Boundary value problems in complex analysis II

Heinrich Begehr

Abstract

This is the continuation of an investigation of basic boundary value problems for first order complex model partial differential equations. Model second order equations are the Poisson and the inhomogeneous Bitsadze equations. Different kinds of boundary conditions are posed as combinations of the Schwarz, the Dirichlet, and the Neumann conditions. Solvability conditions and the solutions are given in explicit form for the unit disc. Exemplarily the inhomogeneous polyanalytic equation is investigated as a model equation of arbitrary order.

1 Boundary value problems for second order equations

There are two basic second order differential operators, the Laplace operator $\partial_z\partial_{\overline{z}}$ and the Bitsadze operator $\partial_{\overline{z}}^2$. The third one, ∂_z^2 is just the complex conjugate of the Bitsadze operator and all formulas and results for this operator can be attained by the ones for the Bitsadze operator through complex conjugation giving dual formulas and results.

For the Laplace and the Poisson, i.e. the inhomogeneous Laplace equation, the Dirichlet and the Neumann boundary value problems are well studied. Before investigating them the Schwarz problem will be studied for both operators.

Theorem 1 *The Schwarz problem for the Poisson equation in the unit disc*

$$w_{z\overline{z}} = f \text{ in } \mathbb{D}\,,\ \operatorname{Re} w = \gamma_0\,,\ \operatorname{Re} w_z = \gamma_1 \text{ on } \partial\mathbb{D}\,,\ \operatorname{Im} w(0) = c_0\,,\ \operatorname{Im} w_z(0) = c_1$$

is uniquely solvable for $f \in L_1(\mathbb{D};\mathbb{C}), \gamma_0, \gamma_1 \in C(\partial\mathbb{D};\mathbb{R}), c_0, c_1 \in \mathbb{R}$. The solution is

$$w(z) = ic_0 + ic_1(z+\overline{z}) - \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma_0(\zeta)\overline{\frac{\zeta+z}{\zeta-z}}\frac{d\overline{\zeta}}{\overline{\zeta}}$$
$$-\frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma_1(\zeta)[\zeta\log(1-z\overline{\zeta})^2 - \overline{\zeta}\log(1-\overline{z}\zeta)^2 + z - \overline{z}]\frac{d\zeta}{\zeta}$$

$$+\frac{1}{\pi}\int\limits_{|\zeta|<1}\{f(\zeta)[\log|\zeta-z|^2-\log(1-\overline{z}\zeta)]-\overline{f(\zeta)}\log(1-\overline{z}\zeta)\}d\xi d\eta \tag{1}$$

$$-\frac{1}{\pi}\int\limits_{|\zeta|<1}\Big\{f(\zeta)\Big[\frac{\log(1-\overline{z}\zeta)}{\zeta^2}+\log|\zeta|\Big]-\overline{f(\zeta)}\Big[\frac{\log(1-z\overline{\zeta})}{\overline{\zeta}^2}+\log|\zeta|\Big]\Big\}d\xi d\eta$$

$$+\frac{1}{\pi}\int\limits_{|\zeta|<1}\Big[\frac{f(\zeta)}{\zeta}+\frac{\overline{f(\zeta)}}{\overline{\zeta}}\Big]\frac{z-\overline{z}}{2}\,d\xi d\eta\ .$$

Proof This result follows from combining the solution

$$w(z)=ic_0+\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma_0(\zeta)\overline{\frac{\zeta+z}{\zeta-z}}\frac{d\zeta}{\zeta}-\frac{1}{2\pi}\int\limits_{|\zeta|<1}\Big[\frac{\omega(\zeta)}{\overline{\zeta}}\overline{\frac{\zeta+z}{\zeta-z}}+\frac{\overline{\omega(\zeta)}}{\zeta}\frac{1+\overline{z}\zeta}{1-\overline{z}\zeta}\Big]d\xi d\eta$$

of the Schwarz problem

$$w_z=\omega \text{ in } \mathbb{D}\ ,\ \operatorname{Re} w=\gamma_0 \text{ on } \partial\mathbb{D}\ ,\ \operatorname{Im} w(0)=c_0$$

with the solution

$$\omega(z)=ic_1+\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma_1(\zeta)\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta}-\frac{1}{2\pi}\int\limits_{|\zeta|<1}\Big[\frac{f(\zeta)}{\zeta}\frac{\zeta+z}{\zeta-z}+\frac{\overline{f(\zeta)}}{\overline{\zeta}}\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}\Big]d\xi d\eta$$

of the Schwarz problem

$$\omega_{\overline{z}}=f \text{ in } \mathbb{D}\ ,\ \operatorname{Re}\omega=\gamma_1 \text{ on } \partial\mathbb{D}\ ,\ \operatorname{Im}\omega(0)=c_1\ .$$

