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Combinatorial 3-Manifolds with 10 Vertices

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Abstract. We give a complete enumeration of all combinatorial 3-manifolds with 10 vertices: There are precisely 247882 triangulated 3-spheres with 10 vertices as well as 518 vertex-minimal triangulations of the sphere product $S^2\times S^1$ and 615 triangulations of the twisted sphere product $S^2 \underline{\times} S^1$.
All the 3-spheres with up to 10 vertices are shellable, but there are 29 vertex-minimal non-shellable 3-balls with 9 vertices.

1. Introduction

Let M be a triangulated 3-manifold with n vertices and face vector $f = (n, f_1, f_2, f_3)$. By Euler's equation, $n - f_1 + f_2 - f_3 = 0$, and by double counting the edges of the ridge-facet incidence graph, $2f_2 = 4f_3$, it follows that

$$f = (n, f_1, 2f_1 - 2n, f_1 - n). \tag{1}$$

A complete characterization of the f-vectors of the 3-sphere S^3, the sphere product $S^2\times S^1$, the twisted sphere product (or 3-dimensional Klein bottle) $S^2 \underline{\times} S^1$, and of the real projective 3-space $\mathbb{R}\mathbf{P}^3$ was given by Walkup.

Theorem 1. (Walkup [31]) *For every 3-manifold M there is an integer $\gamma(M)$ such that*

$$f_1 \geq 4n + \gamma(M) \tag{2}$$

for every triangulation of M with n vertices and f_1 edges. Moreover, there is an integer $\gamma^*(M) \geq \gamma(M)$ such that for every pair (n, f_1) with $n \geq 5$ and

$$\binom{n}{2} \geq f_1 \geq 4n + \gamma^*(M) \tag{3}$$

there is a triangulation of M with n vertices and f_1 edges. In particular,

(a) $\gamma^* = \gamma = -10$ *for* S^3,

(b) $\gamma^* = \gamma = 0$ *for* $S^2 \underline{\times} S^1$,

(c) $\gamma^* = 1$ *and* $\gamma = 0$ *for* $S^2 \times S^1$, *where, with the exception* $(9, 36)$, *all pairs* (n, f_1) *with* $n \geq 5$ *and* $\binom{n}{2} \geq f_1 \geq 4n$ *occur,*

(d) $\gamma^* = \gamma = 7$ *for* $\mathbb{R}\mathbf{P}^3$, *and*

(e) $\gamma^*(M) \geq \gamma(M) \geq 8$ *for all other* 3*-manifolds* M.

By Walkup's theorem, vertex-minimal triangulations of $S^2 \underline{\times} S^1$, $S^2 \times S^1$, and $\mathbb{R}\mathbf{P}^3$ have 9, 10, and 11 vertices, respectively. The 3-sphere can be triangulated vertex-minimally as the boundary of the 4-simplex with 5 vertices. But otherwise, rather little is known on vertex-minimal triangulations of 3- and higher-dimensional manifolds. See [24], [25], and [30] for a discussion, further references, and for various examples of small triangulations of 3-manifolds.

The exact numbers of different combinatorial types of triangulations of S^3, $S^2 \underline{\times} S^1$, and $S^2 \times S^1$ with up to 9 vertices and of neighborly triangulations (i.e., triangulations with complete 1-skeleton) with 10 vertices were obtained by

Grünbaum and Sreedharan [15]	(simplicial 4-polytopes with 8 vertices),
Barnette [8]	(combinatorial 3-spheres with 8 vertices),
Altshuler [2]	(combinatorial 3-manifolds with up to 8 vertices),
Altshuler and Steinberg [5]	(neighborly 4-polytopes with 9 vertices),
Altshuler and Steinberg [6]	(neighborly 3-manifolds with 9 vertices),
Altshuler and Steinberg [7]	(combinatorial 3-manifolds with 9 vertices),
Altshuler [3]	(neighborly 3-manifolds with 10 vertices).

Vertices\Types	All	S^3	$S^2 \times S^1$	$S^2 \underline{\times} S^1$
5	1	1	–	–
6	2	2	–	–
7	5	5	–	–
8	39	39	–	–
9	1297	1296	–	1
10	249015	247882	518	615

Table 1. Combinatorial 3-manifolds with up to 10 vertices

In this paper, the enumeration of 3-manifolds is continued: We completely classify triangulated 3-manifolds with 10 vertices. Moreover, we determine the combinatorial automorphism groups of all triangulations with up to 10 vertices, and we test for all 3-spheres (and all 3-balls) with up to 10 vertices (with up to 9 vertices) whether they are constructible, shellable, or vertex-decomposable. (See [29] for enumeration results for triangulated 3-manifolds with 11 vertices.)

