = \sum_{i \geq r} a_i = \sum_{i \geq s} a'_i, \quad a_i, a'_i \in A_r.$$

Then the $H_r$-component of $g$ is either $a_r$ or $f_r$ from $g = \sum_{i \geq r} a_i$. It follows that the same arises from $g = \sum_{i \geq s} a'_i$, since $\sum H_i$ is a direct sum. Since $f_r$ arises once only from $a_r = f_r + f_{r+1}$ it follows that $r = s$ and $a_r = a'_r$. By repeating this process we see that the expression for $g$ is unique. Thus the sum is direct and since $A_i \neq H_i$ for any $i$, in this case $G$ does not satisfy the simulation theorem.

**Lemma 4.** *Let $G$ be a direct sum of countable many subgroups that are all isomorphic to $Z(2) + Z(2)$. Then $G$ does not satisfy the simulation theorem.*

*Proof.* Let

$$G = \sum_{i=1}^{\infty} H_i,$$

where $H_i$ is isomorphic to $Z(2) + Z(2)$. Let $H_i = \{0, d_i, b_i, d_i + b_i\}$. We define

$$A_i = \{0, d_i, b_i, d_i + b_i + d_{i+1}\}.$$

Then $A_i$ is simulated by $H_i$ and $A_i \neq H_i$.

Exactly as in the previous proof by considering the $H_r$-component of any element $g \in G$ we may show that the sum of the subsets $A_i$ is direct.

Let $g \in G$, $g \neq 0$. We define the length $n(g)$ of $G$ as before. If $n(g) = 0$, then $g \in H_r$ and either $g \in A_r$ or

$$g = (d_r + b_r + d_{r+1}) + d_{r+1} \in A_r + A_{r+1}.$$

In the general case let $r$ be the smallest index such that $g$ has non-zero $H_r$-component. We show that $g \in \sum_{i\geq r} A_i$ by induction on $n(g)$. Suppose that $n(g) = k$ and that the result holds for elements of length less than $k$. Now

$$g = h_r + \cdots + h_{r+k}, \quad h_i \in H_i.$$

If $h_r \in A_r$, then $g = a_r + h$, $a_r \in A_r$, where $n(h) < k$. Hence $h \in \sum_{i>r} A_i$ and so $g \in \sum_{i\geq r} A_i$, as required. If $h_r \notin A_r$, then

$$g = (d_r + b_r + d_{r+1}) + d_{r+1} + h,$$

where $n(d_{r+1} + h) < k$. Hence $g \in A_r + \sum_{i>r} A_i = \sum_{i\geq r} A_i$, as required. It follows that $\sum A_i = G$. Therefore $G$ does not satisfy the simulation theorem.

**Theorem 12.** *If a group $G$ is a direct sum of an infinite family of subgroups, then $G$ does not satisfy the simulation theorem.*

*Proof.* A countable subset indexed by the positive integers may be chosen from this infinite family. By Theorem 11 it suffices to consider this direct summand. So without loss of generality it may be assumed that $G = \sum_{i\geq 1} H_i$, where the sum is direct and each $H_i$ is a subgroup of $G$.

Consider first the case where infinitely many subgroups $H_i$ are not of exponent two. Again, by Theorem 11, it may then be supposed that no subgroup $H_i$ has exponent two. By Lemma 3, $G$ does not satisfy the simulation theorem.

There remains the case where all but a finitely number of subgroups $H_i$ have exponent two. Each subgroup of exponent 2 is a vector space over the field of order 2 so is a direct sum of copies of $Z(2)$. Again by Theorem 11 we may assume that $G = \sum_{i\geq 1} H_i$, where each $H_i$ is isomorphic to $Z(2) + Z(2)$. By Lemma 4, $G$ does not satisfy the simulation theorem.

This completes the proof.

It is now clear from Theorem 9 that if $G = \sum_{i=1}^{k} A_i$, where $A_i$ are subsets simulated by subgroups $H_i$, then $G = \sum_{i=1}^{k} H_i$. We have not been able to extend this result to infinite direct sums. So there is a gap between Theorem 10 and Theorem 12 which we have not been able to close. We can, though, deduce that certain groups do satisfy the simulation theorem.