Here the relations

$$\frac{1}{2\pi}\int\limits_{|\zeta|<1}\frac{\tilde{\zeta}+\zeta}{\tilde{\zeta}-\zeta}\overline{\frac{\zeta+z}{\zeta-z}}\frac{d\xi d\eta}{\overline{\zeta}}=2\tilde{\zeta}\log|\tilde{\zeta}-z|^2-2\tilde{\zeta}\log(1-\overline{z}\tilde{\zeta})-\tilde{\zeta}\log|\tilde{\zeta}|^2+z\ ,$$

$$\frac{1}{2\pi}\int\limits_{|\zeta|<1}\overline{\frac{\tilde{\zeta}+\zeta}{\tilde{\zeta}-\zeta}}\frac{1+\overline{z}\zeta}{1-\overline{z}\zeta}\frac{d\xi d\eta}{\zeta}=-2\overline{\tilde{\zeta}}\log(1-\overline{z}\tilde{\zeta})+\overline{\tilde{\zeta}}\log|\tilde{\zeta}|^2-\overline{z}\ ,$$

$$\frac{1}{2\pi}\int\limits_{|\zeta|<1}\frac{1+\zeta\overline{\tilde{\zeta}}}{1-\zeta\overline{\tilde{\zeta}}}\overline{\frac{\zeta+z}{\zeta-z}}\frac{d\xi d\eta}{\overline{\zeta}}=\frac{2}{\overline{\tilde{\zeta}}}\log(1-z\overline{\tilde{\zeta}})+z\ ,$$

$$\frac{1}{2\pi}\int\limits_{|\zeta|<1}\frac{1+\overline{\zeta}\tilde{\zeta}}{1-\overline{\zeta}\tilde{\zeta}}\frac{1+\overline{z}\zeta}{1-\overline{z}\zeta}\frac{d\xi d\eta}{\zeta}=-\frac{2}{\tilde{\zeta}}\log(1-\overline{z}\tilde{\zeta})-\overline{z}\ ,$$

and

$$\frac{1}{2\pi}\int\limits_{|\zeta|<1}\Big[\frac{1}{\overline{\zeta}}\overline{\frac{\zeta+z}{\zeta-z}}-\frac{1}{\zeta}\frac{1+\overline{z}\zeta}{1-\overline{z}\zeta}\Big]d\xi d\eta=-z-\overline{z}$$

are needed. More simple than this is to verify that (1) is the solution.

The uniqueness of the solution can easily be seen. In case w_1 and w_2 are two solutions then $\omega = w_1 - w_2$ would be a harmonic function with homogeneous data,

$$\omega_{z\overline{z}} = 0 \text{ in } \mathbb{D}\ ,\ \text{Re}\ \omega = 0\ ,\ \text{Re}\ \omega_z = 0 \text{ on } \partial\mathbb{D}\ ,\ \text{Im}\ \omega(0) = 0\ ,\ \text{Im}\ \omega_z(0) = 0\ .$$

As ω_z is analytic, say φ' in $\mathbb{D}$, then integrating the equation $\omega_z = \varphi'$ means $\omega = \varphi + \overline{\psi}$ where ψ is analytic in $\mathbb{D}$.
Then Re $\omega_z = 0$ on $\partial\mathbb{D}$, Im $\omega_z(0) = 0$ means Re $\varphi' = 0$, Im $\varphi'(0) = 0$. From [3], Theorem 6 then φ' is seen to be identically zero, i.e. φ a constant, say a. Then from Re $\omega = 0$ on $\partial\mathbb{D}$, Im $\omega(0) = 0$ it follows Re $\psi = -$Re a and Im $\psi(0) =$ Im a. Thus again [3], Theorem 6 shows $\psi(z) = -\overline{a}$ identically in $\mathbb{D}$. This means ω vanishes identically in $\mathbb{D}$.

There is a dual result to Theorem 1 where the roles of z and $\overline{z}$ are interchanged. This can be attained by setting $W = \overline{w}$ and complex conjugating (1).

Theorem 1′ *The Schwarz problem for the Poisson equation in the unit disc*

$$w_{z\overline{z}} = f \textit{ in } \mathbb{D}\ ,\ \text{Re}\ w = \gamma_0\ ,\ \text{Re}\ w_{\overline{z}} = \gamma_1 \textit{ on } \partial\mathbb{D}\ ,\ \text{Im}\ w(0) = c_0\ ,\ \text{Im}\ w_{\overline{z}}(0) = c_1\ ,$$

for $f \in L_1(\mathbb{D};\mathbb{C}), \gamma_0, \gamma_1 \in C(\partial\mathbb{D};\mathbb{R}), c_0, c_1 \in \mathbb{R}$ *is uniquely solvable by*

$$w(z) = ic_0 + ic_1(z+\overline{z}) + \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma_0(\zeta)\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta}$$

$$+\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma_1(\zeta)[\zeta\log(1-z\overline{\zeta})^2-\overline{\zeta}\log(1-\overline{z}\zeta)^2+z-\overline{z}]\frac{d\zeta}{\zeta}$$

$$+\frac{1}{\pi}\int\limits_{|\zeta|<1}\{f(\zeta)[\log|\zeta-z|^2-\log(1-z\overline{\zeta})]-\overline{f(\zeta)}\log(1-z\overline{\zeta})\}d\xi d\eta \qquad (1')$$

$$-\frac{1}{\pi}\int\limits_{|\zeta|<1}\Big\{f(\zeta)\Big[\frac{\log(1-z\overline{\zeta})}{\overline{\zeta}^2}+\log|\zeta|\Big]-\overline{f(\zeta)}\Big[\frac{\log(1-\overline{z}\zeta)}{\zeta^2}+\log|\zeta|\Big]\Big\}d\xi d\eta$$

$$-\frac{1}{\pi}\int\limits_{|\zeta|<1}\Big[\frac{f(\zeta)}{\zeta}+\frac{\overline{f(\zeta)}}{\overline{\zeta}}\Big]\frac{z-\overline{z}}{2}\,d\xi d\eta\ .$$