2. Enumeration

We used a backtracking approach, described as mixed-lexicographic enumeration in [26], to determine all triangulated 3-manifolds with 10 vertices: The vertex-links of a triangulated 3-manifold with 10 vertices are triangulated 2-spheres with up to 9 vertices. Altogether, there are 73 such 2-spheres, which are processed in decreasing size. As a first vertex-star of a 3-manifold that we are going to build we take the cone over one of the respective 2-spheres and then add further tetrahedra (in lexicographic order) as long as this is possible. If, for example, a triangle of a partial complex that we built is contained in three tetrahedra, then this violates the *pseudo-manifold property*, which requires that in a triangulated 3-manifold every triangle is contained in exactly two tetrahedra. We backtrack, remove the last tetrahedron that we added, and try to add to our partial complex the next tetrahedron (with respect to the lexicographic order). See [26] for further details on the enumeration.

f-vector\Types	All	S^3	$S^2\times S^1$	$S^2 \,\underline{\times}\, S^1$
(10,30,40,20)	30	30	–	–
(10,31,42,21)	124	124	–	–
(10,32,44,22)	385	385	–	–
(10,33,46,23)	952	952	–	–
(10,34,48,24)	2142	2142	–	–
(10,35,50,25)	4340	4340	–	–
(10,36,52,26)	8106	8106	–	–
(10,37,54,27)	13853	13853	–	–
(10,38,56,28)	21702	21702	–	–
(10,39,58,29)	30526	30526	–	–
(10,40,60,30)	38575	38553	10	12
(10,41,62,31)	42581	42498	37	46
(10,42,64,32)	39526	39299	110	117
(10,43,66,33)	28439	28087	162	190
(10,44,68,34)	14057	13745	145	167
(10,45,70,35)	3677	3540	54	83
Total:	249015	247882	518	615

Table 2. Combinatorial 3-manifolds with 10 vertices

Theorem 2. *There are precisely* 249015 *triangulated* 3*-manifolds with* 10 *vertices:* 247882 *of these are triangulated* 3*-spheres,* 518 *are vertex-minimal triangulations of the sphere product* $S^2\times S^1$*, and* 615 *are triangulations of the twisted sphere product* $S^2 \,\underline{\times}\, S^1$*.*

n	Manifold	$\|G\|$	G	Types
5	S^3	120	S_5, transitive	1
6	S^3	48	$O^* = \mathbb{Z}_2 \wr S_3$	1
		72	$S_3 \wr \mathbb{Z}_2$, transitive	1
7	S^3	8	D_4	2
		12	$S_3 \times \mathbb{Z}_2$	1
		14	D_7, transitive	1
		48	$D_4 \times D_3$	1
8	S^3	1	trivial	3
		2	$\mathbb{Z}_2$	13
		4	$\mathbb{Z}_4$	1
			$\mathbb{Z}_2 \times \mathbb{Z}_2$	9
		6	S_3	1
		8	$\mathbb{Z}_2^3$	1
			D_4	3
		12	$S_3 \times \mathbb{Z}_2$	4
		16	$D_4 \times \mathbb{Z}_2$	1
			D_8, transitive	1
		60	$D_5 \times D_3$	1
		384	$\mathbb{Z}_2 \wr S_4$, transitive	1
9	S^3	1	trivial	889
		2	$\mathbb{Z}_2$	319
		3	$\mathbb{Z}_3$	3
		4	$\mathbb{Z}_4$	6
			$\mathbb{Z}_2 \times \mathbb{Z}_2$	46
		6	$\mathbb{Z}_6$	1
			S_3	8
		8	$\mathbb{Z}_2^3$	3
			D_4	5
		12	$S_3 \times \mathbb{Z}_2$	10
		18	D_9, transitive	1
		24	$T^* = S_4$	3
		72	$D_6 \times D_3$	1
		80	$D_5 \times D_4$	1
	$S^2 \underline{\times} S^1$	18	D_9, transitive	1
10	S^3	1	trivial	240683
		2	$\mathbb{Z}_2$	6675
		3	$\mathbb{Z}_3$	10
		4	$\mathbb{Z}_4$	53
			$\mathbb{Z}_2 \times \mathbb{Z}_2$	358
		5	$\mathbb{Z}_5$	1
		6	$\mathbb{Z}_6$	1
			S_3	19
		8	$\mathbb{Z}_2^3$	15
			D_4	31
		10	$\mathbb{Z}_{10}$, transitive	1
			D_5	4
		12	$S_3 \times \mathbb{Z}_2$	15
		16	$D_4 \times \mathbb{Z}_2$	3
		20	D_{10}, transitive	1
			$AGL(1,5)$, transitive	2
		24	$T^* = S_4$	1
			$D_6 \times \mathbb{Z}_2$	2
		48	$O^* = \mathbb{Z}_2 \wr S_3$	2
		84	$D_7 \times D_3$	1
		96	$D_6 \times D_4$	1
		120	S_5	1
		200	$D_5 \wr \mathbb{Z}_2$, transitive	1
		240	$S_5 \times \mathbb{Z}_2$, transitive	1
	$S^2 \times S^1$	1	trivial	420
		2	$\mathbb{Z}_2$	95
		10	$\mathbb{Z}_{10}$, transitive	1
		16	$\langle 2,2,2 \rangle_2$	1
		20	D_{10}, transitive	1
	$S^2 \underline{\times} S^1$	1	trivial	469
		2	$\mathbb{Z}_2$	127
		4	$\mathbb{Z}_2 \times \mathbb{Z}_2$	14
		8	D_4	2
		10	D_5	1
		20	D_{10}, transitive	2

