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Normal Factorization in $SL(2,\mathbb{Z})$ and the Confluence of Singular Fibers in Elliptic Fibrations

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Abstract. In this article we obtain a result about the uniqueness of factorization in terms of conjugates of the matrix $U = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, of some matrices representing the conjugacy classes of those elements of $SL(2,\mathbb{Z})$ arising as the monodromy around a singular fiber in an elliptic fibration (i.e. those matrices that appear in Kodaira's list). Namely we prove that if M is a matrix in Kodaira's list, and $M = G_1 \cdots G_r$ where each G_i is a conjugate of U in $SL(2,\mathbb{Z})$, then after applying a finite sequence of Hurwitz moves the product $G_1 \cdots G_r$ can be transformed into another product of the form $H_1 \cdots H_n G'_{n+1} \cdots G'_r$ where $H_1 \cdots H_n$ is some fixed shortest factorization of M in terms of conjugates of U, and $G'_{n+1} \cdots G'_r = Id_{2\times 2}$. We use this result to obtain necessary and sufficient conditions under which a relatively minimal elliptic fibration without multiple fibers $\phi : S \to D = \{z \in \mathbb{C} : |z| < 1\}$, admits a weak deformation into another such fibration having only one singular fiber.

1. Introduction

The purpose of this article is twofold. On the one hand we begin the study of the extent to which a given element of the mapping class group of an oriented torus (i.e. $SL(2,\mathbb{Z})$) factors uniquely as a product of right handed Dehn twists, i.e. conjugates of the matrix

$$U = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Our first main result (Theorem 19) addresses this question, and gives an affirmative answer for those elements in $SL(2,\mathbb{Z})$ which arise as the monodromy around a singular fiber in an elliptic fibration. As far as we know this subject has two predecessors. The first one is a well known result due to R. Livne and Moishezon [7], which says that any factorization of the identity matrix in $SL(2,\mathbb{Z})$ in terms of r conjugates of U can be transformed by applying a finite sequence of Hurwitz moves, into a standard factorization $(VU)^{6s}$ where $s \geq 0$, $r = 12s$, and

$$V = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

The second one arose in the study of branched covers of 2-manifolds, and was initiated by Hurwitz, Clebsch and Luroth, and more recently continued by several other authors (see [2] and the references therein). These authors study the analogous problem when one replaces $SL(2,\mathbb{Z})$ by the symmetric group S_n, and right handed Dehn twists by transpositions. For instance, Natanzon's result (see [9]) claims that if $\sigma \in S_n$, and $\sigma = \tau_1 \cdots \tau_k = \tau_1' \cdots \tau_k'$ are two factorizations in terms of transpositions, such that the subgroups $\langle \tau_1, \dots, \tau_k \rangle$ and $\langle \tau_1', \dots, \tau_k' \rangle$ act transitively on the n symbols, then there exists a sequence of Hurwitz moves which transforms the product $\tau_1 \cdots \tau_k$ into the product $\tau_1' \cdots \tau_k'$. In particular, this implies a result (which parallels Theorem 19) saying that if one picks a particular shortest transitive factorization $\mu_1 \cdots \mu_{s_\sigma}$ of σ in terms of transpositions (i.e. such that $\langle \mu_1, \dots, \mu_{s_\sigma} \rangle$ acts transitively on the n symbols), then any transitive factorization $\tau_1 \cdots \tau_k$ of σ in terms of transpositions, transforms after a finite sequence of Hurwitz moves into a factorization of the form $\sigma = \mu_1 \cdots \mu_{s_\sigma} \tau_{s_\sigma+1}' \cdots \tau_k'$. The proof of Theorem 19 is based on the careful study of the description of $PSL(2,\mathbb{Z})$ as the direct product $\mathbb{Z}_2 * \mathbb{Z}_3$ developed by R. Livne (see [7]).

On the other hand, we study the problem of when an elliptic fibration over a disk can be deformed into another elliptic fibration over a disk having only one singular fiber. Our result in this direction (Theorem 21) provides necessary and sufficient conditions under which a given relatively minimal elliptic fibration over a disk without multiple fibers can be weakly deformed into another such fibration having only one singular fiber (Definition 7). Weak deformation allows the passing from one deformation family to another whenever there exists a member of each family being topologically equivalent with each other. This result is obtained as an application of our normal factorization result.

This type of problem was posed by Naruki in [8]. In [8] that author considers the confluence of three singular fibers F_1, F_2 and F_3, of types I_a, I_b and I_c in an elliptic fibration into one singular fiber F after a deformation, and studies in depth the necessary condition for the existence of the confluence that if M_1, M_2 and M_3 are the monodromies around F_1, F_2 and F_3, and M is the monodromy around the singular fiber F they coalesce to, then $M = M_1 M_2 M_3$ and $\chi(F) =$

$\chi(F_1)+\chi(F_2)+\chi(F_3)$. He solves the algebraic problem of classifying up to the braid group action, those triples (M_1,M_2,M_3) such that each M_i is a conjugate of U^{a_i}, $M=M_1M_2M_3$ is a matrix in Kodaira's list, and $a_1+a_2+a_3=\chi(F_M)$ (see Table 1). The opposite problem, namely that of finding necessary and sufficient conditions under which a singular fiber in a fibration of arbitrary fiber genus admits a deformation which splits it into several ones has been intensely studied (see for example [10] and the references therein). Moishezon completely solved this problem in the elliptic case (cf. Theorem 6).

2. Basic definitions and facts

Throughout this article, S will denote a complex manifold with complex dimension 2 and D will denote the open unit disk $\{z\in\mathbb{C}:|z|<1\}$.

Definition 1. *By an elliptic fibration we will mean a triple (ϕ,S,D) where $\phi: S\to D$ is a proper surjective holomorphic map with a finite number (possibly zero) of critical values $q_1,\dots,q_k\in D$, such that the preimage of each regular value is a (compact) connected Riemann surface of genus 1.*

We will say that an elliptic fibration is *singular* if it has at least one singular fiber. A singular fiber $\phi^{-1}(q_i)$ is said to be of *Lefschetz type* if

$$C_i:=\{p\in\phi^{-1}(q_i): p \text{ is a critical point of } \phi\}$$

is finite, and for each $p\in C_i$ there exist holomorphic charts around p and q_i relative to which ϕ takes the form $(z_1,z_2)\to z_1^2+z_2^2$. If a fiber of Lefschetz type contains exactly one critical point, it will be said to be *simple*. Every fiber $\phi^{-1}(q)$ of an elliptic fibration can be regarded as an effective divisor $w_{1,q}X_{1,q}+\cdots+w_{r_q,q}X_{r_q,q}$. A (necessarily) singular fiber $\phi^{-1}(q)$ is called a *multiple fiber* if $gcd(w_{1,q},\dots,w_{r_q,q})>1$, and it is said to be of *smooth multiple type* if it is of the form $w_{1,q}X_{1,q}$ with $w_{1,q}>1$ and $X_{1,q}$ is a smooth submanifold of S. An elliptic fibration is said to be *relatively minimal* if no fiber contains an embedded sphere with selfintersection -1.

All elliptic fibrations in this article will be assumed to be relatively minimal.

The Euler characteristic of the domain of an elliptic fibration can be calculated using the following formula which is analogous to the Riemann-Hurwitz formula

$$\chi(S)=\sum_{i=1}^{k}\chi(\phi^{-1}(q_i)).$$

Next we define when two elliptic fibrations will be regarded as being (topologically) the same.

Definition 2. *Two elliptic fibrations (ϕ_1,S_1,D) and (ϕ_2,S_2,D) are said to be topologically equivalent if there exist orientation preserving diffeomorphisms $h: S_1\to S_2$ and $h': D\to D$ such that $\phi_2\circ h=h'\circ\phi_1$. In this case we write $(\phi_1,S_1,D)\sim(\phi_2,S_2,D)$ or simply $\phi_1\sim\phi_2$.*

Definition 3. *By a family of elliptic fibrations we will mean a triple* $(\Phi, \mathcal{S}, D \times D_\epsilon)$ *where* $\mathcal{S}$ *is a three-dimensional complex manifold,* $D_\epsilon = \{z \in \mathbb{C} : |z| < \epsilon\}$ *and* $\Phi : \mathcal{S} \to D \times D_\epsilon$ *is a surjective proper holomorphic map, such that*

1. *if for each* $t \in D_\epsilon$, $D_t := D \times \{t\}$, $\mathcal{S}_t := \Phi^{-1}(D_t)$ *and* $\Phi_t := \Phi|_{\mathcal{S}_t} : \mathcal{S}_t \to D_t$, *then each* $(\Phi_t, \mathcal{S}_t, D_t)$ *is an elliptic fibration;*
2. *the composition* $\mathcal{S} \xrightarrow{\Phi} D \times D_\epsilon \xrightarrow{pr_2} D_\epsilon$ *does not have critical points.*

A family of elliptic fibrations $(\Phi, \mathcal{S}, D \times D_\epsilon)$ *is said to be a deformation of a given elliptic fibration* (ϕ, S, D), *if* (ϕ, S, D) *is biholomorphically equivalent to* $(\Phi_0, \mathcal{S}_0, D_0)$, *i.e. there exist biholomorphic maps* $h : S \to \mathcal{S}_0$ *and* $h' : D \to D$ *such that* $\Phi_0 \circ h = h' \circ \phi$.