Theorem 2 *The Schwarz problem for the inhomogeneous Bitsadze equation in the unit disc*

$$w_{\overline{z}\overline{z}} = f \textit{ in } \mathbb{D}\ ,\ \mathrm{Re}\, w = \gamma_0\ ,\ \mathrm{Re}\, w_{\overline{z}} = \gamma_1 \textit{ on } \partial\mathbb{D}\ ,\ \mathrm{Im}\, w(0) = c_0\ ,\ \mathrm{Im}\, w_{\overline{z}}(0) = c_1\ ,$$

for $f \in L_1(\mathbb{D};\mathbb{C}), \gamma_0, \gamma_1 \in C(\partial\mathbb{D};\mathbb{R}), c_0, c_1 \in \mathbb{R}$ *is uniquely solvable through*

$$\begin{aligned} w(z) &= ic_0 + i(z+\overline{z}) + \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma_0(\zeta)\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta} \\ &\quad -\frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma_1(\zeta)\frac{\zeta+z}{\zeta-z}\ (\zeta - z + \overline{\zeta - z})\frac{d\zeta}{\zeta} \\ &\quad +\frac{1}{2\pi}\int\limits_{|\zeta|<1} \Big(\frac{f(\zeta)}{\zeta}\frac{\zeta+z}{\zeta-z} + \frac{\overline{f(\zeta)}}{\overline{\zeta}}\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}\Big)(\zeta - z + \overline{\zeta - z})d\xi d\eta\ . \end{aligned} \tag{2}$$

Proof Rewriting the problem as the system

$$w_{\overline{z}} = \omega \text{ in } \mathbb{D}\ ,\ \mathrm{Re}\, w = \gamma_0 \text{ on } \partial\mathbb{D}\ ,\ \mathrm{Im}\, w(0) = c_0\ ,$$

$$w_{\overline{z}} = f \text{ in } \mathbb{D}\ ,\ \mathrm{Re}\, \omega = \gamma_1 \text{ on } \partial\mathbb{D}\ ,\ \mathrm{Im}\, \omega(0) = c_1\ ,$$

and combining its solutions

$$w(z) = ic_0 + \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma_0(\zeta)\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta} - \frac{1}{2\pi}\int\limits_{|\zeta|<1} \Big(\frac{\omega(\zeta)}{\zeta}\frac{\zeta+z}{\zeta-z} + \frac{\overline{\omega(\zeta)}}{\overline{\zeta}}\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}\Big)d\xi d\eta\ ,$$

$$\omega(z) = ic_1 + \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma_1(\zeta)\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta} - \frac{1}{2\pi}\int\limits_{|\zeta|<1} \Big(\frac{f(\zeta)}{\zeta}\frac{\zeta+z}{\zeta-z} + \frac{\overline{f(\zeta)}}{\overline{\zeta}}\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}\Big)d\xi d\eta\ ,$$

formula (2) is obtained. Here the relations

$$
\begin{aligned}
\frac{1}{2\pi}\int\limits_{|\zeta|<1}\Big(\frac{1}{\zeta}\frac{\zeta+z}{\zeta-z}-\frac{1}{\overline{\zeta}}\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}\Big)d\xi d\eta &= -z-\overline{z}\ ,\\
\frac{1}{2\pi}\int\limits_{|\zeta|<1}\frac{\tilde{\zeta}+\zeta}{\tilde{\zeta}-\zeta}\frac{1}{\zeta}\frac{\zeta+z}{\zeta-z}\ d\xi d\eta &= \frac{\tilde{\zeta}+z}{\tilde{\zeta}-z}\ (\overline{\tilde{\zeta}-z})\ ,\\
\frac{1}{2\pi}\int\limits_{|\zeta|<1}\frac{1+\overline{\zeta}\tilde{\zeta}}{1-\overline{\zeta}\tilde{\zeta}}\frac{1}{\overline{\zeta}}\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}\ d\xi d\eta &= \tilde{\zeta}+z\ ,\\
\frac{1}{2\pi}\int\limits_{|\zeta|<1}\frac{\overline{\tilde{\zeta}+\zeta}}{\overline{\tilde{\zeta}-\zeta}}\frac{1}{\overline{\zeta}}\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}\ d\xi d\eta &= \frac{1+z\overline{\tilde{\zeta}}}{1-z\overline{\tilde{\zeta}}}\ (\tilde{\zeta}-z)\ ,\\
\frac{1}{2\pi}\int\limits_{|\zeta|<1}\frac{1+\zeta\overline{\tilde{\zeta}}}{1-\zeta\overline{\tilde{\zeta}}}\frac{1}{\zeta}\frac{\zeta+z}{\zeta-z}\ d\xi d\eta &= \frac{1+z\overline{\tilde{\zeta}}}{1-z\overline{\tilde{\zeta}}}\ (\overline{\tilde{\zeta}-z})
\end{aligned}
$$

are used. The uniqueness of the solution follows from the unique solvability of the Schwarz problem for analytic functions, Theorem 6 and Theorem 9 in [3].

