Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 50 (2009), No. 1, 1-9.

A Characterization of $L_2(2^f)$ in Terms of the Number of Character Zeros*

Guohua Qian **Wujie Shi**

Department of Mathematics, Changshu Institute of Technology
Changshu, Jiangsu, 215500, P. R. China
e-mail: ghqian2000@yahoo.com.cn

School of Mathematics, Suzhou University
Suzhou, Jiangsu, 215006, P. R. China
e-mail: wjshi@suda.edu.cn

Abstract. The aim of this paper is to show that $L_2(2^f)$ are the only nonsolvable groups in which every irreducible character of even degree vanishes on just one conjugacy class.

MSC 2000: 20C15
Keywords: finite group, character

1. Introduction

For an irreducible character χ of a finite group G, we know that $v(\chi) := \{g \in G \mid \chi(g) = 0\}$ is a union of some conjugacy classes of G. An old theorem of Burnside asserts that $v(\chi)$ is not empty for any nonlinear $\chi \in \mathrm{Irr}(G)$. It is natural to consider the structure of a finite group provided that the number of character zeros in its character table is very small (see [1], [11], [12] for a few examples). In Berkovich and Kazarin's paper [1], they posed the following question.

Question. *Is it true that $L_2(2^f), f \geq 2$ are the only nonabelian simple groups in which every irreducible character of even degree vanishes on just one conjugacy class?*

*Project supported by the NNSF of China (10571128) and the NSF of Jiangsu Educational Committee (05KJB110002).

Our answer to the question is affirmative.

Theorem A. *Let G be a finite group. If every $\chi \in \mathrm{Irr}(G)$ of even degree vanishes on just one conjugacy class, then G is just one of the following groups:*

(1) *G possesses a normal and abelian Sylow 2-subgroup.*

(2) *G is a Frobenius group with a complement of order 2.*

(3) *$G \cong SL(2,3)$.*

(4) *$G \cong L_2(2^f), f \geq 2$.*

In particular, $L_2(2^f)$ ($f \geq 2$) are the only nonsolvable groups, and therefore the only nonabelian simple groups satisfying the hypothesis.

Instead of proving Theorem A directly, we will study the finite nonsolvable groups G satisfying the following property

($*$) every nonlinear $\chi \in \mathrm{Irr}(G)$ of even degree vanishes on at most two conjugacy classes of G.

Theorem B. *If G is a finite nonsolvable group with no nontrivial solvable normal subgroup, then G has the property ($*$) if and only if $G \cong L_2(7)$ or $L_2(2^f)$ where $f \geq 2$.*

In this paper, G always denotes a finite group, a class always means a conjugacy class. We denote by x^G the conjugacy class of G in which x lies. For a subset A of G, let $k_G(A)$ be the minimal integer l such that A is a subset of a union of l conjugacy classes of G. For $N \lhd G$, we put $\mathrm{Irr}(G|N) = \mathrm{Irr}(G) - \mathrm{Irr}(G/N)$; and for $\lambda \in \mathrm{Irr}(N)$, the inertia subgroup of λ in G is denoted by $\mathrm{I}_G(\lambda)$.

Let $\mathrm{Irr}_2(G)$ be the set of irreducible characters of G with even degree. Our proof depends on the classification theorem of finite simple groups.

2. Theorem B

We begin to list some easy results which will be used later.

Lemma 2.1. *Let $N \lhd G$ and set $\overline{G} = G/N$. Then the following results are true.*

(1) *For any $x \in G$, $\overline{x}^{\overline{G}}$, viewed as a subset of G, is a union of some classes of G; furthermore, $k_G(\overline{x}^{\overline{G}}) = 1$ if and only if $\chi(x) = 0$ for any $\chi \in \mathrm{Irr}(G|N)$.*

(2) *If G has the property ($*$), then so has G/N.*

Proof. (1) See [11, Lemma 3(1)].