Remark 4. It can be seen that if $(\Phi, \mathcal{S}, D \times D_\epsilon)$ is a family of elliptic fibrations, the (oriented) diffeomorphism type of $\mathcal{S}_t$ is independent of $t \in D_\epsilon$. In particular, $\chi(\mathcal{S}_t)$ is also independent of $t \in D_\epsilon$.

Definition 5. *Let* (ϕ, S, D) *be an elliptic fibration. A deformation* $(\Phi, \mathcal{S}, D \times D_\epsilon)$ *of* (ϕ, S, D) *will be said to be a morsification of* (ϕ, S, D), *if for each* $t \neq 0$, *each singular fiber of* $\Phi_t : \mathcal{S}_t \to D_t$ *is either of simple Lefschetz type or of smooth multiple type.*

The following fundamental result is due to Moishezon (see [7]).

Theorem 6. *Every elliptic fibration admits a morsification. Moreover, if the elliptic fibration does not have multiple fibers, then it admits a morsification such that none of its members contains a multiple fiber.*

The following definition is introduced in order to state one of our main results.

Definition 7. *Two elliptic fibrations* (ϕ_1, S_1, D) *and* (ϕ_2, S_2, D) *will be said to be weakly deformation equivalent whenever there exist a finite collection of families of elliptic fibrations* $(\Phi^1, \mathcal{S}^1), \dots, (\Phi^k, \mathcal{S}^k)$, *and* $s_i, t_i \in D_{\epsilon_i}$ *for each* $i = 1, \dots, k$, *such that* $\phi_1 \sim \Phi^1_{s_1}$, $\Phi^i_{t_i} \sim \Phi^{i+1}_{s_{i+1}}$ *for* $i = 1, \dots, k-1$, *and* $\Phi^k_{t_k} \sim \phi_2$.

We now turn to the combinatorial description of elliptic fibrations. Let (ϕ, S, D) be an elliptic fibration and let $q_1, \dots, q_k$ be its critical values. Take $0 < r < 1$ such that the open disk D_r having center 0 and radius r contains the points $q_1, \dots, q_k$. Let us fix a point $q_0 \in \partial D_r$. Notice that q_0 is a regular value. Let us also fix an orientation preserving diffeomorphism j between the genus 1 Riemann surface $\phi^{-1}(q_0)$ and the genus 1 Riemann surface $\mathbb{C}/\mathbb{Z}^2$. These choices uniquely determine an antihomomorphism

$$\lambda_{r,q_0,j} : \pi_1(D - \{q_1, \dots, q_k\}, q_0) \to SL(2, \mathbb{Z})$$

where $SL(2,\mathbb{Z})$ is the group formed by all 2×2 integral matrices whose determinant is 1. Such antihomomorphism is said to be a *representation monodromy* of (ϕ, S, D). In order to make the presentation more standard, we turn the monodromy representation into a homomorphism by regarding $\pi_1(D - \{q_1, \dots, q_k\}, q_0)$

as the group whose binary operation $\star$ is defined by $[\gamma_1] \star [\gamma_2] := [\gamma_2].[\gamma_1]$, where "." denotes the usual composition of homotopy classes of paths. The matrix $\lambda_{r,q_0,j}([C_r])$, where C_r denotes the path $q_0 \exp(2\pi\sqrt{-1}t)$, $0 \leq t \leq 1$, will be called *the total monodromy of* (ϕ, S, D).

Remark 8. The conjugacy class of $\lambda_{r,q_0,j}([C_r])$ in $SL(2,\mathbb{Z})$ is independent of the choices r, q_0 and j, and if $(\Phi, \mathcal{S}, D \times D_\epsilon)$ is a family of elliptic fibrations and $t_1, t_2 \in D_\epsilon$ then the conjugacy classes of the total monodromies of $(\Phi_{t_1}, \mathcal{S}_{t_1}, D)$ and $(\Phi_{t_2}, \mathcal{S}_{t_2}, D)$ are the same.

The group $\pi_1(D - \{q_1, \ldots, q_k\}, q_0)$ is free and has rank k. We now describe a method for obtaining free bases for this group. The bases obtained by this method will be called *special bases*. Pick closed disks $\overline{D}_1, \ldots, \overline{D}_k$ contained in D_r, centered at $q_1, \ldots, q_k$, respectively, and mutually disjoint. Pick simple paths $\beta_1, \ldots, \beta_k$ whose interiors are mutually disjoint and contained in $D_r - \cup \overline{D}_i$, with $\beta_1(0) = \cdots = \beta_k(0) = q_0$ and $q_i^0 := \beta_i(1) \in \partial D_i$ for each $i = 1, \ldots, k$, and such that their initial velocity vectors $\beta_1'(0), \ldots, \beta_k'(0)$ are all nonzero and $0 < \theta_1 < \cdots < \theta_k < \pi$ where θ_i is the angle between the vectors $\beta_i'(0)$ and $\sqrt{-1}q_0$. Let γ_i be a path which starts at q_0, follows β_i until it reaches q_i^0, then traverses once and positively the circle ∂D_i, and finally comes back to q_0 following β_i in the opposite direction. Then $\{[\gamma_1], \ldots, [\gamma_k]\}$ is a basis for the free group $\pi_1(D - \{q_1, \ldots, q_k\}, q_0)$. Notice that $[C_r] = [\gamma_1] \cdots [\gamma_k] = [\gamma_k] \star \cdots \star [\gamma_1]$ and therefore the total monodromy $\lambda([C_r])$ equals $\lambda([\gamma_k]) \cdots \lambda([\gamma_1])$.

The following proposition is standard. Its statement requires the concept of Hurwitz move which we define next.

Definition 9. *Let G be a group and let $g_1 \cdots g_k$ be a product of elements of G. Another such product $g_1' \cdots g_k'$ is said to be obtained from $g_1 \cdots g_k$ by applying a Hurwitz move if for some $1 \leq i \leq k-1$, $g_j' = g_j$ for $j \notin \{i, i+1\}$, and either $g_i' = g_{i+1}$, $g_{i+1}' = g_{i+1}^{-1} g_i g_{i+1}$ or $g_i' = g_i g_{i+1} g_i^{-1}$, $g_{i+1}' = g_i$. We will also say that an ordered set $\{g_1', \ldots, g_k'\}$ is obtained from another ordered set $\{g_1, \ldots, g_k\}$ by applying one Hurwitz move, if the same relations hold between the g_i''s and the g_i's.*

It is important to remark that the Hurwitz moves

$$g_1 \cdots g_i g_{i+1} \cdots g_k \to g_1 \cdots g_{i+1}(g_{i+1}^{-1} g_i g_{i+1}) \cdots g_k$$

and

$$g_1 \cdots g_i g_{i+1} \cdots g_k \to g_1 \cdots (g_i g_{i+1} g_i^{-1}) g_i \cdots g_k$$

are inverse of each other.

Proposition 10. *Let (ϕ, S, D) and (ϕ', S', D) be relatively minimal elliptic fibrations without multiple fibers and having the same number of singular fibers. Let $q_1, \ldots, q_k$ (resp. $q_1', \ldots, q_k'$) be the critical values of (ϕ, S, D) (resp. (ϕ', S', D)). Let λ (resp. λ') be a monodromy representation for (ϕ, S, D) (resp. (ϕ', S', D)). The following statements are equivalent*