It is well known that the Dirichlet problem for the Poisson equation

$$w_{z\overline{z}} = f \text{ in } \mathbb{D}\ ,\ w = \gamma \text{ on } \partial\mathbb{D}$$

is well posed, i.e. it is solvable for any $f \in L_1(\mathbb{D};\mathbb{C}), \gamma \in C(\partial\mathbb{D};\mathbb{C})$ and the solution is unique. That the solution is unique is easily seen.

Lemma 1 *The Dirichlet problem for the Laplace equation*

$$w_{z\overline{z}} = 0 \textit{ in } \mathbb{D}\ ,\ w = 0 \textit{ on } \partial\mathbb{D}$$

is only trivially solvable.

Proof From the differential equation w_z is seen to be analytic. Integrating this quantity $w = \varphi + \overline{\psi}$ is seen where φ and ψ are both analytic in $\mathbb{D}$. Without loss of generality $\psi(0) = 0$ may be assumed. From the boundary condition $\varphi = -\overline{\psi}$ on $\partial\mathbb{D}$ follows. This Dirichlet problem is solvable if and only if, see [3], Theorem 7,

$$0 = \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\overline{\psi(\zeta)}\frac{\overline{z}d\zeta}{1-\overline{z}\zeta} = \overline{\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\psi(\zeta)\frac{d\zeta}{\zeta-z}} - \overline{\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\psi(\zeta)\frac{d\zeta}{\zeta}} = \overline{\psi(z)}\ .$$

This also implies $\varphi = 0$ on $\mathbb{D}$ so that $w = 0$ in $\mathbb{D}$.

As Bitsadze [6] has realized such a result is not true for the equation $w_{\overline{z}\overline{z}} = 0$.

Lemma 2 *The Dirichlet problem for the Bitsadze equation*

$$w_{\overline{z}\overline{z}} = 0 \text{ in } \mathbb{D}\ ,\ w = 0 \text{ on } \partial\mathbb{D}$$

has infinitely many linearly independent solutions.

Proof Here $w_{\overline{z}}$ is an analytic function in $\mathbb{D}$. Integrating gives $w(z) = \varphi(z)\overline{z} + \psi(z)$ with some analytic functions in $\mathbb{D}$. On the boundary $\varphi(z) + z\psi(z) = 0$. As this is an analytic function this relations hold in $\mathbb{D}$ too, see [3], Theorem 7. Hence, $w(z) = (1- \mid z\mid^2)\psi(z)$ for arbitrary analytic ψ. In particular $w_k(z) = (1- \mid z\mid^2)z^k$ is a solution of the Dirichlet problem for any $k \in \mathbb{N}_0$ and these solutions are linearly independent over $\mathbb{C}$.

Because of this result the Dirichlet problem as formulated above is ill-posed for the inhomogeneous Bitsadze equation.

With regard to the Dirichlet problem for the Poisson equation the representation formula [3], $(15')$ is improper as is also [3], $(15''')$. The middle terms are improper. They can easily be eliminated by applying the Gauss Theorem, see [3]. For the respective term in [3], $(15')$ in the case $D = \mathbb{D}$

$$\begin{aligned}
&\frac{1}{2\pi i}\int\limits_{|\zeta|=1} w_{\overline{\zeta}}(\zeta)\log|\zeta - z|^2\, d\overline{\zeta} = \frac{1}{2\pi i}\int\limits_{|\zeta|=1} w_{\overline{\zeta}}(\zeta)\log|1 - z\overline{\zeta}|^2\, d\overline{\zeta}\\
&= -\frac{1}{\pi}\int\limits_{|\zeta|<1} \partial_\zeta[w_{\overline{\zeta}}(\zeta)\log|1 - z\overline{\zeta}|^2]d\xi d\eta\\
&= -\frac{1}{\pi}\int\limits_{|\zeta|<1} w_{\zeta\overline{\zeta}}(\zeta)\log|1 - z\overline{\zeta}|^2\, d\xi d\eta + \frac{1}{\pi}\int\limits_{|\zeta|<1} w_{\overline{\zeta}}(\zeta)\frac{\overline{z}}{1 - \overline{z}\zeta}\, d\xi d\eta
\end{aligned}$$

follows. Applying the Gauss Theorem again shows

$$\begin{aligned}
&\frac{1}{\pi}\int\limits_{|\zeta|<1} w_{\overline{\zeta}}(\zeta)\frac{\overline{z}}{1 - \overline{z}\zeta}\, d\xi d\eta = \frac{1}{\pi}\int\limits_{|\zeta|<1} \partial_{\overline{\zeta}}\Big[w(\zeta)\frac{\overline{z}}{1 - \overline{z}\zeta}\Big]d\xi d\eta\\
&= \frac{1}{2\pi i}\int\limits_{|\zeta|=1} w(\zeta)\frac{\overline{z}}{1 - \overline{z}\zeta}\, d\zeta = \frac{1}{2\pi i}\int\limits_{|\zeta|=1} w(\zeta)\frac{\overline{z}}{\overline{\zeta - z}}\frac{d\zeta}{\zeta}\ .
\end{aligned}$$