(2) The result follows directly from (1). □

Lemma 2.2. *For any nonlinear $\chi \in \mathrm{Irr}(G)$, we have:*

(1) *If G is nonsolvable and $k_G(v(\chi)) \leq 2$, then $\chi_{G'}$ is irreducible.*

(2) *If $v(\chi) \subset N$ for some $N \lhd G$, then $gcd(\chi(1), |G/N|) = 1$. In particular, χ_N is irreducible.*

Proof. (1) Suppose that $\chi_{G'}$ is reducible. By [7, Theorem 6.28], we can find a normal subgroup M of G with $G' \leq M < G$ and an irreducible character ψ of M such that $\chi = \psi^G$. It follows that χ vanishes on $G - M$, and thus $k_G(G - M) \leq 2$. By [13, Theorem 2.2] G is solvable, a contradiction.

(2) See [12, Lemma 2.2]. □

Next, we need the following Lemma 2.3. An irreducible character χ of G is called p-defect zero for some prime p if $\chi(1)_p = |G|_p$, that is, the p-part of the degree $\chi(1)$ equals the p-part of the order of G. It is well-known that if $\chi \in \mathrm{Irr}(G)$ is p-defect zero then $\chi(x) = 0$ whenever $x \in G$ is of order a multiple of p.

Lemma 2.3. *Let G be a nonabelian simple group. Then there exists $\chi \in \mathrm{Irr}_2(G)$ such that χ is of p-defect zero for some prime divisor p of $|G|$.*

Proof. It suffices to consider the nonabelian simple group G with no irreducible character of 2-defect zero. By [15, Corollary], we may assume $G \cong A_n$ or $G \cong M_{12}$, M_{22}, M_{24}, J_2, HS, Suz, Ru, Co_1, Co_3, or B. Suppose that G is isomorphic to $A_n, n \leq 8$ or one of the above sporadic simple groups. Then the result follows by [2]. Suppose that $G \cong A_n, n \geq 9$. By [9, Proposition], there is $\chi \in \mathrm{Irr}(G)$ such that $2p|\chi(1)$, where p is the maximal prime not exceeding n. Clearly, χ is of p-defect zero since $|G|_p = p$. □

Now we are ready to prove Theorem B.

Proof of Theorem B. Let N be a minimal normal subgroup of G. Since G has no nontrivial solvable normal subgroup, N is nonsolvable.

Step 1. G is almost simple, that is, N is a nonabelian simple group with $N \leq G \leq Aut(N)$.

Clearly $N = N_1 \times \cdots \times N_s$ is a direct product of isomorphic simple groups $N_i, 1 \leq i \leq s$. Suppose that $s \geq 2$. Let $\theta_i \in \mathrm{Irr}_2(N_i)$ be of p-defect zero (Lemma 2.3), and set $\theta = \theta_1 \times \cdots \times \theta_s$. Then θ is an irreducible character of N, also $\theta^g \in \mathrm{Irr}(N)$ is of p-defect zero for any $g \in G$. Let χ_0 be an irreducible constituent of θ^G, let $x_1 \in N_1, x_2 \in N_2$ be of order p, and $y_2 \in N_2$ be of a prime order q $(q \neq p)$. Now for any $g \in G$, we have

$$\theta^g(x_1) = \theta^g(x_1 x_2) = \theta^g(x_1 y_2) = 0,$$

and this implies that $\chi_0(x_1) = \chi_0(x_1 x_2) = \chi_0(x_1 y_2) = 0$. Since x_1, x_1x_2, x_1y_2 lie in distinct conjugacy classes, we obtain a contradiction. Thus N is simple.