1. $(\phi, S, D) \sim (\phi', S', D)$;
2. *there exist an orientation preserving diffeomorphism* $h : D \to D$ *with* $h(\{q_1, \ldots, q_k\}) = \{q'_1, \ldots, q'_k\}$, $h(q_0) = q'_0$, *and a matrix* $A \in SL(2,\mathbb{Z})$, *such that* $c_A \circ \lambda = \lambda' \circ h_*$, *where* c_A *denotes the automorphism of* $SL(2,\mathbb{Z})$ *defined by* $c_A(B) = A^{-1}BA$, *and*
$$h_* : \pi_1(D - \{q_1, \ldots, q_k\}, q_0) \to \pi_1(D - \{q'_1, \ldots, q'_k\}, q'_0)$$
is the group isomorphism induced by h;
3. *there exist an isomorphism*
$$\psi : \pi_1(D - \{q_1, \ldots, q_k\}, q_0) \to \pi_1(D - \{q'_1, \ldots, q'_k\}, q'_0)$$
sending $[C_r]$ *to* $[C_{r'}]$, *and a matrix* $A \in SL(2,\mathbb{Z})$, *such that* $c_A \circ \lambda = \lambda' \circ \psi$, *where* c_A *denotes the automorphism of* $SL(2,\mathbb{Z})$ *defined by* $c_A(B) = A^{-1}BA$;
4. *there exist special bases* $\{[\gamma_1], \ldots, [\gamma_k]\}$ *and* $\{[\gamma'_1], \ldots, [\gamma'_k]\}$ *for the groups* $\pi_1(D - \{q_1, \ldots, q_k\}, q_0)$ *and* $\pi_1(D - \{q'_1, \ldots, q'_k\}, q'_0)$, *respectively, and a matrix* $A \in SL(2,\mathbb{Z})$ *such that the product* $\lambda([\gamma_k]) \cdots \lambda([\gamma_1])$ *becomes the product* $\lambda'([\gamma'_k]) \cdots \lambda'([\gamma'_1])$ *after the application of a (finite) number of Hurwitz moves, followed by the conjugation of all the elements in the resulting product by* A;
5. *for any pair of special bases* $\{[\gamma_1], \ldots, [\gamma_k]\}$ *and* $\{[\gamma'_1], \ldots, [\gamma'_k]\}$ *for the groups* $\pi_1(D - \{q_1, \ldots, q_k\}, q_0)$ *and* $\pi_1(D - \{q'_1, \ldots, q'_k\}, q'_0)$, *respectively, there exists a matrix* $A \in SL(2,\mathbb{Z})$ *such that the product* $\lambda([\gamma_k]) \cdots \lambda([\gamma_1])$ *becomes the product* $\lambda'([\gamma'_k]) \cdots \lambda'([\gamma'_1])$ *after the application of a (finite) number of Hurwitz moves, followed by the conjugation of all the elements in the resulting product by* A.

In the rest of this section $\pi_1(D - \{q_1, \ldots, q_k\}, q_0)$ (resp. $\pi_1(D - \{q'_1, \ldots, q'_k\}, q'_0)$) will be abbreviated by π_1 (resp. π'_1).

The equivalence $1 \Leftrightarrow 2$ is a particular case of the result mentioned immediately after the statement of Theorem 2.4 of [6].

$2 \Rightarrow 3$ is immediate, but its reciprocal is less obvious. Let $f : D \to D$ be an orientation preserving diffeomorphism such that $f(q'_i) = q_i$ for $i = 0, \ldots, k$. It is enough to prove that the automorphism $f_* \circ \psi : \pi_1 \to \pi_1$ (which preserves $[C_r]$) equals h_* for some orientation preserving diffeomorphism $h : D \to D$ such that $h(\{q_1, \ldots, q_k\}) = \{q_1, \ldots, q_k\}$ and $h(q_0) = q_0$. Actually, let us see that every automorphism φ of π_1 such that $\varphi([C_r]) = [C_r]$ is induced by some orientation preserving diffeomorphism h with the properties described in the last sentence. Let $\{[\gamma_1], \ldots, [\gamma_k]\}$ be a special basis for the group π_1, let $F(x_1, \ldots, x_k)$ be the free group in the alphabet $\{x_1, \ldots, x_k\}$ and $\nu : F(x_1, \ldots, x_k) \to \pi_1$ the isomorphism sending x_i to $[\gamma_i]$ for $i = 1, \ldots, k$. Notice that $\nu(x_k \cdots x_1) = [\gamma_k] \star \cdots \star [\gamma_1] = [C_r]$. The well known fact (see [4]) that the group of automorphisms of $F(x_1, \ldots, x_k)$ which send the product $x_k \cdots x_1$ to itself, is generated by the elementary automorphisms $\{\phi_1, \ldots, \phi_{k-1}\}$ such that $\phi_i(x_j) = x_j$ if $j \notin \{i, i+1\}$, and $\phi_i(x_i) = x_i^{-1}x_{i+1}x_i$, $\phi_i(x_{i+1}) = x_i$, allows us to reduce the problem to proving that each automorphism of π_1 defined as $\varphi_i := \nu \circ \phi_i \circ \nu^{-1}$ for $i = 1, \ldots, k-1$, is induced by some orientation preserving diffeomorphism $h_i : D \to D$ with $h_i(\{q_1, \ldots, q_k\}) = \{q_1, \ldots, q_k\}$ and $h_i(q_0) = q_0$. h_i is explicitly constructed as a half twist performed on an appropriately chosen annulus containing the points q_i and q_{i+1}.

$2 \Rightarrow 4$ is an immediate consequence of the fact that if $\{[\gamma_1], \ldots, [\gamma_k]\}$ is a special basis for π_1 then for any orientation preserving diffeomorphism $h : D \to D$ with $h(\{q_1, \ldots, q_k\}) = \{q'_1, \ldots, q'_k\}$ and $h(q_0) = q'_0$, $h_*([\gamma_1]), \ldots, h_*([\gamma_k])$ is a special basis for π'_1.

$4 \Rightarrow 5$ Let $\{[\delta_1], \ldots, [\delta_k]\}$ (resp. $\{[\delta'_1], \ldots, [\delta'_k]\}$) be special bases for π_1 (resp. π'_1). We have that $[\delta_k]\star\cdots\star[\delta_1] = [\gamma_k]\star\cdots\star[\gamma_1]$ and $[\delta'_k]\star\cdots\star[\delta'_1] = [\gamma'_k]\star\cdots\star[\gamma'_1]$. The well known fact from [4] invoked above is equivalent to the fact that if $y_1, \ldots, y_k$ and $z_1, \ldots, z_k$ are free bases for $F(x_1, \ldots, x_k)$, such that $y_k \cdots y_1 = z_k \cdots z_1$, then there exists a finite sequence of Hurwitz moves that transforms the product $y_k \cdots y_1$ into the product $z_k \cdots z_1$. Applied to our situation this gives the existence of a finite sequence of Hurwitz moves which transforms the product $[\delta_k] \star \cdots \star [\delta_1]$ into the product $[\gamma_k] \star \cdots \star [\gamma_1]$, and another finite sequence of Hurwitz moves transforming the product $[\gamma'_k] \star \cdots \star [\gamma'_1]$ into the product $[\delta'_k] \star \cdots \star [\delta'_1]$. Combining this with the existence of a sequence of Hurwitz moves and a conjugation transforming the product $\lambda([\gamma_k]) \cdots \lambda([\gamma_1])$ into the product $\lambda'([\gamma'_k]) \cdots \lambda'([\gamma'_1])$ allows us to conclude that there exists a finite sequence of Hurwitz moves and a conjugation transforming the product $\lambda([\delta_k]) \cdots \lambda([\delta_1])$ into the product $\lambda'([\delta'_k]) \cdots \lambda'([\delta'_1])$.

$5 \Rightarrow 3$ Let $\{[\gamma_1], \ldots, [\gamma_k]\}$ (resp. $\{[\gamma'_1], \ldots, [\gamma'_k]\}$) be a special basis for π_1 (resp. π'_1). Then the product $\lambda([\gamma_k]) \cdots \lambda([\gamma_1])$ can be transformed to the product $\lambda'([\gamma'_k]) \cdots \lambda'([\gamma'_1])$ by applying a sequence $\mu_1, \ldots, \mu_l$ of Hurwitz moves, followed by the conjugation of all the elements in the resulting product by a matrix $A \in SL(2, \mathbb{Z})$. Let $\{[\gamma''_1], \ldots, [\gamma''_k]\}$ be the special basis for π_1 obtained by applying the sequence $\mu_l^{-1}, \ldots, \mu_1^{-1}$ of Hurwitz moves to $\{[\gamma'_1], \ldots, [\gamma'_k]\}$. Let $\psi : \pi \to \pi'$ be the isomorphism determined $\psi([\gamma_i]) = [\gamma''_i]$ for each $i = 1, \ldots, k$. It can easily verified that $c_A \circ \lambda = \lambda' \circ \psi$.

3. Kodaira's list

Let (ϕ, S, D) be an elliptic fibration and let $q \in D$. It is a well known fact that the fiber $\phi^{-1}(q)$ is a triangulable topological space. In [3] Kodaira studied the problem of classifying fibers in elliptic fibrations under the following equivalence relation which takes into account not only the topological structure of the fiber but also the structure of the map ϕ in a regular neighborhood of it.

Definition 11. *Let (ϕ, S, D) and (ϕ', S', D) be elliptic fibrations and let $q, q' \in D$. Let $\sum m_i X_i$ and $\sum n_j Y_j$ be the effective divisors associated to $\phi^{-1}(q)$ and $(\phi')^{-1}(q')$, respectively. The fibers $\phi^{-1}(q)$ and $(\phi')^{-1}(q')$ are said to be of the same type if there is a homeomorphism $f : \phi^{-1}(q) \to (\phi')^{-1}(q')$, so that the induced map $f_* : H_2(\phi^{-1}(q); \mathbb{Z}) \to H_2((\phi')^{-1}(q'); \mathbb{Z})$ sends the class $\sum m_i[X_i]$ to the class $\sum n_j[Y_j]$.*

We will rely heavily on the following classical result due to Kodaira (see [3]).