The basic divisible groups are the Prüferian groups $Z(p^\infty)$ and the group $Q$ of the rationals. Every divisible group may be expressed uniquely as a direct sum of these groups.

**Theorem 13.** *If an abelian group $G$ is contained in a finite direct sum of basic divisible groups, then $G$ satisfies the simulation theorem.*

*Proof.* Let $D$ be a direct sum of $k$ basic divisible subgroups. Let a subgroup $H$ of $D$ be a direct sum of $r$ non-zero subgroups $H_i$. Then the divisible hull $E_i$ of $H_i$ may be assumed to be a subgroup of $D$ and the sum $E$ of the subgroups $E_i$ is direct and is the divisible hull of $H$. Since $E$ is divisible there is a subgroup $F$ of

$D$ such that $D = E + F$, $E \cap F = \{0\}$ and $F$ is divisible. Then $k = s + t$, where $E$ is the direct sum of $s$ basic divisible subgroups and $F$ of $t$ such subgroups. Hence $r \leq s \leq k$.

Now if $G = \sum_{i\in I} A_i$ is a factorization of $G$ into simulated factors $A_i$ and $|I|$ is infinite or $|I| > k$ then it follows from Theorem 9 that $G$ contains a subgroup which is a direct sum of $k+1$ non-zero subgroups. This is not possible. Hence $|I| \leq k$. By Theorem 10 it follows that $G$ satisfies the simulation theorem.

## 4. Rédei's Theorem

Since the factors in Hajós' theorem may be assumed to have prime order it follows that any infinite group satisfying Rédei's theorem must also satisfy Hajós' theorem. The following example shows that not all groups satisfying Hajós' theorem also satisfy Rédei's theorem. For any odd prime $p$, Hajós' theorem holds for $\sum Z(p)$, but by Lemma 3 Rédei's theorem does not hold if the sum is infinite.

**Theorem 14.** *If $H$ is a subgroup of a group $G$ and Rédei's theorem holds for $G$ it also holds for $H$.*

*Proof.* The proof of this result for Hajós' theorem in [5, 85.3] shows that any counterexample for $H$ extends to a counterexample for $G$ using cyclic subsets. Since these may be assumed to have prime order, the same proof applies to Rédei's theorem.

**Theorem 15.** *If the group $G$ satisfies Rédei's theorem, then $G$ is of the form*

$$G = F + \sum_{i=1}^{s} Z(p_i^\infty) + \sum_{\mu} Z(2),$$

*where $F$ is a finite group and $\mu$ is any cardinal.*

*Proof.* As has already been stated if a group $G$ satisfies Rédei's theorem it must satisfy Hajós' theorem and so be of the form given by Fuchs [4, 5]. From Lemma 3 it follows that $\sum_\mu Z(p)$ does not satisfy Rédei's theorem for $p > 2$ and $\mu$ being infinite. Hence $G$ must be of the given form.

**Theorem 16.** *If*

$$G = F + \sum_{\mu} Z(2),$$

*where $F$ is finite and $\mu$ is any cardinal, then $G$ satisfies Rédei's theorem.*

*Proof.* Let $G = \sum_{i\in I} A_i$, where each $A_i$ has prime order, say $p_i$. If $p_i \neq 2$, then $(A_i)_{p_i} \subseteq F$. Since $F$ is finite and the sum $\sum (A_i)_{p_i}$ is direct it follows that the set of $i$ with $p_i \neq 2$ is finite.