Suppose that $C_G(N) > 1$. Then $C_G(N)$ contains a minimal normal subgroup M of G. Set $T = M \times N$. Arguing on $M \times N$ as in the above paragraph, we conclude that M, N are nonabelian simple groups, and we can find $\psi \in \mathrm{Irr}_2(M)$, $\theta \in \mathrm{Irr}_2(N)$ so that ψ is of q-defect zero, and θ is of p-defect zero, where q, p are prime divisors of $|M|$ and $|N|$ respectively. Let $x \in M$, $y \in N$ be of order q, p respectively. Then for any irreducible constituent χ of $(\psi \times \theta)^G$, we see that $\chi(x) = \chi(y) = \chi(xy) = 0$. Clearly, x, y, xy lie in distinct classes of G, a contradiction. Thus $C_G(N) = 1$, so $N \le G \le Aut(N)$, and then G is an almost simple group.

Step 2. N is a simple group of Lie type.

Suppose that $N \cong A_n$ for some $n \ge 8$. Let π be the permutation character of N, and δ be the mapping of N into $\{0, 1, 2, \cdots\}$ such that $\delta(g)$ is the number of 2-cycles in the standard composition of g. Set

$$\lambda = \frac{(\pi-1)(\pi-2)}{2} - \delta,\ \rho = \frac{\pi(\pi-3)}{2} + \delta.$$

By [5, V, Theorem 20.6], both λ and ρ are irreducible characters of N. Observe that either $\lambda(1) = (n-1)(n-2)/2$ or $\rho(1) = n(n-3)/2$ is even. Let χ_0 be an irreducible constituent of τ^G, where $\tau \in \{\lambda, \rho\}$ is of even degree. Since $G/N \le Out(N) = Out(A_n) = Z_2$ ($n \ge 8$, see [2]), it follows that $N = G'$. Now Lemma 2.2 (1) implies that $(\chi_0)_N = \tau$.

For even n, set

$a_1 = (1, \ldots, n-1)$, $a_2 = (1, \ldots, n-2)(n-1, n)$,
$a_3 = (1, \ldots, n-5)(n-4, n-3, n-2)$;
$b_1 = (1, \ldots, n-3), b_2 = (1, 2, \ldots n-3)(n-2, n-1, n)$,
$b_3 = (1, \ldots, n-4)(n-3, n-2)$.

For odd n, set

$a_1 = (1, \ldots, n-2)$, $a_2 = (1, \ldots, n-4)(n-3, n-2, n-1)$,
$a_3 = (1, \ldots, n-5)(n-4, n-3)$;
$b_1 = (1, \ldots, n)$, $b_2 = (1, \ldots, n-3)(n-2, n-1)$,
$b_3 = (1, \ldots, n-6)(n-5, n-4, n-3)$.

We see that $\lambda(a_i) = 0 = \rho(b_i)$ for any $i = 1, 2, 3$. Therefore, either $\chi_0(a_1) = \chi_0(a_2) = \chi_0(a_3) = 0$ or $\chi_0(b_1) = \chi_0(b_2) = \chi_0(b_3) = 0$. Clearly a_1, a_2, a_3 (or b_1, b_2, b_3) lie in distinct classes of G. We obtain a contradiction.

Suppose that N is isomorphic to A_7 or one of the sporadic simple groups. Assume $G = N$. We obtain a contradiction by [2]. Assume $G > N$. Since G has no nontrivial solvable normal subgroup, $G \le Aut(N)$. It follows by [2] that $|Out(N)| \le 2$, and so $|G/N| = 2$ and $N = G'$. By Lemma 2.2, every $\theta \in \mathrm{Irr}_2(N)$ is extendable to $\chi \in \mathrm{Irr}(G)$, and that $k_G(v(\theta)) = k_G(v(\chi) \cap N) \le 1$. By [2], we also get a contradiction. .

Note that $A_5 \cong L_2(4) \cong L_2(5)$, $A_6 \cong L_2(9)$. By the classification theorem of finite simple groups, N must be a simple group of Lie type.

Remarks and notation: Since N is one of the simple groups of Lie type, by [15] N has an irreducible character χ_0 of 2-defect zero. Let σ_0 be an irreducible constituent of χ_0^G. Observe that $\chi_0^g(x) = 0$ for any $g \in G$ and any $x \in N$ of even order. It follows that $\sigma_0(x) = 0$ whenever $x \in N$ is of even order.