Theorem 12. *Let (ϕ, S, D) be a relatively minimal elliptic fibration and let q_i be a critical value of ϕ. Then*

1. *the fiber $\phi^{-1}(q_i)$ is of the same type of one and only one of the following pairs:*

   $w\mathrm{I}_0$: wX_0, $w > 1$ *where X_0 is a non-singular elliptic curve;*

   $w\mathrm{I}_1$: wX_0, $w \geq 1$ *where X_0 is a rational curve with an ordinary double point;*

   $w\mathrm{I}_2$: $wX_0 + wX_1$, $w \geq 1$ *where X_0 and X_1 are non-singular rational curves with intersection $X_0 \cdot X_1 = p_1 + p_2$;*

   II: $1X_0$ *where X_0 is a rational curve with one cusp;*

   III: $X_0 + X_1$ *where X_0 and X_1 are non-singular rational curves with $X_0 \cdot X_1 = 2p$;*

   IV: $X_0 + X_1 + X_2$, *where X_0, X_1, X_2 are non-singular rational curves and $X_0 \cdot X_1 = X_1 \cdot X_2 = X_2 \cdot X_0 = p$.*

   The rest of the types are denoted by wI_b, $b \geq 3$, I_b^*, II^*, III^*, IV^* *and are composed of non-singular rational curves $X_0, X_1, \ldots, X_s, \ldots$ such that $X_s \cdot X_t \leq 1$ (i.e. X_s and X_t have at most one simple intersection point) for $s < t$ and $X_r \cap X_s \cap X_t$ is empty for $r < s < t$. These types are therefore described completely by showing all pairs X_s, X_t with $X_s \cdot X_t = 1$ together with $\sum w_i X_i$.*

   $w\mathrm{I}_b$: $wX_0 + wX_1 + \cdots + wX_{b-1}$, $w = 1, 2, 3, \ldots$, $b = 3, 4, 5, \ldots$, $X_0 \cdot X_1 = X_1 \cdot X_2 = \cdots = X_s \cdot X_{s+1} = \cdots = X_{b-2} \cdot X_{b-1} = X_{b-1} \cdot X_0 = 1;$

   I_b^*: $X_0 + X_1 + X_2 + X_3 + 2X_4 + \cdots + 2X_{4+b}$ *where* $b \geq 0$, *and* $X_0 \cdot X_4 = X_1 \cdot X_4 = X_2 \cdot X_{4+b} = X_3 \cdot X_{4+b} = X_4 \cdot X_5 = X_5 \cdot X_6 = \cdots = X_{3+b} \cdot X_{4+b} = 1;$

   II^*: $X_0 + 2X_1 + 3X_2 + 4X_3 + 5X_4 + 6X_5 + 4X_6 + 3X_7 + 2X_8$, *where* $X_0 \cdot X_1 = X_1 \cdot X_2 = X_2 \cdot X_3 = X_3 \cdot X_4 = X_4 \cdot X_5 = X_5 \cdot X_7 = X_5 \cdot X_6 = X_6 \cdot X_8 = 1;$

   III^*: $X_0 + 2X_1 + 3X_2 + 4X_3 + 3X_4 + 2X_5 + 2X_6 + X_7$, *where* $X_0 \cdot X_1 = X_1 \cdot X_2 = X_2 \cdot X_3 = X_3 \cdot X_5 = X_3 \cdot X_4 = X_4 \cdot X_6 = X_6 \cdot X_7 = 1;$

   IV^*: $X_0 + 2X_1 + 3X_2 + 2X_3 + 2X_4 + X_5 + X_6$, *where* $X_0 \cdot X_1 = X_1 \cdot X_2 = X_2 \cdot X_3 = X_2 \cdot X_4 = X_3 \cdot X_5 = X_4 \cdot X_6 = 1.$

2. *the conjugacy class of $\lambda([\gamma_i])$, where $[\gamma_i]$ is the i^{th} term in any special basis for $\pi_1(D - \{q_1, \ldots, q_k\}, q_0)$, depends only on the type of the fiber $\phi^{-1}(q_i)$.*
3. *for each type T above there exists a relatively minimal elliptic fibration (ϕ_T, S_T, D) with $F_T := \phi_T^{-1}(0)$ as its unique singular fiber, and having type T.*

The following table contains for each type T, a matrix representative M_T of the conjugacy class of the total monodromy of (ϕ_T, S_T, D), the Euler characteristic of S_T (which is the same as the Euler characteristic of F_T) (see [11]), and a particular factorization of M_T in $SL(2,\mathbb{Z})$ which will play a central role in next section.

T	M_T	$\chi(S_T)$	$m.n.f.$
$w\mathrm{I}_n$ $(w \geq 1, n \geq 0)$	$\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$	n	U^n
II	$\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$	2	VU
III	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$	3	VUV
IV	$\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$	4	$(VU)^2$
I_n^* $(n \geq 0)$	$\begin{bmatrix} -1 & -n \\ 0 & -1 \end{bmatrix}$	$n+6$	$U^n(VU)^3$ $(= -U^n)$
II*	$\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$	10	$VU(VU)^3$ $(= -VU)$
III*	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	9	$VUV(VU)^3$ $(= -VUV)$
IV*	$\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$	8	$(VU)^2(VU)^3$ $(= -(VU)^2)$

Table 1

4. Factorization of Kodaira's matrices in terms of conjugates of U

In this section we recall some basic facts about the group $SL(2,\mathbb{Z})$, formed by all 2×2 matrices with integral entries and determinant 1, and about the modular group $PSL(2,\mathbb{Z})$ defined as the quotient $SL(2,\mathbb{Z})/\{\pm Id_{2\times 2}\}$, and prove some uniqueness results (Theorems 19 and 20) about the factorization in terms of conjugates of the matrix

$$U = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

of the matrices appearing in Kodaira's list (second column of Table 1).

4.1. Study of $PSL(2,\mathbb{Z})$ as $\langle w, b \mid w^2 = b^3 = 1 \rangle$

Although a significant part of the material in this section can be found in references [7], [1], [5], for the sake of completeness we have included complete proofs of those results that are more specialized.

In what follows we will refer to particular elements (classes) in $PSL(2,\mathbb{Z})$ by specifying one of its representatives. We will use capital letters for the elements of $SL(2,Z)$ and the corresponding lower case letters for their images in $PSL(2,\mathbb{Z})$. For example, since $U = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ then u denotes the class $\pm U$.

It is a well known fact that the modular group is isomorphic to the free product $\mathbb{Z}_2 * \mathbb{Z}_3$ via an isomorphism taking a generator of $\mathbb{Z}_2$ to $w = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, and a generator of $\mathbb{Z}_3$ to $b = wu$. Hence,

$$G = PSL(2,\mathbb{Z}) \cong \left\langle w, b \,\middle|\, w^2 = b^3 = 1 \right\rangle .$$

From this we see that the abelianization of G is $\mathbb{Z}_2 \times \mathbb{Z}_3$ with any conjugate of u being sent to 1. Consequently, the abelianization of $SL(2,\mathbb{Z})$ is $\mathbb{Z}_{12}$, with any conjugate of the matrix U being sent to 1.

It also follows that each element $a \neq id_{2\times 2}$ in this group can be written uniquely as a product $a = t_k \cdots t_1$, where each t_i is either w, b, or b^2 and no consecutive pair $t_{i+1}t_i$ is formed either by two powers of b or two copies of w. We call the product $t_k \cdots t_1$ the *reduced expression of* a, and k the *length* of a, which we will denote it by $l(a)$. Let $c = t'_1 \cdots t'_l$ be the reduced expression of an element $c \neq id_{2\times 2}$. If exactly the first $m \geq 1$ terms of c cancel with those of a, i.e. $t'_i = t_i^{-1}$, for $1 \leq i \leq m$, and if $m < \min(k,l)$, then $ac = t_k \cdots t_{m+1} t'_{m+1} \cdots t'_l$ and $t_{m+1} t'_{m+1}$ has to be equal to a non trivial power of b. This is because if t_{m+1} were not a power of b then it would have to be w and therefore t_m would be a first or second power of b, and so would be t'_m. Hence, t'_{m+1} would also have to be w but in this case there would be $m+1$ instead of m cancellations at the juncture of a and c. Thus, t_{m+1} and t'_{m+1} are both powers of b and since there are exactly m cancellations their product must be non trivial. Thus, the reduced expression for ac is of the form

$$ac = t_k \cdots t_{m+2} b^e t'_{m+2} \cdots t'_l, \; e = 1 \text{ or } 2, \;\text{ if } m < \min(k,l). \tag{1}$$

Let s_1 denote the element bwb. The shortest conjugates of s_1 in G are precisely $s_0 = b^2(bwb)b = wb^2$ with length 2, $s_2 = b(bwb)b^2 = b^2w$ with length 2, and s_1 itself with length 3. It can easily be seen that any other conjugate g of s_1 has length greater than 3, and that its reduced expression is of the form $q^{-1}s_1q$, where q is a reduced word that begins with w (see [1]), therefore $l(g) = 2l(q) + 3$. A conjugate g of s_1 will be called *short* if $g \in \{s_0, s_1, s_2\}$, and it will be called *long* otherwise.