Since $F$ is finite and each $f \in F$ is contained in a finite sum of factors there exists a finite subset, say $\{1, \ldots, n\}$ of $I$ with $F \subseteq \sum_{i=1}^{n} A_i$. We may assume that all subsets $A_i$ with $|A_i| \neq 2$ are included here. For each $i$ with $|A_i| = 2$ let

$A_i = \{0, b_i\}$. Let $J = \langle A_1 \cup \cdots \cup A_n \rangle$. Then $J$ is finite and, as above, there is a finite subset, say $\{1, \ldots, n+k\}$, of $I$ such that $J \subseteq \sum_{i=1}^{n+k} A_i$. Let $K = \langle A_1 \cup \cdots \cup A_{n+k} \rangle$. Let $b \in K$. Since $G = \sum A_i$ there exists $l$ such that $b = \sum_{i=1}^{n+k+l} a_i$, with $a_i \in A_i$. Let $c = b - \sum_{i=1}^{n+k} a_i$. Then $\sum_{i=n+k+1}^{n+k+l} a_i = c \in K$. Since $c \in K$ there exist integers $r_{i,j}$ such that

$$c = \sum_{i=1}^{n+k} \sum_{j} r_{i,j} a_{i,j}, \quad a_{i,j} \in A_i.$$

Now

$$\sum_{i=1}^{n} \sum_{j} r_{i,j} a_{i,j} \in \langle A_1 \cup \cdots \cup A_n \rangle = J.$$

Therefore there exist $a_i' \in A_i$ such that

$$\sum_{i=1}^{n} \sum_{j} r_{i,j} a_{i,j} = \sum_{i=1}^{n+k} a_i'.$$

For $i \geq n+1$, $A_i = \{0, b_i\}$. Hence $A_i = \langle b_i \rangle$ and so $\sum r_{i,j} a_{i,j}$ may be replaced by $r_i b_i$. Also

$$\sum_{i=n+1}^{n+k} a_i' = \sum_{i=n+1}^{n+k} t_i b_i, \quad 0 \leq t_i \leq 1.$$

Let

$$r_i + t_i = 2u_i + s_i, \quad 0 \leq s_i \leq 1.$$

Since $2G \subseteq F$ we have that

$$-\sum 2u_i b_i = \sum_{i=1}^{n} a_i'', \quad a_i'' \in A_i.$$

Hence

$$\sum_{i=n+k+1}^{n+k+l} a_i + \sum_{i=1}^{n} a_i'' = \sum_{i=1}^{n} a_i' + \sum_{i=n+1}^{n+k} s_i b_i.$$

Since $\sum A_i$ is direct it follows that $a_i = 0$, $n+k+1 \leq i \leq n+k+l$. Hence $c = 0$ and so $b \in \sum_{i=1}^{n+k} A_i$. It follows that $K = \sum_{i=1}^{n+k} A_i$. Since $K$ is finite, Rédei's theorem implies that some $A_i$ is a subgroup. Therefore $G$ satisfies Rédei's theorem.

**Theorem 17.** *If*

$$G = F + \sum_{i=1}^{r} Z(p_i^{\infty}),$$

*where $F$ is a finite group and $p_1, \ldots, p_r$ are distinct primes not dividing $|F|$, then Rédei's theorem holds for $G$.*

*Proof.* Let $G = \sum_{i \in I} A_i$ be a factorization in which each subset $A_i$ has prime order. Let $A_i$ have order $q$. Then $q$ divides $\big|\langle A_i \rangle\big|$ and so either $q$ divides $|F|$ or

$q = p_j$ for some $j$. Since $p_1, \dots, p_r$ do not divide $|F|$ it follows by the remark after the proof of Theorem 3 that the set of $i$ such that $|A_i|$ divides $|F|$ is finite and that $F = \sum (A_i)_{q_i}$, where $|A_i| = q_i$ and the sum is taken over all $i$ such that $q_i$ divides $|F|$.

Since $G$ is countable we may choose the positive integers as the index set $I$. Thus we may suppose that $F = \sum_{i=1}^{k} (A_i)_{q_i}$. The remaining subsets $A_i$ have cardinality taken from the set of primes $\{p_1, \dots, p_r\}$. Let $I_j = \{i \in I : |A_i| = p_j\}$ for $1 \le j \le r$. Then, for $i \in I_j$, $(A_i)_{p_j} \subseteq Z(p_j^\infty)$ and $\big|(A_i)_{p_j}\big| = p_j$.

By Theorem 2 we may replace each subset $A_i$, $1 \le i \le k$, by a subset $D_i$ such that $(A_i)_F = (D_i)_F$ and the components in $\sum_{j=1}^{r} Z(p_j^\infty)$ of elements of $D_i$ are zero or are of prime order. Without renaming the subsets we shall assume that this replacement has been made.