Let $P \in Syl_2(N)$, and $\Delta = \cup_{g \in G}(P^g - \{1\})$. We have

$$\Delta \subseteq v(\sigma_0),\ \text{and so } k_G(\Delta) \le 2.$$

Step 3. If $G = N \cong L_2(q)$ for some odd $q = p^f > 5$, then $G \cong L_2(7)$.

Note that all irreducible characters of $L_2(q)$ are listed in [6, XI, Theorem 5.5, 5.6, 5.7]. Let $\eta \in \mathrm{Irr}_2(G)$ be of degree $p^f + 1$, and C be a Singer cycle of G, and $\Xi = \cup_{g\in G}(C^g - 1)$. For any $v \in \Xi$, if p is a prime divisor of element order $o(v)$, then either σ_0 or η is p-defect zero, and so either $\sigma_0(v) = 0$ or $\eta(v) = 0$. This implies that $k_G(\Xi) \le 4$. Since $k_G(\Xi) = (q-1)/4$ (see [5, II, Theorem 8.5]), we have that $q = 7, 9, 11, 13, 17$. By [2], we conclude that $q = 7$ and $G \cong L_2(7)$.

Step 4. If a Sylow 2-subgroup P of N is nonabelian, then $G \cong L_2(7)$.

In this case, since P has an element of order 4, we conclude that $v(\sigma_0) = \Delta \subset N$ and $k_G(v(\sigma_0)) = 2$. By Lemma 2.2 (2), $|G/N|$ is odd and $(\sigma_0)_N = \chi_0$. Therefore σ_0 is of 2-defect zero, and $\sigma_0(x) = 0$ for any $x \in G$ of even order. This implies that all elements of even order are contained in Δ, thus $C_G(t)$ is a 2-group for any involution t of G. Since P is nonabelian, by [14, III, Theorem 5] we conclude that G is isomorphic to one of the following groups: $Sz(q), q = 2^{2m+1}$, $L_2(q)$ where q is a Fermat prime or Mersenne prime, $L_3(4)$, $L_2(9)$, M_{10}.

By [2], neither M_{10} nor $L_3(4)$ nor $L_2(9)$ has the property $(*)$. Note that all elements of order 4 in $Sz(2^{2m+1})$ constitute two conjugacy classes , which can be easily verified by [6, XI, Theorem 3.10]. Therefore $G \cong Sz(2^{2m+1})$ is not the case. Now by step 3, we conclude that $G \cong L_2(7)$.

Step 5. If a Sylow 2-subgroup P of N is abelian, then $G \cong L_2(2^f), f \ge 2$.

Since P is abelian, by [6, XI, Theorem 13.7], N is one of the following groups: $L_2(2^f)$; $L_2(q)$ where $q = 3, 5 \pmod 8$; ${}^2G_2(q), q = 3^{2m+1}$. Recall that $\sigma_0(x) = 0$ whenever $x \in N$ is of even order.

Suppose that $N \cong {}^2G_2(q)$. Then all elements of even order in N lie in at least three classes of G (see [6, XI, Theorem 13.4]), a contradiction.

Therefore $N \cong L_2(q)$, where $q = 2^f$ or $q = 3, 5 \pmod 8$. Then $Aut(N) = N\langle\phi, \delta\rangle$, where $\langle\phi\rangle$ is the group of field automorphisms of N, $\langle\delta\rangle$ is the group of diagonal automorphisms of N.

Case 1. Suppose that $N = L_2(q)$ where $q > 5, q = 3, 5 \pmod 8$.

In this case, we have $|N|_2 = 4$. Since ϕ and δ commute modulo $Inn(N)$, we have $N = G'$. Let $\theta \in \mathrm{Irr}_2(N)$ be such that if $q = 3 \pmod 8$ then $\theta(1) = q + 1$, and if $q = 5 \pmod 8$ then $\theta(1) = q - 1$. By Lemma 2.2 (1), θ is extendable to an irreducible character μ of G.