Let $c_b(a) = b^{-1}ab$ denote conjugation by b. This is an automorphism of G that sends $u = wb$ to b^2wb^2. The map $\varphi : \mathbb{Z}_2 * \mathbb{Z}_3 \to \mathbb{Z}_2 * \mathbb{Z}_3$ defined by sending w to itself, and b to b^2, that is, $\varphi = Id * \psi$, where ψ is the automorphism of $\mathbb{Z}_3$ that sends b to b^2, is an automorphism that maps b^2wb^2 to s_1. Hence the composite $\varphi \circ c_b$ of these two automorphisms is an automorphism ρ that sends u and v into s_1 and s_0, respectively, and takes conjugates of u into conjugates of s_1.

The following notion is the key ingredient for understanding the reduced expression of a product of conjugates of s_1.

Definition 13. *We will say that two conjugates g and h of s_1 join well if*

$$l(gh) \geq \max(l(g), l(h)).$$

In [1] (Lemma 4.10) the following result is proved.

Lemma 14. *Suppose that $g = t_k \cdots t_1$ and $h = t'_1 \cdots t'_l$ are the reduced expressions of two conjugates of s_1 that join well. When gh is calculated either:*

1. *no cancellation occurs, and in this case $t_k \cdots t_1 t'_1 \cdots t'_l$ is the reduced expression of gh, or*
2. *exactly the first $m \geq 1$ terms of g and h cancel out, in which case*

$$m < \min(k, l). \tag{2}$$

Moreover, if g is short or h is short, then both are short and they are s_2 and s_0, respectively. If both are long with reduced expressions of the form $g = q_1^{-1} s_1 q_1$ and $h = q_2^{-1} s_1 q_2$, hence with lengths $2l(q_i) + 3$, then the reduced expression of gh is of the form

$$gh = t_k \cdots t_{m+2} b^e t'_{m+2} \cdots t'_l, \; e = 1 \text{ or } 2,$$

and the inequality (2) *can be improved to*

$$m < \min((k-1)/2, (l-1)/2)$$

which implies that $m \leq \min(l(q_1), l(q_2))$.

Now suppose that g and h are two conjugates of s_1 that do not join well, and are not both short. The next lemma shows that in this case there exist g' and h' conjugates of s_1 such that $gh = g'h'$ and $l(g') + l(h') < l(g) + l(h)$ ([1], Proposition 4.15).

Lemma 15. *Suppose that g and h are conjugates of s_1 which satisfy the inequality $l(gh) < \max(l(g), l(h))$, and assume that at least one of them is long. Then $l(g) \neq l(h)$. If $l(g) < l(h)$, the elements $g' = ghg^{-1}$, $h' = g$ are conjugates of s_1 and satisfy:*

1. *$gh = g'h'$, and*
2. *$l(g') + l(h') < l(g) + l(h)$.*

If instead, $l(h) < l(g)$, then the same conclusion holds taking $g' = h$, and $h' = h^{-1}gh$.

Notice that in the previous proof, the pair (g', h') is obtained from the pair (g, h) by performing one Hurwitz move.

Using the previous lemma we can prove that a product $g_1 \cdots g_r$ of conjugates of s_1 can always be transformed by applying a finite number of Hurwitz moves into a product $g'_1 \cdots g'_r$ of conjugates of s_1 in which each pair of consecutive terms joins well. Notice that if $g'_1 \cdots g'_s$ is obtained from $g_1 \cdots g_r$ by applying a finite number of Hurwitz moves, then $s = r$, $g'_1 \cdots g'_s = g_1 \cdots g_r$ and $\{C(g'_1), \dots, C(g'_r)\} = \{C(g_1), \dots, C(g_r)\}$ where $C(g)$ denotes the conjugacy class of g.

Proposition 16. *Let $g_1 \cdots g_r$ be a product of r conjugates of s_1. Then after a finite number of Hurwitz moves one can obtain a new product $g'_1 \cdots g'_r$ of conjugates of s_1, such that either they are all short, or any pair of consecutive factors $g'_i g'_{i+1}$ join well.*

Before proving this proposition we need to know how to handle pairs of consecutive short conjugates of s_1 that do not join well.

Proposition 17. *Let $p = s_{i_1} s_{i_2} \cdots s_{i_l}$ with $l \geq 2$ be a product of short conjugates of s_1, where there is at least one pair of consecutive terms that do not join well. Then after a finite number of Hurwitz moves, p can be written as a product $s_{j_1} s_{j_2} \cdots s_{j_l}$ of short conjugates of s_1 (with the same number of terms) where s_{j_1} can be chosen arbitrarily from the set $\{s_0, s_1, s_2\}$. In a similar way, $s_{i_1} s_{i_2} \cdots s_{i_l}$ can be transformed by applying a finite number of Hurwitz moves into another product of short conjugates of s_1 with the same number of terms, where the last conjugate can be chosen arbitrarily.*

Proof. We use induction on l. For $l = 2$ a direct computation shows that the pairs $s_2 s_0$, $s_0 s_1$, and $s_1 s_2$ are the only ones that do not join well, and that each product equals b. From this the claim follows by noticing that each product can be changed into any other by a Hurwitz move: $s_2 s_0 = s_0(s_0^{-1} s_2 s_0) = s_0 s_1$, $s_0 s_1 = s_1(s_1^{-1} s_0 s_1) = s_1 s_2$, and $s_1 s_2 = s_2(s_2^{-1} s_1 s_2) = s_2 s_0$.

Now let $l > 2$. If the first pair does not join well, the same argument as before could be applied. Hence, we may assume that there is a consecutive pair in the product $s_{i_2} \cdots s_{i_l}$ which does not join well. Then, by induction we can change this product by a new product $s_{j_2} \cdots s_{j_l}$, where s_{j_2} can be made to be any short conjugate. Consequently, if s_{i_1} is s_0 (resp. s_1, s_2) then we may choose s_{j_2} to be s_1 (resp. s_2, s_0) so that the first pair does not join well and therefore can be changed again by a pair whose first term can be chosen arbitrarily.

The proof of the second part is analogous. □

Proof of Proposition 16. Among all products $g'_1 \cdots g'_r$ obtained by Hurwitz moves from the product $g_1 \cdots g_r$ we may choose one such that the sum $\sum_{i=1}^r l(g'_i)$ is as small as possible. If all g'_i are short we are done. If not, any g'_i which is long has to join well with any term (if any) before or after it, for otherwise, by Lemma 15, the corresponding pair could be transformed by a Hurwitz move into another one making the sum $\sum_{i=1}^r l(g'_i)$ smaller.

On the other hand, let $s_{i_1} \cdots s_{i_l}$ be a product of consecutive short conjugates that appears in $g'_1 \cdots g'_r$. If this product precedes a long conjugate g'_k, i.e., if

$s_{i_1}\cdots s_{i_l}g'_k$ is a segment of the product $g'_1\cdots g'_r$, then, by the previous lemma, either any pair of consecutive elements of $s_{i_1}\cdots s_{i_l}$ joins well or we can transform this product via Hurwitz moves into $s_{j_1}s_{j_2}\cdots s_{j_l}$, where s_{j_l} can be chosen arbitrarily. If the reduced expression of g'_k is of the form $wb^e t_3\cdots t_m$, $e=1$ or 2, we may choose $s_{j_l}=s_2$ so that s_{j_l} and g'_k do not join well. In this situation Lemma 15 guarantees that by applying one Hurwitz move we would obtain a new product whose length sum is smaller than that of $g'_1\cdots g'_r$. But this is a contradiction. Similarly, if the reduced expression of g'_k is of the form $b^e wt_3\cdots t_m$, with $e=1$ or 2, then if $e=1$ (resp. $e=2$), we could choose s_{j_l} to be s_2 (resp. s_1) so that s_{j_l} and g'_k do not join well. As before, this leads to a contradiction. We conclude that any pair of consecutive elements of $s_{i_1}\cdots s_{i_l}$ must join well. The argument is essentially the same in case $g'_k s_{i_1}\cdots s_{i_l}$ is a segment of the product $g'_1\cdots g'_r$. Thus, we see that if a product $g'_1\cdots g'_r$ obtained from $g_1\cdots g_r$ via Hurwitz moves and minimizing the sum $\sum_{i=1}^r l(g'_i)$ among such products, contains at least one long conjugate then all consecutive pairs in it must join well. This proves the proposition. □

Let us define the *left end* of a conjugate g of s_1, denoted by $\mathrm{left}(g)$, as follows: If g is long of the form $g=q^{-1}s_1q$, define $\mathrm{left}(g)=q^{-1}s_1$. If g is $s_0=wb^2, s_1=bwb$, or $s_2=b^2w$, we define its left end as w, b, b^2, respectively.