Let $H_j$ be the unique subgroup of order $p_j$ in $Z(p_j^\infty)$. Let $f \in H_j$, $f \ne 0$. We may choose an ascending family of finite subsets $K_{j,l}$ of $I_j$ such that $\cup_l K_{j,l} = I_j$. Then

$$\sum_{i \in K_{j,l}} (A_i)_{p_j} + \Big( \sum_{i \notin K_{j,l}} A_i \cap Z(p_j^\infty) \Big) = Z(p_j^\infty).$$

By the remark following the proof of Theorem 6, $f$ is a period of one of these factors. If no subset $(A_i)_{p_j}$, $i \in I_j$, is periodic, then $f \in \big(\sum_{i \notin K_{j,l}} A_i\big)$ for all $l$. Since these sums are direct it follows that

$$f \in \bigcap_l \Big( \sum_{i \in K_{j,l}} A_i \Big) = \sum_{i \notin I_j} A_i.$$

Let $f \in \sum_{i \notin I_j} a_i$. Then $f = \sum (a_i)_{p_j}$ and $\sum (a_i)_{p_j'} = 0$. For $i \notin I_j$, $\sum (A_i)_{p_j}$ is direct. Hence $(a_i)_{p_j} = 0$ for each $i$. Also $|A_i| = \big|(A_i)_{p_j}\big|$ for $i \notin I_j$. Hence $a_i = 0$ and so $f = 0$. This is false. Therefore there exists $i \in I_j$ such that $f$ is a period of $(A_i)_{p_j}$. Since $\big|(A_i)_{p_j}\big| = |A_i| = p_j$ it follows that $(A_i)_{p_j} = H_j$.

Let these subsets $A_i$ be $A_{k+1}, \dots, A_{k+r}$ with $(A_{k+j})_{p_j} = H_j$. We may replace these subsets $A_{k+j}$ by subsets $B_{k+j}$ such that $(A_{k+j})_{p_j} = (B_{k+j})_{p_j}$ and such that the $p_j'$-components in $B_{k+j}$ have prime order. We shall assume that this replacement has been made without renaming the subsets. Let $H = F + H_1 + \dots + H_r$. Then $H$ contains all elements of $G$ of prime order. Hence $A_1, \dots, A_k, A_{k+1}, \dots, A_{k+r}$ are contained in $H$. Since $|A_1 + \dots + A_k| = |F|$ and $|A_{k+1}| = |K_J|$ it follows that $\sum_{i=1}^{k+r} A_i = H$. Since $H$ is finite it follows by Rédei's theorem that some $A_i$ is a subgroup.

Therefore $G$ satisfies Rédei's theorem.

In the case of finite cyclic groups there is a generalization of Rédei's theorem. In Theorem 1 [9], it is shown that if the order of each factor is a prime power and the finite group is cyclic, then one factor must be periodic. This result extends to the infinite case as follows.

**Theorem 18.** *Let*

$$G = F + \sum_{j=1}^{r} Z(p_j^\infty),$$

*where $F$ is a finite cyclic group and $p_1, \dots, p_r$ are distinct primes not dividing $|F|$. Then if $G = \sum_{i \in I} A_i$ is a factorization in which each $|A_i|$ is a prime power, one of the factors is periodic.*

*Proof.* Let $J = \{i \in I : |A_i| \text{ divides } |F|\}$. By the remark after the proof to Theorem 3, $F = \sum_{i \in I} (A_i)_F$. By Theorem 2 we may replace each of this finite set of factors $A_i$ by a factor $D_i$ such that $(A_i)_F = (D_i)_F$ and $D_i \subseteq F + \sum_{j=1}^{r} K_j$, where $K_j$ is the subgroup of $Z(p_j^\infty)$ of order $p_j$. Without renaming the factors we shall assume that this replacement has been made.