Thus inserting these in [3], $(15')$ leads to the representation

$$w(z) = \frac{1}{2\pi i}\int\limits_{|\zeta|=1} w(\zeta)\Big(\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - 1\Big)\frac{d\zeta}{\zeta} - \frac{1}{\pi}\int\limits_{|\zeta|<1} w_{\zeta\overline{\zeta}}(\zeta)\log\Big|\frac{1 - z\overline{\zeta}}{\zeta - z}\Big|^2\, d\xi d\eta. \tag{3}$$

The kernel function in the boundary integral is the Poisson kernel, the one in the area integral is called Green function for the unit disc with respect to the Laplace operator.

Definition 1 *The function* $G(z,\zeta) = (1/2)G_1(z,\zeta)$ *with*

$$G_1(z,\zeta) = \log \left| \frac{1-z\overline{\zeta}}{\zeta - z} \right|^2 \ , \ z,\zeta \in \mathbb{D} \ , \ z \neq \zeta \ , \tag{4}$$

is called Green function of the Laplace operator for the unit disc.

Remark The Green function has the following properties. For any fixed $\zeta \in \mathbb{D}$ as a function of z

(1) $G(z,\zeta)$ is harmonic in $\mathbb{D} \setminus \{\zeta\}$,
(2) $G(z,\zeta) + \log|\zeta - z|$ is harmonic in $\mathbb{D}$,
(3) $\lim_{z\to t} G(z,\zeta) = 0$ for all $t \in \partial\mathbb{D}$,
(4) $G(z,\zeta) = G(\zeta,z)$ for $z,\zeta \in \mathbb{D}, z \neq \zeta$.

They can be checked by direct calculations.

Green functions exist for other domains than just the unit disc. The existence is related to the solvability of the Dirichlet problem for harmonic functions in the domain. The Riemann mapping theorem can be used to find it e.g. for regular simply connected domains. Having the Green function [3], (15′) and [3], (15‴) can be altered as above leading to the Green representation formula, see e.g. [1]. Green functions exist also in higher dimensional spaces and for other strongly elliptic differential operators.

For the unit disc the following result is shown.

Theorem 3 *Any* $w \in C^2(\mathbb{D};\mathbb{C}) \cap C^1(\bar{\mathbb{D}};\mathbb{C})$ *can be represented as*

$$w(z) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} w(\zeta)\Big(\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - 1\Big)\frac{d\zeta}{\zeta} - \frac{1}{\pi}\int\limits_{|\zeta|<1} w_{\zeta\overline{\zeta}}(\zeta) G_1(z,\zeta)\, d\xi d\eta \ , \tag{3′}$$

where $G_1(z,\zeta)$ *is defined in* (4).

Formulas [3], (15′) and [3], (15‴) are both unsymmetric. Adding both gives some symmetric formula which is for the unit disc

$$\begin{aligned} w(z) \;=\; & \frac{1}{4\pi i} \int\limits_{|\zeta|=1} w(\zeta)\Big(\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}}\Big)\frac{d\zeta}{\zeta} \\ & - \frac{1}{4\pi i} \int\limits_{|\zeta|=1} (\zeta w_\zeta(\zeta) + \overline{\zeta} w_{\overline{\zeta}}(\zeta)) \log|\zeta - z|^2 \, \frac{d\zeta}{\zeta} \\ & + \frac{1}{\pi} \int\limits_{|\zeta|<1} w_{\zeta\overline{\zeta}}(\zeta) \log|\zeta - z|^2 \, d\xi d\eta \ . \end{aligned} \tag{5}$$

Motivated by the procedure before, the Gauss Theorems are applied in a symmetric way to

$$\frac{1}{\pi}\int\limits_{|\zeta|<1} w_{\zeta\overline{\zeta}}(\zeta)\log|1-z\overline{\zeta}|^2\,d\xi d\eta$$
$$=\frac{1}{2\pi}\int\limits_{|\zeta|<1}\Big\{\partial_\zeta[w_{\overline{\zeta}}(\zeta)\log|1-z\overline{\zeta}|^2]+\partial_{\overline{\zeta}}[w_\zeta(\zeta)\log|1-z\overline{\zeta}|^2]$$
$$+\partial_{\overline{\zeta}}\Big[w(\zeta)\frac{\overline{z}}{1-\overline{z}\zeta}\Big]+\partial_\zeta\Big[w(\zeta)\frac{z}{1-z\overline{\zeta}}\Big]\Big\}d\xi d\eta$$
$$=\frac{1}{4\pi i}\int\limits_{|\zeta|=1}\log|1-z\overline{\zeta}|^2\,[\zeta w_\zeta(\zeta)+\overline{\zeta}w_{\overline{\zeta}}(\zeta)]\frac{d\zeta}{\zeta}$$
$$+\frac{1}{4\pi i}\int\limits_{|\zeta|=1}w(\zeta)\Big[\frac{\overline{z}\zeta}{1-\overline{z}\zeta}+\frac{z\overline{\zeta}}{1-z\overline{\zeta}}\Big]\frac{d\zeta}{\zeta}$$