Lemma 18. *If in a product $p=g_1\cdots g_r$ of conjugates of s_1 all pairs of consecutive factors join well, then the reduced expression of p is of the form* $\mathrm{left}(g_1)t_1\cdots t_l$, *where each t_i is one of b, b^2 or w.*

Proof. We prove this by induction on r, the assertion being trivial for $r=1$. We distinguish several cases.

1. $g_1=s_0$. Since g_1 and g_2 join well, Lemma 14 implies that either g_2 is short or no cancellation occurs when the product g_1g_2 is calculated. If g_2 is short, then it should be equal to s_0 or to s_2. If $g_2=s_0$, by the induction hypothesis, we must have that the reduced expression for $g_2\cdots g_r$ is of the form $wt_1\cdots t_k$, and consequently $g_1\cdots g_r=wb^2wt_1\cdots t_k=\mathrm{left}(s_0)t'_1\cdots t'_l$. On the other hand, if $g_2=s_2$ then $g_2\cdots g_r=b^2t_1\cdots t_k$ and the result also holds, since $g_1\cdots g_r=wbt_1\cdots t_k=\mathrm{left}(s_0)t'_1\cdots t'_l$. It only rests to consider the case in which no cancellation occurs when g_1g_2 is calculated. By the induction hypothesis, we know that the reduced expression of $g_2\cdots g_r$ has the form $\mathrm{left}(g_2)t_1\cdots t_k$. On the other hand, when $g_1\mathrm{left}(g_2)$ is calculated no cancellation occurs. We conclude that the reduced expression of $g_1\cdots g_r$ is of the form $\mathrm{left}(g_1)t'_1\cdots t'_l$, and the results holds.
2. If $g_1=s_2$ or $g_1=s_1$. In these cases the argument is exactly the same as in the previous case.
3. g_1 is long. By Lemma 14, either g_2 is short and no cancellation occurs when g_1g_2 is calculated, or g_2 is long. In the first case, by the induction hypothesis, $g_2\cdots g_r=\mathrm{left}(g_2)t_1\cdots t_k$. On the other hand, since no cancellation occurs when g_1g_2 is calculated we have that no cancellation occurs when $g_1\mathrm{left}(g_2)$

is calculated. We conclude that the reduced expression of $g_1 \cdots g_r$ is of the form left$(g_1)t'_1 \cdots t'_l$ and the result holds. Let us assume now that g_2 is long. If $g_1 = q_1^{-1}s_1q_1$ and $g_2 = q_2^{-1}s_1q_2$ then, by Lemma 14, either no cancellation occurs, or the number of terms that cancel out in the product g_1g_2 is $\leq \min(l(q_1), l(q_2))$. By induction $g_2 \cdots g_r = q_2^{-1}s_1t_1 \cdots t_k$, and in either case the reduced expression of $g_1 \cdots g_r$ starts with $q_1^{-1}s_1$. $\square$

4.2. Uniqueness of factorization results

Let us set

$$W = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

and

$$V = W^{-1}UW = -WUW = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

The last column of Table 1 contains a particular factorization of the corresponding monodromy matrix in terms of conjugates of U (V is a conjugate of U). This factorization will be called the *minimal normal factorization* (which we will abbreviate as m.n.f.) of the corresponding matrix. In this section we intend to prove the following theorem.

Theorem 19. *Let M be one of the matrices in Table* 1. *If $M = G_1 \cdots G_r$ is a factorization of M in terms of conjugates of U in $SL(2,\mathbb{Z})$, then r is greater than or equal to the number of factors in the m.n.f. of M. Moreover, if n is such number then after a finite number of Hurwitz moves the product $G_1 \cdots G_r$ transforms into a product $C_1 \cdots C_nD_{n+1} \cdots D_r$ where*

- *in cases* $\mathrm{wI}_n - \mathrm{IV}$, *$C_1 \cdots C_n$ is the m.n.f. of M and $D_{n+1} \cdots D_r$ is equal to the identity matrix $Id_{2\times 2}$, and*
- *in cases* $\mathrm{I}_n^* - \mathrm{IV}^*$, *$C_1 \cdots C_n$ is the m.n.f. of $-M$ and $D_{n+1} \cdots D_r$ is equal to $-Id_{2\times 2}$.*

For instance, for

$$M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \text{(case III}^*\text{)}$$

any factorization $M = G_1 \cdots G_r$ can be transformed using Hurwitz moves into $M = VUVD_4 \cdots D_r$ where $D_4 \cdots D_r = -Id_{2\times 2}$. By a well known theorem of Moishezon [7] we also know that any factorization of the identity in terms of conjugates of U, can be transformed using Hurwitz moves into a product of the form $(VU)^{6s}$ with $s \geq 0$, from which we can strengthen the theorem above as follows.

Theorem 20. *Let M be a matrix that corresponds to the monodromy of a singular fiber in an elliptic fibration. If $M = G_1 \cdots G_r$ is a factorization of M in terms of conjugates of U, then r is greater than or equal to the number of factors in the m.n.f. of M. Moreover, if n is such number, then after a finite number of*

Hurwitz moves the product $G_1 \cdots G_r$ *becomes the product* $C_1 \cdots C_n(VU)^{6s}$*, where* $C_1 \cdots C_n$ *is the m.n.f. of* M *and* $s = (r-n)/12$*, in cases* $\mathrm{wI}_n - \mathrm{IV}$*, and into* $C_1 \cdots C_n(VU)^{6s+3}$*, where* $C_1 \cdots C_n$ *is the m.n.f. of* $-M$ *and* $s = (r-n-6)/12$*, in cases* $\mathrm{I}_n^* - \mathrm{IV}^*$.

We now present the proof of Theorem 19.

Proof. We deal with cases wI_n–IV first, and once we have established these, a rather trivial argument takes care of the remaining cases I_n^*–IV^*. In what follows, $\pi : SL(2,\mathbb{Z}) \to PSL(2,\mathbb{Z})$ will be the canonical homomorphism.

Claim: Let M be one of the matrices in cases $wI_n - IV$. Suppose that $m := \pi(M) = g_1 \cdots g_r$ is a factorization in terms of conjugates of $\pi(U) = u$. Then after a finite number of Hurwitz moves the product $g_1 \cdots g_r$ transforms into a new one of the form $(g'_1 \cdots g'_n)(g'_{n+1} \cdots g'_r)$ where if $G'_1 \cdots G'_n$ is the m.n.f. of M, then $g'_i = \pi(G'_i)$ for $i = 1, \dots, n$, and $g'_{n+1} \cdots g'_r = \pi(Id_{2\times 2})$.

Assuming this claim we can prove cases wI_n–IV of the theorem as follows. A product $H_1 \cdots H_s$ in $SL(2,\mathbb{Z})$ will be said to be a *lift* of a product $k_1 \cdots k_s$ in $PSL(2,\mathbb{Z})$ if $\pi(H_i) = k_i$ for each $i = 1, \dots, s$. It can be immediately verified that if a product $H_1 \cdots H_s$ is a lift of a product $k_1 \cdots k_s$, then the product $H'_1 \cdots H'_s$ obtained by applying a Hurwitz move to $H_1 \cdots H_s$, is a lift of the product $k'_1 \cdots k'_s$ obtained by applying the same Hurwitz move to the product $k_1 \cdots k_s$. Let M now be one of the matrices in cases wI_n–IV, and let $G_1 \cdots G_r$ be a factorization of M in terms of conjugates of U. Let $m = g_1 \cdots g_r$ where $m = \pi(M)$ and $g_i = \pi(G_i)$ for each $i = 1, \dots, r$. Since each G_i is a conjugate of U, then each g_i is a conjugate of $\pi(U)$. Also by definition the product $G_1 \cdots G_r$ is a lift of the product $g_1 \cdots g_r$. The claim guarantees the existence of a finite sequence of Hurwitz moves which transforms $g_1 \cdots g_r$ into a new product $(g'_1 \cdots g'_n)(g'_{n+1} \cdots g'_r)$. It follows that if $G'_1 \cdots G'_r$ is the product obtained from $G_1 \cdots G_r$ by applying the same sequence of Hurwitz moves, then $G'_1 \cdots G'_r$ is a lift of $(g'_1 \cdots g'_n)(g'_{n+1} \cdots g'_r)$ where each G_i is a conjugate of U. Now, by observing that U and $-U$ have different traces and therefore do not belong to the same conjugacy class, we conclude that $G'_1 \cdots G'_n$ has to be the m.n.f. of M, and therefore that $G'_{n+1} \cdots G'_r = Id_{2\times 2}$.