Let $I_j = \{i \in I : |A_i| \text{ is a power of } p_j\}$, $1 \le j \le r$. Since $Z(p_j^\infty)$ is countable we may assume, for some chosen $j$, that $I_j$ is the set of positive integers. For each $k$ let

$$B_k = Z(p_j^\infty) \cap \Big( \sum_{i \notin I_j} A_i + \sum_{i > k} A_i \Big).$$

By Theorem 1 we may replace the finite set of factors $A_i$, $1 \le i \le k$, by the factors $(A_i)_{p_j}$. Hence

$$B_k + \sum_{i=1}^{k} (A_i)_{p_j} = Z(p_j^\infty).$$

By the Corollary to Theorem 4 one of these factors is periodic. If no factor $(A_i)_{p_j}$ is periodic, then $K_j$ is a group of periods of $B_k$ for all $k$. Let $c \in K_j$, $c \ne 0$. Then $c \in \cap_{k \ge 1} B_k$. Since $\sum A_i$ is a direct sum it is easily seen that

$$\bigcap B_k = Z(p_j^\infty) \cap \Big( \sum_{i \notin I_j} A_i \Big).$$

By the remark at the end of the proof of Theorem 2 this intersection is $\{0\}$, where $K = Z(p_j^\infty)$, $A = \sum_{i \notin I_j} A_i$. Thus $c \in B_k$ for all $k$ is not possible and so some subset $(A_i)_{p_j}$ has $K_j$ as a group of periods. Let the corresponding value of $i$ be denoted by $i(j)$. Again, without renaming the factors, we may assume that

$$A_{i(j)} \subseteq F + K_1 + \cdots + K_r + Z(p_j^\infty).$$

Let $L_j = \langle (A_{i(j)})_{p_j} \rangle$. Then $K_j \subseteq L_j$ and, for all $i \in J$, $A_i \subseteq F + \sum_{j=1}^{r} L_j$. Let $D_j = L_j \cap \big( \sum_{i \ne i(j)} A_i \big)$. Consider the sum

$$\sum_{i \in J} A_i + \sum_{j=1}^{r} A_{i(j)} + \sum_{j=1}^{r} D_j.$$

We claim that this sum is direct. Let

$$\sum_{i \in J} a_i + \sum_{j=1}^{r} a_{i(j)} + \sum_{j=1}^{r} d_j = \sum_{i \in J} a'_i + \sum_{j=1}^{r} a'_{i(j)} + \sum_{j=1}^{r} d'_j, \quad a_i, a'_i \in A_i, \quad d_j, d'_j \in D_j.$$

Choose $m$ with $1 \le m \le r$. Then

$$\sum_{i\in J}(a_i)_{p_m} + \sum_{j=1}^{r}(a_{i(j)})_{p_m} + d_m = \sum_{i\in J}(a_i')_{p_m} + \sum_{j=1}^{r}(a'_{i(j)})_{p_m} + d'_m.$$

Now $(a_i)_{p_m} \in K_m$, $i \in J$ and $(a_{i(j)})_{p_m} \in K_m$ for $j \neq m$. $K_m$ is a group of periods of $(A_{i(m)})_{p_m}$. It follows that

$$\sum_{i\in J}(a_i)_{p_m} + \sum_{j=1}^{r}(a_{i(j)})_{p_m} \in (A_{i(m)})_{p_m}$$

and similarly

$$\sum_{i\in J}(a_i')_{p_m} + \sum_{j=1}^{r}(a'_{i(j)})_{p_m} \in (A_{i(m)})_{p_m}.$$

Now from $(A_{i(m)})_{p_m} + \sum_{i\neq i(m)} A_i = G$ and $A_{i(m)} \subseteq L_m$ it follows that $D_m + (A_{i(m)})_{p_m} = L_m$ and that the sum is direct. Therefore $d_m = d'_m$. This is true for each $m$, $1 \le m \le r$. Hence

$$\sum_{i\in J} a_i + \sum_{j=1}^{r} a_{i(j)} = \sum_{i\in J} a_i' + \sum_{j=1}^{r} a'_{i(j)}.$$