$$=\frac{1}{4\pi i}\int\limits_{|\zeta|=1}\log|\zeta-z|^2\,[\zeta w_\zeta(\zeta)+\overline{\zeta}w_{\overline{\zeta}}(\zeta)]\frac{d\zeta}{\zeta}$$
$$+\frac{1}{4\pi i}\int\limits_{|\zeta|=1}w(\zeta)\Big[\frac{z}{\zeta-z}+\frac{\overline{z}}{\overline{\zeta-z}}\Big]\frac{d\zeta}{\zeta}\ .$$

Here are two possibilities. At first the second term in (5) can be eliminated giving

$$w(z)=\frac{1}{2\pi i}\int\limits_{|\zeta|=1}w(\zeta)\Big(\frac{\zeta}{\zeta-z}+\frac{\overline{\zeta}}{\overline{\zeta-z}}-1\Big)\frac{d\zeta}{\zeta}-\frac{1}{\pi}\int\limits_{|\zeta|<1}w_{\zeta\overline{\zeta}}(\zeta)\log\Big|\frac{1-z\overline{\zeta}}{\zeta-z}\Big|^2\,d\xi d\eta\ ,$$

i.e. (3). Next the first term in (5) is simplified so that

$$\begin{aligned} w(z) \;=\; & \frac{1}{2\pi i}\int\limits_{|\zeta|=1}w(\zeta)\frac{d\zeta}{\zeta}-\frac{1}{2\pi i}\int\limits_{|\zeta|=1}(\zeta w_\zeta(\zeta)+\overline{\zeta}w_{\overline{\zeta}}(\zeta))\log|\zeta-z|^2\,\frac{d\zeta}{\zeta} \\ & +\frac{1}{\pi}\int\limits_{|\zeta|<1}w_{\zeta\overline{\zeta}}(\zeta)\log|(\zeta-z)(1-z\overline{\zeta})|^2\,d\xi d\eta\ . \end{aligned} \tag{6}$$

Here the normal derivative appears in the second term while a new kernel function arises in the area integral.

Definition 2 *The function $N(z,\zeta)=-(1/2)N_1(z,\zeta)$ with*

$$N_1(z,\zeta)=\log|(\zeta-z)(1-z\overline{\zeta})|^2\ ,\ z,\zeta\in\mathbb{D}\ ,\ z\neq\zeta\ , \tag{7}$$

is called Neumann function of the Laplace operator for the unit disc.
Remark The Neumann function, sometimes [7] also called Green function of second kind or second Green function, has the properties

(1) $N(z,\zeta)$ is harmonic in $z \in \mathbb{D} \setminus \{\zeta\}$,

(2) $N(z,\zeta) + \log|\zeta - z|$ is harmonic in $z \in \mathbb{D}$ for any $\zeta \in \overline{\mathbb{D}}$,

(3) $\partial_\nu N(z,\zeta) = -1$ for $z \in \partial\mathbb{D}, \zeta \in \mathbb{D}$,

(4) $N(z,\zeta) = N(\zeta,z)$ for $z,\zeta \in \mathbb{D}, z \neq \zeta$.

(5) $\displaystyle \frac{1}{2\pi} \int_{|\zeta|=1} N(z,\zeta)\frac{d\zeta}{\zeta} = 0$.

They can be checked by direct calculations.
The last result may therefore be formulated as follows.
Theorem 4 *Any $w \in C^2(\mathbb{D};\mathbb{C}) \cap C^1(\overline{\mathbb{D}};\mathbb{C})$ can be represented as*

$$w(z) = \frac{1}{2\pi i} \int_{|\zeta|=1} w(\zeta)\frac{d\zeta}{\zeta} - \frac{1}{2\pi i} \int_{|\zeta|=1} \partial_\nu w(\zeta) \log|\zeta - z|^2 \frac{d\zeta}{\zeta}$$

$$+\frac{1}{\pi} \int_{|\zeta|<1} w_{\zeta\overline{\zeta}}(\zeta) N_1(z,\zeta) d\xi d\eta \ . \qquad (6')$$

This formula can also be written as

$$w(z) = \frac{1}{4\pi i} \int_{|\zeta|=1} [w(\zeta)\partial_{\nu_\zeta} N_1(z,\zeta) - \partial_\nu w(\zeta) N_1(z,\zeta)] \frac{d\zeta}{\zeta} + \frac{1}{\pi} \int_{|\zeta|<1} w_{\zeta\overline{\zeta}}(\zeta) N_1(z,\zeta) d\xi d\eta \ . \qquad (6'')$$