Observe that $\theta(1) = 4k$ for some odd $k > 1$, and that θ is of r-defect zero for any prime divisor r of $\theta(1)$. Let $x = x_1x_2 \in N$ be of order $2k$, where $o(x_1) = 2, o(x_2) = k$. We have that $\theta(x_1) = \theta(x_2) = \theta(x) = 0$, and so $\mu(x_1) = \mu(x_2) = \mu(x) = 0$, a contradiction.

Case 2. Suppose that $N \cong L_2(2^f)$, $f \ge 2$.

In this case, $Out(N) = \langle\phi\rangle$. We need to prove that $G = N$. Observe that if $N \cong L_2(4)(\cong L_2(5))$, then $G = N$ by [2]. Thus we may assume that $f \ge 3$. Suppose that $G > N$. Then $G = G \cap N\langle\phi\rangle$. Following [3, §38] and using the notation of that table for the characters of $L_2(q)$, we may take $\theta_1 \in \mathrm{Irr}(N)$ of degree $2^f - 1$ such that the stabilizer of θ in $Aut(N)$ is N. Thus θ_1 induces to an irreducible character of G. This implies by Lemma 2.2 that θ_1^G is of odd degree.

In particular, G/N is a cyclic group of odd order.

Recall that $\chi_0 \in \mathrm{Irr}(N)$ is of degree 2^f, and σ_0 is an extension of χ_0 to G. Since $\Delta = \cup_{g\in G}(P^g - 1)$ is a class of N, it forces Δ to be also a class of G. This implies that $|C_G(t)| = |G/N||P|$ for any $t \in P - \{1\}$, and so that $C_G(t) = PA$, where $A \cap N = 1$, $A \cong G/N$. Observe that $\sigma_0(g) = 0$ whenever $g \in G$ is of even order and that G/N is a cyclic group of odd order. Suppose that there are primes r_1, r_2 such that r_1r_2 divides $|A|$. We can find $x_1, x_2 \in C_G(t) = PA$ of order $2r_1, 2r_1r_2$ respectively, and then $\sigma_0(t) = \sigma(x_1) = \sigma(x_2) = 0$. However, t, x_1, x_2 lie in distinct classes of G, a contradiction. Hence $|G/N|$ is an odd prime q. Also, we see that Θ, the set of elements of order $2q$, forms a class of G. Let $w \in A$ be of order q, and $y = wt$. Since Θ is a class of G, all cyclic subgroups of order $2q$ are conjugate to $\langle y\rangle$. Note that distinct subgroups of order $2q$ have no common element of order $2q$, it follows that

$$|\Theta| = |G : N_G(\langle y\rangle)|(q - 1).$$

As $N_G(\langle w\rangle) = \langle w\rangle N_N(\langle w\rangle) = \langle w\rangle \times (N_G(\langle w\rangle) \cap N) = C_G(w)$, we have

$$N_G(\langle y\rangle) = N_G(\langle w\rangle) \cap N_G(\langle t\rangle) = C_G(w) \cap C_G(t) = C_G(y).$$

Then

$$|G : C_G(y)|(q - 1) = |G : N_G(\langle y\rangle)|(q - 1) = |\Theta| = |y^G| = |G : C_G(y)|,$$

a contradiction. Thus $G = N = L_2(2^f)$ as desired. □

3. Theorem A

Lemma 3.1. *Let $N \lhd G$ and H/N be a Hall π-subgroup of G/N. If $\eta \in \mathrm{Irr}(H)$ induces to an irreducible character χ of G, then $\chi(x) = 0$ for any π'-element $x \in G - N$.*

Proof. It follows directly from the definition of induced character. □

Lemma 3.2. *Let $G \cong L_2(2^f), f \geq 3$ and $P \in Syl_2(G)$. If H is a proper subgroup of G with $P \leq H$, then $H \leq N_G(P)$.*