Table 3. Symmetry groups of triangulated 3-manifolds with up to 10 vertices

Table 1 gives the total numbers of all triangulations with up to 10 vertices. The numbers of 10-vertex triangulations are listed in detail in Table 2. All triangulations can be found online at [22]. The topological types were determined with the bistellar flip program BISTELLAR [23]; see [10] for a description.

For a given triangulation, it is a purely combinatorial task to determine its combinatorial symmetry group. We computed the respective groups with a program written in GAP [14].

Corollary 3. *There are exactly* 1, 1, 5, 36, 408, *and* 7443 *triangulated* 3*-manifolds with* 5, 6, 7, 8, 9, *and* 10 *vertices, respectively, that have a non-trivial combinatorial symmetry group.*

The symmetry groups along with the numbers of combinatorial types of triangulations that correspond to a particular group are listed in Table 3. Altogether, there are 14 examples that have a vertex-transitive symmetry group; see [18].

All simplicial 3-spheres with up to 7 vertices are polytopal. However, there are two non-polytopal 3-spheres with 8 vertices, the Grünbaum and Sreedharan sphere [15] and the Barnette sphere [8]. The classification of triangulated 3-spheres with 9 vertices into polytopal and non-polytopal spheres was started by Altshuler and Steinberg [5], [6], [7] and completed by Altshuler, Bokowski, and Steinberg [4] and Engel [13]. For neighborly simplicial 3-spheres with 10 vertices the numbers of polytopal and non-polytopal spheres were determined by Altshuler [3], Bokowski and Garms [11], and Bokowski and Sturmfels [12].

Problem 4. *Classify all simplicial* 3*-spheres with* 10 *vertices into polytopal and non-polytopal spheres.*

3. 3-Balls

Along with the enumeration of triangulated 3-spheres with up to 10 vertices we implicitly enumerated all triangulated 3-balls with up to 9 vertices: Let B^3_{n-1} be a triangulated 3-ball with $n-1$ vertices and let v_n be a new vertex. Then the union $B^3_{n-1} \cup (v_n * \partial B^3_{n-1})$ of B^3_{n-1} with the cone $v_n * \partial B^3_{n-1}$ over the boundary ∂B^3_{n-1} with respect to v_n is a triangulated 3-sphere. Thus there are at most as many combinatorially distinct 3-spheres with n vertices as there are combinatorially distinct 3-balls with $n-1$ vertices. If, on the contrary, we delete the star of a vertex from a triangulated 3-sphere S^3_n with n vertices, then, obviously, we obtain a 3-ball with $n-1$ vertices. If we delete the star of a different vertex from S^3_n then we might or might not obtain a combinatorially different ball. Let $\#B^3(n-1)$ and $\#S^3(n)$ be the numbers of combinatorially distinct 3-balls and 3-spheres with $n-1$ and n vertices, respectively. Then

$$\#S^3(n) \leq \#B^3(n-1) \leq n \cdot \#S^3(n).$$

For the explicit numbers of simplicial 3-balls with up to 9 vertices see Table 4.

Vertices\Types	All	Non-Shellable	Not Vertex-Decomposable
4	1	–	–
5	3	–	–
6	12	–	–
7	167	–	2
8	10211	–	628
9	2451305	29	623819

Table 4. Combinatorial 3-balls with up to 9 vertices

4. Vertex-decomposability, shellability, and constructibility

The concepts of vertex-decomposability, shellability, and constructibility describe three particular ways to assemble a simplicial complex from the collection of its facets (cf. Björner [9] and see the surveys [16], [19], and [32]). The following implications are strict for (pure) simplicial complexes:

$$\text{vertex decomposable} \implies \text{shellable} \implies \text{constructible.}$$

It follows from Newman's and Alexander's fundamental works on the foundations of combinatorial and PL topology from 1926 [27] and 1930 [1] that a constructible d-dimensional simplicial complex in which every $(d-1)$-face is contained in exactly two or at most two d-dimensional facets is a PL d-sphere or a PL d-ball, respectively.