Cases I_n^*–IV^* can be dealt with as follows. Let M be one of the matrices in cases I_n^*–IV^*. Suppose that $G_1 \cdots G_r$ is a factorization of M in terms of conjugates of U. By applying the homomorphism π we obtain a factorization $g_1 \cdots g_r$ of $\pi(M)$ in terms of conjugates of $u = \pi(U)$. Since $\pi(M) = \pi(-M)$ and $-M$ is one of the matrices in cases wI_n–IV, we can apply the claim to $\pi(M) = g_1 \cdots g_r$. We conclude that there exists a sequence of Hurwitz moves which transforms the product $g_1 \cdots g_r$ into a product $(g'_1 \cdots g'_n)(g'_{n+1} \cdots g'_r)$ where the m.n.f. of $-M$ is a lift of $g'_1 \cdots g'_n$ and $g'_{n+1} \cdots g'_r = \pi(Id_{2\times 2})$. Let $(G'_1 \cdots G'_n)(G'_{n+1} \cdots G'_r)$ be the product obtained from $G_1 \cdots G_r$ by applying the same sequence of Hurwitz moves. By the observations made immediately after the claim we know that the facts that $G'_1 \cdots G'_n$ is a lift of $g'_1 \cdots g'_n$ and that each G'_i is a conjugate of U imply that $G'_1 \cdots G'_n$ has to be the m.n.f. of $-M$, and therefore that $G'_{n+1} \cdots G'_r = -Id_{2\times 2}$ since $M = G'_1 \cdots G'_r = (-M)(G'_{n+1} \cdots G'_r)$. This finishes the proof of the theorem. □

In order to prove the claim we may first apply the automorphism ρ (defined in Remark 4.1) and then prove the equivalent claim for $\rho(m)$. Notice that after doing this the image of the canonical factorization of M becomes a factorization of $\rho(m)$ in terms of the elements $\rho(u) = s_1$ and $\rho(v) = s_0$. Now we prove the claim by analyzing each of the four possible cases.

Case 1: $m = u^n$ hence $\rho(m) = \rho(u)^n = s_1^n$. It suffices to prove that for each $n \geq 0$, if $s_1^n = g_1 \cdots g_r$ where each g_i is a conjugate of s_1, then $r \geq n$ and there exists a sequence of Hurwitz moves which transforms the product $g_1 \cdots g_r$ into a product $(g'_1 \cdots g'_n)(g'_{n+1} \cdots g'_r)$ where the m.n.f. of M is a lift of $g'_1 \cdots g'_n$ and $g'_{n+1} \cdots g'_r = \pi(Id_{2\times 2})$. In the case $n = 0$ the m.n.f. of $Id_{2\times 2}$ is taken to an empty product. We proceed by induction on n. The result is immediate when $n = 0$. Let us suppose that $s_1^n = g_1 \cdots g_r$. By Proposition 16, after applying a finite number of Hurwitz moves one arrives at a new product $s_1^n = g'_1 \cdots g'_r$ in which either any pair of consecutive g_i's in this product join well or all factors are short conjugates of s_1.

In the first case, by Lemma 18 we know that the reduced expression of this product must be of the form $\text{left}(g'_1)t_1 \cdots t_l$. On the other hand, g'_1 cannot be a long conjugate $q^{-1}s_1q$. This is because the reduced expression of s_1^n is $b(wb^2)^{n-1}wb$, the reduced expression of $\text{left}(q^{-1}s_1q) = q^{-1}s_1$ has the form $l_1 \cdots l_s bwb$ and the sequence bwb does not appear in the reduced expression of s_1^n. In a similar way, g'_1 cannot be s_0 or s_2 since $\text{left}(s_0) = w$ and $\text{left}(s_2) = b^2$ but the reduced expression of s_1^n starts with the element b. Hence $g'_1 = s_1$, and we can cancel out this element on both sides of $s_1^n = g'_1 \cdots g'_r$ and apply the induction hypothesis to obtain the result.

In the second case, i.e. when all the g'_i's are short, we may assume that there is at least one pair of consecutive elements that do not join well, for otherwise we would be in the previous case. By Proposition 17, after a finite number of Hurwitz moves one arrives at a product $g''_1 \cdots g''_r$ with $g''_1 = s_1$. Cancelling out this element in equation $s_1^n = g''_1 \cdots g''_r$ and applying the induction hypothesis one obtains the result.

Case 2: $m = vu$, hence $\rho(m) = s_0s_1$. Let us suppose that $s_0s_1 = g_1 \cdots g_r$. Again, by Proposition 16, after a finite number of Hurwitz moves we arrive at a new product $s_0s_1 = g'_1 \cdots g'_r$ (e.1) in which either any pair of consecutive g'_i's join well or all factors are short conjugates of s_1. In the first case, since $s_0s_1 = b$, g'_1 can neither be long nor equal to s_0 or s_2, for exactly the same reason as in the previous case. We conclude that $g'_1 = s_1$. Since $s_0s_1 = s_1s_2$, we can cancel out s_1 on both sides of (e.1) in order to obtain $s_2 = g'_2 \cdots g'_r$. Again, by Lemma 18 we know that the reduced expression of this product must be of the form $\text{left}(g'_2)t_1 \cdots t_k$ and this must be equal to $\text{left}(s_2) = b^2$. This rules out the possibility of g'_2 being a long conjugate or equal to s_1 or s_0. Thus, $g'_2 = s_2$ and after applying a finite sequence of Hurwitz moves (e.1) can be written in the form $s_0s_1 = s_1s_2g''_3 \cdots g''_r$. As in the proof of Proposition 17, one extra Hurwitz move allows us to write s_1s_2 as s_0s_1 and the claim follows.

In the second case, i.e. when all the g'_i's are short we may assume that at least one pair of consecutive elements does not join well. By Proposition 17 after applying a finite sequence of Hurwitz moves one arrives at a product $s_0s_1 = g''_1\cdots g''_r$ (e.2) with $g''_1 = s_0$. Cancelling out s_0 on both sides of (e.2) we obtain $s_1 = g''_2\cdots g''_r$. But Case 1 (with $n = 1$) implies that this product can be changed using Hurwitz moves into a new one $g'''_2\cdots g'''_r$ with $g'''_2 = s_1$. The claim follows.

The strategy of the proof for the remaining cases will be the same. In order to make it shorter, we will abbreviate by Case ∗i, the case where all pairs of consecutive elements in a product join well, and by Case ∗ii, the case where all elements in a product are short and at least one pair of consecutive elements does not join well. We will also abbreviate the expression "after a finite number of Hurwitz moves" by "after H.m.".

Case 3: $m = vuv$, hence $\rho(m) = s_0s_1s_0$. Suppose that $s_0s_1s_0 = g_1\cdots g_r$. After H.m. we arrive at the product $s_0s_1s_0 = g'_1\cdots g'_r$ (e.3) falling in one of the following cases.

Case 3i: Since $s_0s_1s_0 = bwb^2$, the comparison of the left sides of the reduced expressions of both sides of (e.3) implies that $g'_1 = s_1$. Since $s_0s_1s_0 = s_1s_2s_0$ we have $s_1s_2s_0 = g'_1\cdots g'_r$ and we can cancel out s_1 in this equation to obtain $s_2s_0 = g'_2\cdots g'_r$. Now, $s_2s_0 = s_0s_1$ implies that $s_0s_1 = g'_2\cdots g'_r$ and we are in the previous case. We know that after H.m. the product $g'_2\cdots g'_r$ can be changed to a new one of the form $s_0s_1g''_4\cdots g''_r$ with $g''_4\cdots g''_r = \pi(Id_{2\times2})$, and consequently $s_0s_1s_0 = s_1s_0s_1g''_4\cdots g''_r$. After H.m. the right hand side can be transformed into $s_1s_2s_0g''_4\cdots g''_r$ and this into $s_0s_1s_0g''_4\cdots g''_r$. The claim follows.

Case 3ii: Proposition 17 allows us to assume that $g'_r = s_0$ and if we cancel it out in (e.3) we obtain $s_0s_1 = g'_1\cdots g'_{r-1}$ and ending up again in the previous case. After H.m. the product $g'_1\cdots g'_{r-1}$ transforms into a new product of the form $s_0s_1g''_3\cdots g''_{r-1}$ with $g''_3\cdots\ g''_{r-1} = \pi(Id_{2\times2})$. Thus $s_0s_1s_0 = s_0s_1g''_3\cdots g''_{r-1}s_0$ which after H.m. becomes $s_0s_1(gs_0g^{-1})g''_3\cdots g''_{r-1}$ where $g = g''_3\cdots g_{r-1} = \pi(Id_{2\times2})$. The latter product is therefore $s_0s_1s_0g''_3\cdots g''_{r-1}$ and the claim follows.