Proof. It is enough to investigate the maximal subgroups of $L_2(2^f)$ (see [5, II, Theorem 8.27]). □

Proof of Theorem A. We need only to prove that if every member of $\mathrm{Irr}_2(G)$ has just one class of zeros, then G is one of the types listed in the theorem. Suppose that $\mathrm{Irr}_2(G)$ is empty. By a well-known theorem of Ito and Michler, G possesses a normal abelian Sylow 2-subgroup. In what follows, we assume that $\mathrm{Irr}_2(G)$ is not empty.

Case 1. Suppose that G is nonsolvable.

By Theorem B, there exists a normal solvable subgroup N of G such that $G/N \cong L_2(7)$ or $L_2(2^f)$. Clearly $G/N \cong L_2(7)$ is not the case, and so $G/N \cong$

$L_2(2^f)$. Suppose that $N > 1$. To reach a contradiction, we may assume that N is a minimal normal subgroup of G, and thus N is an elementary abelian q-group for some prime q. Let $\chi_0 \in \mathrm{Irr}(G/N)$ be of degree 2^f. Let $P \leq G$ be such that $P/N \in Syl_2(G/N)$, and $\Delta = \cup_{g\in G}(P^g - N)$. Then

$$\Delta = v(\chi_0), \ \ k_G(\Delta) = 1.$$

For any $\chi \in \mathrm{Irr}(G|N)$, by Lemma 2.1 we conclude that χ vanishes on Δ, and then by [13, Lemma 1.1] we see that $\chi(1)$ is even. Let us consider the subgroup P. For any $t \in P - N$, we have

$$|C_G(t)| = |C_{G/N}(tN)| = |P/N|.$$

If q is odd, then the above equation yields that P is a Frobenius group with N as its kernel, and this leads to the contradiction that P/N, as an abelian Frobenius complement is cyclic. Thus N is a 2-group. Since N is a nontrivial normal subgroup of the 2-group P, we can take a non-principal $\lambda_0 \in \mathrm{Irr}(N/N_1) \subseteq \mathrm{Irr}(N)$, where N/N_1 is a chief factor of P. Clearly λ_0 is P-invariant. Note that if χ is an irreducible constituent of λ_0^G, then $\chi \in \mathrm{Irr}(G|N)$ and then $\chi(1)$ is even.

Assume that $\mathrm{I}_G(\lambda_0) = G$. Observe that N can be viewed as an irreducible G-module over a field F_2 of 2 elements. Then $\mathrm{Irr}(N)$ has a natural G-module structure induced by the conjugate action of G on N, and since N is irreducible, $\mathrm{Irr}(N)$ is also an irreducible G-module (see Section 1.6 of [8]). Let W be the set of all G-invariant linear character of N. Then W is a nontrivial G-submodule of $\mathrm{Irr}(N)$, and this implies that $W = \mathrm{Irr}(N)$. Now applying [7, Theorem 6.32] we conclude that $N \leq Z(G)$. By [2, Page xvi, Table 5], we have either $G \cong L_2(2^f)\times N$ or $G \cong SL(2,5)$. If $G \cong L_2(2^f) \times N$, then $|C_G(t)| > |P/N|$ for any $t \in P - N$, a contradiction. If $G \cong SL(2,5)$, then we also obtain a contradiction by [2].

Assume that $\mathrm{I}_G(\lambda_0) < G$. By Lemma 3.2, we have that $P \leq \mathrm{I}_G(\lambda_0) \leq H$, where $H/N = N_{G/N}(P/N)$. Let ψ_0 be an irreducible constituent of λ_0^H. By Clifford theorem, ψ_0 induces to an irreducible character η_0 of G. Observe that $\eta_0 \in \mathrm{Irr}(G|N)$ vanishes on Δ and is of even degree. Since H/N is a Hall subgroup of G/N, it follows by Lemma 3.1 that $\eta_0 = \psi_0^G$ vanishes on some element outside Δ. Thus η_0 vanishes on at least two classes of G, a contradiction.