A *shelling* of a triangulated d-ball or d-sphere is a linear ordering of its f_d facets $F_1, \dots, F_{f_d}$ such that if we remove the facets from the ball or sphere in this order, then at every intermediate step the remaining simplicial complex is a simplicial ball. A simplicial ball or sphere is *shellable* if it has a shelling; it is *extendably shellable* if any partial shelling $F_1, \dots, F_i$, $i < f_d$, can be extended to a shelling; and it is *strongly non-shellable* if it has no *free* facet that can be removed from the triangulation without loosing ballness.

A triangulated d-ball or d-sphere is *constructible* if it can be decomposed into two constructible d-balls of smaller size (with a single d-simplex being constructible) and if, in addition, the intersection of the two balls is a constructible ball of dimension $d-1$. A triangulated d-ball or d-sphere is *vertex-decomposable* if we can remove the star of a vertex v such that the remaining complex is a vertex-decomposable d-ball (with a single d-simplex being vertex-decomposable) and such that the link of v is a vertex-decomposable $(d-1)$-ball or a vertex-decomposable $(d-1)$-sphere, respectively.

We tested vertex-decomposability and shellability with a straightforward backtracking implementation.

Corollary 5. *All triangulated 3-spheres with $n \leq 10$ vertices are shellable and therefore constructible.*

An example of a non-constructible and thus non-shellable 3-sphere with 13 vertices was constructed in [19], whereas all 3-spheres with 11 vertices are shellable [29].

It remains open whether there are non-shellable respectively non-constructible 3-spheres with 12 vertices.

Corollary 6. *All triangulated* 3*-balls with* $n \leq 8$ *vertices are shellable and therefore extendably shellable.*

Examples of non-shellable 3-balls can be found at various places in the literature (cf. the references in [19], [20], and [32]) with the smallest previously known non-shellable 3-ball by Ziegler [32] with 10 vertices.

Corollary 7. *There are precisely* 29 *vertex-minimal non-shellable simplicial* 3*-balls with* 9 *vertices, ten of which are strongly non-shellable. The twenty-nine balls have between* 18 *and* 22 *facets, with one unique ball* $B\_3\_9\_18$ *having* 18 *facets and* f*-vector* $(9, 33, 43, 18)$.

A list of the facets and a visualization of the ball $B\_3\_9\_18$ is given in [20].

The cone over a simplicial d-ball with respect to a new vertex is a $(d+1)$-dimensional ball. It is shellable respectively vertex-decomposable if and only if the original ball is shellable respectively vertex-decomposable (cf. [28]).

Corollary 8. *There are non-shellable* 3*-balls with* $d+6$ *vertices and* 18 *facets for* $d \geq 3$.

Each of the 29 non-shellable 3-balls with 9 vertices can be split into a pair of shellable balls.

Corollary 9. *All triangulated* 3*-balls with* $n \leq 9$ *vertices are constructible.*

Klee and Kleinschmidt [17] showed that all simplicial d-balls with up to $d+3$ vertices are vertex-decomposable.

Corollary 10. *There are not vertex-decomposable* 3*-balls with* $d+4$ *vertices and* 10 *facets for* $d \geq 3$.

In fact, there are exactly two not vertex-decomposable 3-balls with 7 vertices; see [21] for a visualization of these two balls. One of the examples has 10 tetrahedra, the other has 11 tetrahedra.

For the numbers of not vertex-decomposable 3-balls with up to 9 vertices see Table 4.

Corollary 11. *All triangulated* 3*-spheres with* $n \leq 8$ *vertices are vertex-decomposable.*

Klee and Kleinschmidt [17] constructed an example of a not vertex-decomposable polytopal 3-sphere with 10 vertices.

Corollary 12. *There are precisely* 7 *not vertex-decomposable* 3*-spheres with* 9 *vertices, which are all non-polytopal. Moreover, there are* 14468 *not vertex-decomposable* 3*-spheres with* 10 *vertices.*

Four of the seven examples with 9 vertices are neighborly with 27 tetrahedra, the other three have 25, 26, and 26 tetrahedra, respectively. The 25 tetrahedra of the smallest example are:

1234	1235	1246	1257	1268
1278	1345	1456	1567	1679
1689	1789	2348	2359	2378
2379	2468	2579	3458	3568
3569	3689	3789	4568	5679.

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