Since the sum $\sum A_i$ is direct it follows that $a_i = a_i'$, $i \in J$, $a_{i(j)} = a'_{i(j)}$, $1 \le j \le r$. Thus the original claim is correct. Now

$$\sum_{i\in J} A_i + \sum_{j=1}^{r} A_{i(j)} + \sum_{j=1}^{r} D_j \subseteq F + \sum_{j=1}^{r} L_j.$$

Also $|F| = \prod_{i\in J} |A_i|$,

$$|L_j| = |D_j|\big|(A_{i(j)})_{p_j}\big| = |D_j||A_{i(j)}|.$$

It follows that

$$F + \sum_{j=1}^{r} L_j = \sum_{i\in J} A_i + \sum_{j=1}^{r} A_{i(j)} + \sum_{j=1}^{r} D_j.$$

Since this group is finite it follows by Theorem 2 [9] that some factor is periodic. Since $D_j \subseteq L_j \subseteq Z(p_j^\infty)$, if $D_j$ is periodic then $K_j$ is a group of periods of $D_j$. However $K_j$ is a group of periods of $(A_{i(j)})_{p_j}$ and the sum $(A_{i(j)})_{p_j} + D_j$ is direct. Therefore $D_j$ is not periodic. Thus one of the factors $A_i$ is periodic.

This completes the proof.

## 5. Hajós' theorem

This theorem states that if a finite abelian group is a direct sum of cyclic subsets then one of these subsets is a subgroup. Fuchs [4, 5] considered the problem of determining the infinite abelian groups for which this theorem holds. He showed that any group which satisfies Hajós' theorem must belong to the class of groups of the form

$$G = F + \sum_{i=1}^{t} Z(p_i^{\infty}) + \sum_{\mu} Z(p),$$

where $F$ is any finite group, $p, p_1, \dots, p_t$ are prime numbers and $\mu$ is any cardinal. In the opposite direction he showed that it held for the groups $F + \sum_{\mu} Z(p)$. However it seems to us that his proof of this latter case requires also the condition that $p$ does not divide $|F|$. In the extreme case where $F$ is a finite $p$-group the proof would involve $J = K = \{0\}$ and so no deduction could be made. In other cases it is required that $g \in F$ implies $g \in J$ which need not hold. As we now show, however, a modification of his proof will suffice to prove the desired result.

**Theorem 19.** *If*

$$G = F + \sum_{\mu} Z(p),$$

*where $F$ is finite, $p$ is prime and $\mu$ is any cardinal, then $G$ satisfies Hajós' theorem.*

*Proof.* Let $G = \sum_{i \in I} A_i$ be a factorization in which each $A_i$ is cyclic of prime order. Let

$$A_i = \{0, a_i, 2a_i, \dots, (p_i - 1)a_i\}.$$

If $p_i \neq p$, then $(A_i)_{p_i} \subseteq F$. Since $F$ is finite and $\sum (A_i)_{p_i}$ is direct it follows that only finitely many subsets $A_i$ exist with $|A_i| \neq p_i$. Since $F$ is finite there exists a finite subset of $I$, which we may assume to be $\{1, \dots, n\}$ by renaming elements of $I$, such that $F \subseteq \sum_{i=1}^{n} A_i$. We may also assume that the factors $A_i$ with $|A_i| \neq p$ are included here.

Let $J = \langle A_1 \cup \dots \cup A_n \rangle$. Since $G$ is a torsion group, $J$ is a finite subgroup of $G$. As above there exists a finite subset, say $\{1, \dots, n+k\}$ such that $J \subseteq \sum_{i=1}^{n+k} A_i$. Let $K = \langle A_1 \cup \dots \cup A_{n+k} \rangle$. Then $K$ is a finite subgroup of $G$. Let $b \in K$. Then

$$b = \sum_{i=1}^{n+k+m} t_i a_i, \quad 0 \leq t_i \leq p_i - 1.$$

Let $c = b - \sum_{i=1}^{n+k} t_i a_i$. Then $c \in K$ and $c = \sum_{i=n+k+1}^{n+k+m} t_i a_i$. Since $c \in K$ there exist integers $r_i$ such that $c = \sum_{i=1}^{n+k} r_i a_i$. Then $\sum_{i=1}^{n} r_i a_i \in J$. Hence

$$\sum_{i=1}^{n} r_i a_i = \sum_{i=1}^{n+k} w_i a_i, \quad 0 \leq w_i \leq p_i - 1.$$