Case 2. Suppose that G is solvable.

Assume first that there is some $\chi \in \mathrm{Irr}_2(G)$ such that $\chi_{G'}$ is reducible. By [7, Theorem 6.22], there exist a subgroup H with $G' \leq H < G$ and an irreducible character λ of H so that $\chi = \lambda^G$. This implies that χ vanishes on $G - H$, and so $k_G(G - H) = 1$. Now it is easy to verify in this case that G is a Frobenius group with a complement of order 2 (see [11, Lemma 2(2)]).

In what follows, we assume that $\chi_{G'}$ is irreducible for any $\chi \in \mathrm{Irr}_2(G)$, and we will show in this case that $G \cong SL(2,3)$. Since $\chi_{G'}$ is irreducible, χ vanishes at some element of G', and consequently $v(\chi) \subset G'$ because $k_G(v(\chi)) = 1$. It follows by Lemma 2.2 (2) that $gcd(\chi(1), |G/G'|) = 1$ for any $\chi \in \mathrm{Irr}_2(G)$. In particular, $|G/G'|$ is odd.

Let $E \lhd G$ maximal be such that G/E is nonabelian. By [7, Lemma 12.3] G/E is a p-group or a Frobenius group. Suppose that G/E is a p-group and let ψ be a nonlinear irreducible character of G/E. Being a prime divisor of $|G/G'|$, p is coprime to $\chi(1)$ for any $\chi \in \mathrm{Irr}_2(G)$. Then $\chi_0\psi \in \mathrm{Irr}_2(G)$ for some $\chi_0 \in \mathrm{Irr}_2(G)$, and p is a common divisor of $|G/G'|$ and $(\chi_0\psi)(1)$, a contradiction. Therefore G/E is a Frobenius group with a kernel N/E and a cyclic complement.

For any $\tau_0 \in \mathrm{Irr}_2(N)$, by Frobenius reciprocity τ_0 is extendible to some $\chi_0 \in \mathrm{Irr}_2(G)$, and thus [7, Theorem 12.4] implies that both χ_0 and τ_0 vanish on $N-E$, then

$$k_G(N-E)=1,$$

and so

$$k_{G/E}(N/E-E/E)=1.$$

This implies that

$$|N/E|=1+|G/N|.$$

Since $|G/N|$ is odd, N/E is an elementary abelian 2-group. Set $|N/E|=2^r$ and let $t \in N-E$. We have

$$2^r=|C_G(t)|=|C_N(t)|=|N/E|+\sum_{\eta\in\mathrm{Irr}(N|E)}|\eta(t)|^2.$$

This implies that $N'=E$, and that for any $\eta \in \mathrm{Irr}(N|E)$ (that is, for any nonlinear $\eta \in \mathrm{Irr}(N)$), η must vanish on $N-E$, so $\eta(1)$ is even (see [13, Lemma 1.1]), and hence η is extendible to G.

Clearly $E>1$. Let E/F be a chief factor of G. If E/F is of odd order, then the above fact implies that N/F is a Frobenius group with the kernel E/F, and then being a Frobenius complement, the elementary abelian 2-group N/E is of order 2, which is impossible. Thus E/F is a 2-group. Let us investigate the quotient group G/F and let $K \cong G/N$ be a Hall $2'$-subgroup of G/F. Since every nonlinear irreducible character of N/F is of even degree, K acts nontrivially on N/F and fixes every nonlinear irreducible character of N/F. By [10, Lemma 19.2], we conclude that $N'/F=E/F\le Z(G/F)$. Since $N/E, E/F$ are chief factors of G, it is easy to see that

$$E/F=N'/F=Z(N/F)=Z(G/F)=\Phi(N/F).$$

Thus $|E/F|=2$ and N/F is an extraspecial 2-group. Now [4, Ch.5, Theorem 6.5] implies that $2^r-1=|G/N|$ divides 2^e+1 for some integer $e\le r/2$. This yields that $2^r=4$, and so $G/F\cong SL(2,3)$.

