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Banach-Mazur Distance of Central Sections of a Centrally Symmetric Convex Body

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Abstract. We prove that the Banach-Mazur distance between arbitrary two central sections of co-dimension c of any centrally symmetric convex body in E^n is at most $(2c+1)^2$.

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As usual, by a convex body of Euclidean n-space E^n we mean a compact convex set with non-empty interior. Denote by $\mathcal{B}^n$ the family of all centrally symmetric convex bodies of E^n which are centered at the center o of E^n. Let E_1^k and E_2^k be k-dimensional subspaces of E^n, let C_1 be a convex body of E_1^k centered at o and let C_2 be a convex body of E_2^k centered at o. The *Banach-Mazur distance between* C_1 and C_2 is the number

$$\delta(C_1, C_2) \,=\, \inf\{\lambda;\ a(C_2) \subset C_1 \subset \lambda\, a(C_2)\},$$

where a stands for an affine transformation, and λA stands for the image of a set A under the homothety with center o and a positive ratio λ.

Extensive surveys of results on Banach-Mazur distance are given by Thompson [5] and by Tomczak-Jaegermann [6]. See also the last section of the recent book by Brass, Moser, and Pach [1]. The classic paper of Dvoretzky [2] stipulated intensive research on Banach-Mazur distance between central sections of centrally symmetric convex bodies. In particular, Rudelson [4] considers asymptotic behavior of Banach-Mazur distance between k-dimensional sections of bodies of $\mathcal{B}^n$.

Our aim is to prove the upper bound $(2c+1)^2$ of the Banach-Mazur distance between every two $(n-c)$-dimensional central sections of an arbitrary body of $\mathcal{B}^n$. Let us point out that this estimate depends only on the co-dimension c of the sections. So our estimate does not grow when the dimension n tends to infinity.

The proof of our Theorem is based on Lemma whose formulation requires some notation. Let $C \in \mathcal{B}^n$. Let S be a central section of C of co-dimension c, this is of dimension $n-c$. By compactness arguments we see that there exist c segments $I_1, \dots, I_c$ centered at o whose end-points are in the boundary of C such that the convex hull

$$P = \operatorname{conv}(I_1 \cup \dots \cup I_c \cup S) \tag{1}$$

has the maximum volume from amongst all convex hulls of this form, where S is fixed. Clearly, $P \subset C$.

Since the Banach-Mazur distance is invariant with respect to affine transformations, without loss of generality further we assume that I_i is the segment of length 2 contained in the i-th coordinate axis of E^n and centered at o for $i \in \{1, \dots, c\}$, and that S is in the $(n-c)$-dimensional subspace containing the remaining coordinate axes of E^n.

Lemma. *Let $C \in \mathcal{B}^n$ and let S be an $(n-c)$-dimensional central section of C. For the cylinder $K = I_1 \times \dots \times I_c \times (c+1)S$, where $I_1, \dots, I_c$ are defined above, we have*

$$\delta(C, K) \le 2c + 1.$$

Proof. For every $i \in \{1, \dots, c\}$ we denote by g_i and h_i the end-points of I_i. We provide through every g_i the hyperplane G_i parallel to the hyperplane containing S and all the segments from amongst $I_1, \dots, I_c$ which are different from I_i. Analogously, through every h_i we provide the hyperplane H_i parallel to G_i. In order to see that

$$G_1, \dots, G_c, H_1, \dots, H_c \text{ are supporting hyperplanes of } C, \tag{2}$$

assume the opposite. Then the central symmetry of C and of our construction implies that a $j \in \{1, \dots, c\}$ exists such that G_j, H_j are not supporting hyperplanes of C. As a consequence, we can find a segment $J_j \subset C$ centered at o such that its end-points are out of the strip between G_j and H_j. Thus $\operatorname{conv}(J_1 \cup \dots \cup J_c \cup S)$, where $J_m = I_m$ for all $m \in \{1, \dots, c\}$ different from j, has volume greater than P, see (1). So our opposite assumption contradicts the choice of $I_1, \dots, I_c$, see (1). Thus (2) is true.

We intend to show that

$$C \subset K. \tag{3}$$

Assume that this is not true, i.e. assume that there exists a point $u \in C$ such that $u \notin K$. Since u is out of K, from (2) we conclude that $u = (a_1, \dots, a_c, qa_{c+1}, \dots, qa_n)$, where $q > c+1$, such that $|a_1| \le 1, \dots, |a_c| \le 1$ and such that $w = (0, \dots, 0, a_{c+1}, \dots, a_n)$ is a point of the relative boundary of S. We provide the straight line through u and w. Its parametric equation is $x_1 = ta_1, \dots, x_c =$

ta_c, $x_{c+1} = ((q-1)t+1)a_{c+1}, \dots, x_n = ((q-1)t+1)a_n$, where $-\infty < t < \infty$. For $t = -\frac{1}{q-1}$ we get the point $z = (-\frac{1}{q-1}a_1, \dots, -\frac{1}{q-1}a_c, 0, \dots, 0)$. Since $|-\frac{1}{q-1}a_1| + \dots + |-\frac{1}{q-1}a_c| = \frac{1}{q-1}(|a_1| + \dots + |a_c|) \le \frac{c}{q-1} < 1$, we conclude that z is an interior point of P. From $P \subset C$ we see that z is an interior point of C. Hence the assumption that $u \in C$ and the fact that w is a point of the segment uz different from u imply that w is an interior point of C. This contradicts the fact that w is a point of the relative boundary of S. As a consequence, (3) holds true.