Case 4: $m = (vu)^2$ hence $\rho(m) = (s_0s_1)^2$. Suppose $(s_0s_1)^2 = g_1\cdots g_r$. After H.m. we arrive at the product $s_0s_1s_0 = g'_1\cdots g'_r$ (e.4) falling in one of the following cases.

Case 4i: Since $(s_0s_1)^2 = b^2$, the comparison of the left sides of the reduced expressions of both sides of (e.4) implies that $g'_1 = s_2$. Since $(s_0s_1)^2 = s_2s_0s_0s_1$ we have that $s_2s_0s_0s_1 = g'_1\cdots g'_r$. Cancelling out s_2 on both sides of this equation gives $s_0s_0s_1 = g'_2\cdots g'_r$ (e.5). Now $s_0s_0s_1 = w$ and therefore left$(g'_2) = w$. This implies that $g'_2 = s_0$ and by cancelling this term out in equation (e.5) we obtain $s_0s_1 = g'_3\cdots g'_r$. Since this is Case 2, we know that after H.m. the product $g'_3\cdots g'_r$ transforms into a product of the form $s_0s_1g''_5\cdots g''_r$ and we have $(s_0s_1)^2 = s_2s_0s_0s_1g''_5\cdots g''_r$ with $g''_5\cdots g''_r = \pi(Id_{2\times2})$. But $s_2s_0s_0s_1$ can be changed with one extra Hurwitz move into $s_0s_1s_0s_1$ and the claim follows.

Case 4ii: In this case, by Proposition 17, after H.m. the product $g'_1\cdots g'_r$ transforms into a product product $(s_0s_1)^2 = g''_1\cdots g''_r$ (e.6) with $g''_r = s_1$. Cancelling this element in equation (e.6) gives $s_0s_1s_0 = g''_1\cdots g''_{r-1}$. Since this is Case 3, we know that after H.m. the product $g''_1\cdots g''_{r-1}$ transforms into a product of the form $s_0s_1s_0g'''_4\cdots g'''_{r-1}$ with $g'''_4\cdots g'''_{r-1} = \pi(Id_{2\times2})$. We have therefore that

$s_0s_1s_0s_1 = s_0s_1s_0g_4''' \cdots g_{r-1}'''s_1$, and after H.m. this product transform into the product $s_0s_1s_0(gs_1g^{-1})g_4''' \cdots g_{r-1}'''$ with $g = g_4''' \cdots g_{r-1}''' = \pi(Id_{2\times 2})$. The claim follows. □

5. Confluence of singular fibers in elliptic fibrations

In this section we apply Proposition 10 and Theorem 19 to the question of giving necessary and sufficient conditions under which the set of singular fibers in an elliptic fibration can be fused into a unique singular fiber.

Theorem 21. *Let $\phi : S \to D$ be a relatively minimal singular elliptic fibration without multiple fibers. Then ϕ is weakly deformation equivalent to the elliptic fibration $\phi_T : S_T \to D$, if and only if the total monodromy $\lambda_{r,q_0,j}([C_r])$ of ϕ is conjugate with the matrix M_T in Table* 1 *and $\chi(S)$ equals $\chi(S_T)$.*

Proof. Necessity. Let us assume that $\phi : S \to D$ is weakly deformation equivalent to a $\phi_T : S_T \to D$ as described in the statement. Remark 4 implies that the total monodromies $\lambda_{r,q_0,j}([C_r])$ of ϕ, and $\lambda_{r',q_0',j'}([C_{r'}])$ of ϕ_T are conjugate of each other, but the latter is conjugate with M_T. Remark 8 implies that $\chi(S) = \chi(S_T)$.

Sufficiency. By Theorem 6 there exist morsifications $(\Phi, \mathcal{S}, D\times D_\epsilon)$ and $(\Psi, \mathcal{T}, D\times D_\delta)$ of $\phi : S \to D$ and $\phi_T : S_T \to D$, respectively. According to that theorem we may assume that none of the members of either morsification contains a multiple fiber. Let us fix $t \neq 0$ in D_ϵ and $t' \neq 0$ in D_δ and let us consider the elliptic fibrations $(\Phi_t, \mathcal{S}_t, D)$ and $(\Psi_{t'}, \mathcal{T}_{t'}, D)$. We claim that these elliptic fibrations are topologically equivalent. Let us denote by $q_1, \dots, q_k$ (resp. $q_1', \dots, q_l'$) the critical values of Φ_t (resp. $\Psi_{t'}$). We begin to prove the claim by pointing out that k must be equal to l, since $k = \sum_{i=1}^{k} \chi(\Phi_t^{-1}(q_i)) = \chi(\mathcal{S}_t) = \chi(S) = \chi(S_T) = \chi(\mathcal{T}_{t'}) = \sum_{i=1}^{l} \chi(\Psi_t^{-1}(q_i')) = l$ where the first and last equalities are justified by the fact that all singular fibers of Φ_t and $\Psi_{t'}$ are of type I_1 and therefore each one of these fibers has Euler characteristic 1, and the third and fifth equalities are justified by Remark 4. Since $\chi(S_T)$ always equals the number of factors n_T in the m.n.f. of M_T, we conclude that $k = n_T$. Let $\lambda = \lambda_{r,q_0,j}$ and $\lambda' = \lambda_{r',q_0',j'}$ be monodromy representations of Φ_t and $\Psi_{t'}$, respectively, and let $\gamma_1, \dots, \gamma_k$ and $\gamma_1', \dots, \gamma_k'$ be special bases for $\pi_1(D-\{q_1, \dots, q_{n_T}\}, q_0)$ and $\pi_1(D-\{q_1', \dots, q_{n_T}'\}, q_0')$, respectively. Now, the total monodromy $\lambda([C_r])$ of Φ_t is conjugate of a total monodromy of ϕ and the total monodromy $\lambda'([C_{r'}])$ of $\Psi_{t'}$ is conjugate of a total monodromy of ϕ_T, we have that $\lambda([C_r])$ and $\lambda'([C_{r'}])$ are conjugates of each other, say $\lambda([C_r]) = A^{-1}\lambda'([C_{r'}])A$ for some $A \in SL(2,\mathbb{Z})$. On the other hand, $\lambda([C_r]) = \lambda(\gamma_1)\cdots\lambda(\gamma_{n_T})$ and $\lambda'([C_{r'}]) = \lambda'(\gamma_1')\cdots\lambda'(\gamma_{n_T}')$ are factorizations of the corresponding total monodromies in terms of conjugates of U. It is clear that there exist $B, C \in SL(2,\mathbb{Z})$ such that $M_T = (B^{-1}\lambda(\gamma_1)B)\cdots(B^{-1}\lambda(\gamma_{n_T})B) = (C^{-1}\lambda'(\gamma_1')C)\cdots(C^{-1}\lambda'(\gamma_{n_T}')C)$. By Theorem 19 the product $(B^{-1}\lambda(\gamma_1)B)\cdots(B^{-1}\lambda(\gamma_{n_T})B)$ can be transformed into the product $(C^{-1}\lambda'(\gamma_1')C)\cdots(C^{-1}\lambda'(\gamma_{n_T}')C)$ by applying performing a finite number of Hurwitz moves. By using the immediate fact that if in a group a product $g_1 \cdots g_r$ can be transformed by applying Hurwitz moves to another product $h_1 \cdots h_r$, then for any element $g \in G$, the product $(g^{-1}g_1g) \cdots (g^{-1}g_rg)$ can be transformed into the product $(g^{-1}h_1g) \cdots (g^{-1}h_rg)$ by applying Hurwitz moves, we obtain that $\lambda(\gamma_1) \cdots \lambda(\gamma_{n_T})$ can be transformed into the product $(BC^{-1}\lambda'(\gamma'_1)(B\,C^{-1})^{-1}) \cdots (BC^{-1}\lambda'(\gamma'_{n_T})(BC^{-1})^{-1}))$. We conclude that the product $\lambda(\gamma_1) \cdots \lambda(\gamma_{n_T})$ transforms into the product $\lambda'(\gamma'_1) \cdots \lambda'(\gamma'_{n_T})$ by a finite sequence of Hurwitz moves followed by a conjugation of all factors in the product by the same element is $SL(2,\mathbb{Z})$. But Proposition 10 implies that under these circumstances, the elliptic fibrations Φ_t and $\Psi_{t'}$ are topologically equivalent. In conclusion, $(\Phi, \mathcal{S}, D \times D_\epsilon)$ and $(\Psi, \mathcal{T}, D \times D_\delta)$ are deformations of (ϕ, S, D) and (ϕ_M, S_M, D), respectively, such that $(\Phi_t, \mathcal{S}_t, D)$ and $(\Psi_{t'}, \mathcal{T}_{t'}, D)$ are topologically equivalent, and therefore (ϕ, S, D) and (ϕ_M, S_M, D) are weakly deformation equivalent. □

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