Theorem 3 immediately provides the solution to the Dirichlet problem.
Theorem 5 *The Dirichlet problem for the Poisson equation in the unit disc*

$$w_{z\overline{z}} = f \text{ in } \mathbb{D} \ , \ w = \gamma \text{ on } \partial\mathbb{D} \ ,$$

for $f \in L_1(\mathbb{D};\mathbb{C})$ and $\gamma \in C(\partial\mathbb{D};\mathbb{C})$ is uniquely given by

$$w(z) = \frac{1}{2\pi i} \int_{|\zeta|=1} \gamma(\zeta)\Big(\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - 1\Big)\frac{d\zeta}{\zeta} - \frac{1}{\pi} \int_{|\zeta|<1} f(\zeta) G_1(z,\zeta) d\xi d\eta \ . \qquad (8)$$

This is at once clear from the properties of the Poisson kernel and the Green function.
As the Dirichlet problem formulated as for the Poisson equation is not uniquely solvable for the Bitsadze equation another kind Dirichlet problem is considered which is motivated from decomposing this Bitsadze equation in a first order system.

Theorem 6 *The Dirichlet problem for the inhomogeneous Bitsadze equation in the unit disc*

$$w_{\overline{z}\overline{z}} = f \text{ in } \mathbb{D} \ , \ w = \gamma_0 \ , \ w_{\overline{z}} = \gamma_1 \text{ on } \partial\mathbb{D} \ ,$$

for $f \in L_1(\mathbb{D};\mathbb{C}), \gamma_0, \gamma_1 \in C(\partial\mathbb{D};\mathbb{C})$ *is solvable if and only if for* $|z| < 1$

$$\frac{\overline{z}}{2\pi i} \int\limits_{|\zeta|=1} \Big(\frac{\gamma_0(\zeta)}{1-\overline{z}\zeta} - \frac{\gamma_1(\zeta)}{\zeta}\Big) d\zeta + \frac{\overline{z}}{\pi} \int\limits_{|\zeta|<1} f(\zeta) \frac{\overline{\zeta - z}}{1-\overline{z}\zeta} \, d\xi d\eta = 0 \tag{9}$$

and

$$\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma_1(\zeta) \frac{\overline{z} d\zeta}{1-\overline{z}\zeta} - \frac{1}{\pi} \int\limits_{|\zeta|<1} f(\zeta) \frac{\overline{z} d\xi d\eta}{1-\overline{z}\zeta} = 0 \ . \tag{10}$$

The solution then is

$$w(z) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma_0(\zeta) \frac{d\zeta}{\zeta - z} - \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma_1(\zeta) \frac{\overline{\zeta - z}}{\zeta - z} \, d\zeta + \frac{1}{\pi} \int\limits_{|\zeta|<1} f(\zeta) \frac{\overline{\zeta - z}}{\zeta - z} \, d\xi d\eta \ . \tag{11}$$

Proof Decomposing the problem into the system

$$w_{\overline{z}} = \omega \text{ in } \mathbb{D} \ , \ w = \gamma_0 \text{ on } \partial\mathbb{D} \ ,$$

$$\omega_{\overline{z}} = f \text{ in } \mathbb{D} \ , \ \omega = \gamma_1 \text{ on } \partial\mathbb{D} \ ,$$

and composing its solutions

$$w(z) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma_0(\zeta) \frac{d\zeta}{\zeta - z} - \frac{1}{\pi} \int\limits_{|\zeta|<1} \omega(\zeta) \frac{d\xi d\eta}{\zeta - z} \ ,$$

$$\omega(z) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma_1(\zeta) \frac{d\zeta}{\zeta - z} - \frac{1}{\pi} \int\limits_{|\zeta|<1} f(\zeta) \frac{d\xi d\eta}{\zeta - z} \ ,$$

and the solvability conditions

$$\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma_0(\zeta) \frac{\overline{z} d\zeta}{1-\overline{z}\zeta} = \frac{1}{\pi} \int\limits_{|\zeta|<1} \omega(\zeta) \frac{\overline{z} d\xi d\eta}{1-\overline{z}\zeta} \ ,$$

$$\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma_1(\zeta) \frac{\overline{z} d\zeta}{1-\overline{z}\zeta} = \frac{1}{\pi} \int\limits_{|\zeta|<1} f(\zeta) \frac{\overline{z} d\xi d\eta}{1-\overline{z}\zeta} \ ,$$

proves (11) together with (9) and (10). Here

$$\frac{1}{\pi}\int\limits_{|\zeta|<1}\frac{d\xi d\eta}{(\tilde{\zeta}-\zeta)(1-\overline{z}\zeta)} = \frac{\overline{\tilde{\zeta}-z}}{1-\overline{z}\tilde{\zeta}} - \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{\overline{\zeta-z}}{1-\overline{z}\zeta}\frac{d\zeta}{\zeta-\tilde{\zeta}} = \frac{\overline{\tilde{\zeta}-z}}{1-\overline{z}\tilde{\zeta}}$$

and

$$-\frac{1}{\pi}\int\limits_{|\zeta|<1}\frac{d\xi d\eta}{(\zeta-\tilde{\zeta})(\zeta-z)} = -\frac{1}{\pi}\int\limits_{|\zeta|<1}\frac{1}{\tilde{\zeta}-z}\Big(\frac{1}{\zeta-\tilde{\zeta}}-\frac{1}{\zeta-z}\Big)d\xi d\eta = \frac{\overline{\tilde{\zeta}-z}}{\tilde{\zeta}-z}$$