Now we will show that

$$\frac{1}{2c+1}K \subset P. \tag{4}$$

Since every convex body is the convex hull of its extreme points, it is sufficient to show that all extreme points of $\frac{1}{2c+1}K$ are in P. Every extreme point of $\frac{1}{2c+1}K$ has the form $e' = (\frac{1}{2c+1}e_1, \dots, \frac{1}{2c+1}e_n)$, where $e = (e_1, \dots, e_n)$ is an extreme point of K. Then $|e_1| = \dots = |e_c| = 1$ and $(0, \dots, 0, e_{c+1}, \dots e_n)$ is in the relative boundary of $(c+1)S$.

The segment oe has the equation $x_1 = te_1, \dots, x_n = te_n$, where $0 \le t \le 1$. The equation of the boundary bd(P) of P is $|x_1| + \dots + |x_c| + ||(0, \dots, 0, x_{c+1}, \dots, x_n)|| = 1$, where $||\ \ ||$ denotes the norm of the normed space whose unit ball is C. In order to find the point of the intersection of the segment oe with bd(P) we substitute the above equation of oe into the above equation of bd(P). We obtain $ct + ||(0, \dots, 0, te_{c+1}, \dots, te_n)|| = 1$. Since $(0, \dots, 0, e_{c+1}, \dots, e_n)$ belongs to the relative boundary of $(c+1)S$ which is a subset of the boundary of $(c+1)C$, we get $ct + (c+1)t = 1$. Hence for $t' = \frac{1}{2c+1}$ we obtain a common point of oe and bd(P). Substituting t' into the parametric equation of the segment oe, we see that this point is just e'. We conclude that every extreme point e' of $\frac{1}{2c+1}K$ belongs to P. So (4) has been shown.

From (3), (4) and from $P \subset C$ we obtain that

$$\frac{1}{2c+1}K \subset C \subset K.$$

This implies the thesis of Lemma. □

Theorem. *Let S_1 and S_2 be central sections of co-dimension c of a centrally symmetric convex body in E^n. Then*

$$\delta(S_1, S_2) \le (2c+1)^2.$$

Proof. Assume that $S_1 \ne S_2$ and that $S_0 = S_1 \cap S_2$ is $(n-d)$-dimensional. Of course, $d \le 2c$. Clearly S_0 is an $(n-d)$-dimensional central section of S_i, where $i \in \{1, 2\}$. We apply Lemma taking S_i, where $i \in \{1, 2\}$, in the part of C. Since S_i is of co-dimension c, the present $n - c$ plays the part of n from Lemma. Moreover, we take S_0 in the part of S. For the section S_0 of S_i, where $i \in \{1, 2\}$, we define a cylinder K_i analogically like the cylinder K is defined for S in Lemma. Since S_i is $(n-c)$-dimensional, K_i is $(n-c)$-dimensional for $i \in \{1, 2\}$. From $(n-c) - (n-d) = d - c$ and by Lemma we get $\delta(S_1, K_1) \le 2(d-c) + 1$ and

$\delta(S_2, K_2) \leq 2(d-c)+1$. These inequalities, the obvious equality $\delta(K_1, K_2) = 1$ and $0 \leq d \leq 2c$ imply $\delta(S_1, S_2) \leq (2(d-c)+1)^2 \cdot 1 = (2d-2c+1)^2 \leq (2c+1)^2$. □

By John's [3] theorem, $\delta(S_1, S_2) \leq n - c$ under the assumptions of Theorem. Thus the estimate from Theorem is better only when $(2c+1)^2 < n - c$. So for $n > (2c+1)^2 + c$. In particular, for $n > 10$ when $c = 1$, and for $n > 27$ when $c = 2$.

From the proof of Theorem we conclude the following more precise corollary. Theorem is its special case for $d = 2c$.

Corollary. *Let S_1 and S_2 be central sections of co-dimension c of a centrally symmetric convex body in E^n such that $S_1 \cap S_2$ is of co-dimension d. Then*

$$\delta(S_1, S_2) \leq (2d - 2c + 1)^2.$$

The author expects that the estimates from Theorem and Corollary are not the best possible and would not be surprised if the bound $2c+1$ or better holds true. *The problem is to improve the estimate obtained in Theorem.* Especially for $c = 1$. Just for $c = 1$ our Theorem gives the estimate 9, while the author is not able to find an n and a $C \in \mathcal{B}^n$ with two central $(n-1)$-dimensional sections whose Banach-Mazur distance is over 2.

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