Let

$$w_i + r_i = p u_i + s_i, \quad 0 \leq s_i \leq p - 1$$

for $n+1 \leq i \leq n+k$. Since $pG \subseteq F$ it follows that

$$-\sum_{i=n+1}^{n+k} p u_i a_i = \sum_{i=1}^{n} v_i a_i, \quad 0 \leq v_i \leq p_i - 1.$$

Hence

$$c - \sum_{i=n+1}^{n+k} pu_i a_i = \sum_{i=1}^{n} w_i a_i + \sum_{i=n+1}^{n+k} s_i a_i = \sum_{i=n+k+1}^{n+k+m} t_i a_i + \sum_{i=1}^{n} v_i a_i,$$

where $0 \leq w_i, s_i, t_i, v_i \leq p_i - 1$. Since $\sum_{i \in I} A_i$ is direct it follows that $t_i = 0$, $n+k+1 \leq i \leq n+k+m$.

Therefore $b = \sum_{i=1}^{n+k} t_i a_i \in \sum_{i=1}^{n+k} A_i$. It follows that $\sum_{i=1}^{n+k} A_i = K$. Since $K$ is a finite group it follows, by Hajós' theorem, that some subset $A_i$ is a subgroup. Thus Hajós' theorem holds for $G$.

**Theorem 20.** *If*

$$G = F + \sum_{j=1}^{r} Z(p_j^{\infty}),$$

*where $F$ is a finite group and $p_1, \ldots, p_r$ are distinct primes not dividing $|F|$, then $G$ satisfies Hajós' theorem.*

*Proof.* Since cyclic subsets may be assumed to have prime order this follows immediately from Theorem 17.

## References


[1] Amin, K.; Corrádi, K.; Sands, A. D.: *The Hajós property for 2-groups*. Acta Math. Hung. **89** (2000), 189–198. Zbl 1004.20032

[2] de Bruijn, N. G.: *On the factorization of finite abelian groups*. Indag. Math. **15** (1953), 258–264. cf. Nederl. Akad. Wet., Proc., Ser. A **56** (1953), 258–264. Zbl 0050.25704

[3] Corrádi, K.; Sands, A. D.; Szabó, S.: *Simulated factorizations*. J. Algebra **151** (1992), 12–25. Zbl 0774.20030

[4] Fuchs, L.: *On the possibility of extending Hajós' theorem to infinite abelian groups*. Publ. Math. Debrecen **5** (1959), 338–347. Zbl 0080.24804

[5] Fuchs, L.: *Abelian Groups*. Pergamon Press, Oxford-London-New York-Paris 1960. Zbl 0100.02803

[6] Hajós, G.: *Über einfache und mehrfache Bedeckung des $n$-dimensionalen Raumes mit einem Würfelgitter*. Math. Z. **47** (1941), 427–467. Zbl 0025.25401 and JFM 67.0137.04

[7] Rédei, L. *Die neue Theorie der endlichen abelschen Gruppen und Verallgemeinerung des Hauptsatzes von Hajós*. Acta Math. Acad. Sci. Hung. **16** (1965), 329–373. Zbl 0138.26001

[8] Sands, A. D.: *The factorization of abelian groups*. Q. J. Math., Oxf. II. Ser. **10** (1959), 81–91. Zbl 0087.25604

[9] Sands, A. D.: *Factorization of cyclic groups*. Proc. Colloq. on Abelian Groups (Tihany) 1964, 139–146. Zbl 0132.27103

[10] Sands, A. D.: *Replacement of factors by subgroups in the factorization of abelian groups*. Bull. Lond. Math. Soc. **32** (2000), 297–304. Corrigendum ibid. **37**(1) (2005), 80. Zbl 1021.20039

[11] Sands, A. D.; Szabó, S.: *Factorization of periodic subsets*. Acta Math. Hung. **57** (1991), 159–167. Zbl 0754.20014

Received July 4, 2005