To finish the proof of Theorem A, it suffices to show that $F=1$. Suppose that $F>1$. Towards a contradiction we may assume that F is a minimal normal subgroup of G. Assume that F is a 2-group. Since $|C_G(t)|=4$ for any $t\in N-E$, there is $x\in N-E$ of order $|N|/2\ge 8$ ([13, Lemma 1.3]), which leads to the contradiction that $|C_G(x)|\ge 8>4$. Assume that F is a q-group for some odd prime q and set $P\in Syl_2(N)$. Since $|C_G(t)|=|N/E|=4$ for any $t\in N-E$, we see that $C_P(x)\le P'$ for any $1\ne x\in F$. It follows by [10, Lemma 19.1] that $N=PF$ is a Frobenius group with a complement P and that P is either cyclic or isomorphic to Q_8. Then we can find some $\theta_0\in\mathrm{Irr}(N)$ of degree 8. Let χ_0 be an extension of θ_0 to G. We have $N-F\subseteq v(\chi_0)$, a contradiction. Thus $F=1$, and the proof of Theorem A is complete.

References

[1] Berkovich, Y.; Kazarin, L.: *Finite groups in which the zeros of every nonlinear irreducible character are conjugate modulo its kernel.* Houston J. Math. **24**(4) (1998), 619–630. Zbl 0969.20004

[2] Conway, J. H.; Curtis, R. T.; Norton, S. P.; Parker, R. A.; Wilson, R. A.: *Atlas of Finite Groups.* Oxford University Press (Clarendon), Oxford and New York 1985. Zbl 0568.20001

[3] Dornhoff, L.: *Group representation theory, Part A: Ordinary representation theory.* Dekker, New York 1971. Zbl 0227.20002

[4] Gorenstein, D.: *Finite Groups.* Harper and Row, New York, Evanston, London 1968. Zbl 0185.05701

[5] Huppert, B.: *Endliche Gruppen* I. Springer-Verlag, Berlin 1967. Zbl 0217.07201

[6] Huppert, B.; Blackburn, N.: *Finite Groups* III. Springer-Verlag, Berlin, Heidelberg and New York 1982. Zbl 0514.20002

[7] Isaacs, I. M.: *Character Theory of Finite Groups.* Academic Press, New York 1976. Zbl 0337.20005

[8] Landrock, P.: *Finite group algebras and their modules.* Cambridge University Press, 1983. Zbl 0523.20001

[9] Manz, O.; Staszewski, R. S.; Willems, W.: *On the number of components of a graph related to character degrees.* Proc. Am. Math. Soc. **103**(1) (1988), 31–37. Zbl 0645.20005

[10] Manz, O.; Wolf, T. R.: *Representations of Solvable Groups.* Cambridge University Press, Cambridge 1993. Zbl 0928.20008

[11] Qian, G.: *Bounding the Fitting height of a solvable group by the number of zeros in a character table.* Proc. Am. Math. Soc. **130** (2002), 3171–3176. Zbl 1007.20008

[12] Qian, G.: *Finite solvable groups with an irreducible character vanishing on just one class of elements.* Commun. Algebra **35**(7) (2007), 2235–2240. Zbl 1127.20008

[13] Qian, G.; Shi, W.; You, X.: *Conjugacy classes outside a normal subgroup.* Commun. Algebra **32**(12) (2004), 4809–4820. Zbl 1094.20013

[14] Suzuki, M.: *Finite groups with nilpotent centralizers.* Trans. Am. Math. Soc. **99**(3) (1961), 425–470. Zbl 0101.01604

[15] Willems, W.: *Blocks of defect zero in finite simple groups of Lie type.* J. Algebra **113** (1988), 511–522. Zbl 0653.20014

Received November 8, 2007