are used.
This problem can also be considered for the Poisson equation.
Theorem 7 *The boundary value problem for the Poisson equation in the unit disc*

$$w_{z\overline{z}} = f \textit{ in } \mathbb{D}\ ,\ w=\gamma_0\ ,\ w_z=\gamma_1 \textit{ on } \partial\mathbb{D}\ ,$$

for $f\in L_1(\mathbb{D};\mathbb{C}), \gamma_0,\gamma_1\in C(\partial\mathbb{D};\mathbb{C})$ *is uniquely solvable if and only if*

$$-\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma_0(\zeta)\frac{zd\overline{\zeta}}{1-z\overline{\zeta}} + \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma_1(\zeta)\log(1-z\overline{\zeta})d\zeta$$
$$= \frac{1}{\pi}\int\limits_{|\zeta|<1}f(\zeta)\log(1-z\overline{\zeta})d\xi d\eta \tag{12}$$

and

$$\frac{\overline{z}}{2\pi i}\int\limits_{|\zeta|=1}\gamma_1(\zeta)\frac{d\zeta}{1-\overline{z}\zeta} = \frac{\overline{z}}{\pi}\int\limits_{|\zeta|<1}f(\zeta)\frac{d\xi d\eta}{1-\overline{z}\zeta}\ . \tag{13}$$

The solution then is

$$w(z) = -\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma_0(\zeta)\frac{d\overline{\zeta}}{\overline{\zeta-z}} - \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma_1(\zeta)\log(1-z\overline{\zeta})d\zeta$$
$$+\frac{1}{\pi}\int\limits_{|\zeta|<1}f(\zeta)(\log|\zeta-z|^2-\log(1-\overline{z}\zeta))d\xi d\eta\ . \tag{14}$$

Proof The system

$$w_z=\omega\ ,\ \omega_{\overline{z}}=f \text{ in } \mathbb{D}\ ,\ w=\gamma_0\ ,\ \omega=\gamma_1 \text{ on } \partial\mathbb{D}$$

is uniquely solvable according to Theorem 10 if and only if

$$-\frac{z}{2\pi i}\int\limits_{|\zeta|=1}\gamma_0(\zeta)\frac{d\overline{\zeta}}{1-z\overline{\zeta}}=\frac{z}{\pi}\int\limits_{|\zeta|<1}\omega(\zeta)\frac{d\xi d\eta}{1-z\overline{\zeta}},$$

$$\frac{\overline{z}}{2\pi i}\int\limits_{|\zeta|=1}\gamma_1(\zeta)\frac{d\zeta}{1-\overline{z}\zeta}=\frac{\overline{z}}{\pi}\int\limits_{|\zeta|<1}f(\zeta)\frac{d\xi d\eta}{1-\overline{z}\zeta}\ .$$

The solution then is

$$w(z)=-\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma_0(\zeta)\frac{d\overline{\zeta}}{\overline{\zeta-z}}-\frac{1}{\pi}\int\limits_{|\zeta|<1}\omega(\zeta)\frac{d\xi d\eta}{\overline{\zeta-z}}\ ,$$

$$\omega(z)=\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma_1(\zeta)\frac{d\zeta}{\zeta-z}-\frac{1}{\pi}\int\limits_{|\zeta|<1}f(\zeta)\frac{d\xi d\eta}{\zeta-z}\ .$$

Inserting ω into the first condition gives (12) for

$$\begin{aligned}\frac{1}{\pi}\int\limits_{|\zeta|<1}\omega(\zeta)\frac{d\xi d\eta}{1-z\overline{\zeta}} \quad &= \quad \frac{1}{2\pi i}\int\limits_{|\tilde{\zeta}|=1}\gamma_1(\tilde{\zeta})\frac{1}{\pi}\int\limits_{|\zeta|=1}\frac{d\xi d\eta}{(\tilde{\zeta}-\zeta)(1-z\overline{\zeta})}\ d\tilde{\zeta}\\ &\quad -\frac{1}{\pi}\int\limits_{|\tilde{\zeta}|<1}f(\tilde{\zeta})\frac{1}{\pi}\int\limits_{|\zeta|<1}\frac{d\xi d\eta}{(\tilde{\zeta}-\zeta)(1-z\overline{\zeta})}\ d\tilde{\xi}d\tilde{\eta}\end{aligned}$$

with

$$\begin{aligned}-\frac{1}{\pi}\int\limits_{|\zeta|<1}\frac{zd\xi d\eta}{(\zeta-\tilde{\zeta})(1-z\overline{\zeta})} \quad &= \quad -\log(1-z\overline{\tilde{\zeta}})-\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\log(1-z\overline{\zeta})\frac{d\zeta}{\zeta-\tilde{\zeta}}\\ &= \quad -\log(1-z\overline{\tilde{\zeta}})+\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{\log(1-z\overline{\zeta})}{1-\tilde{\zeta}\overline{\zeta}}\frac{d\overline{\zeta}}{\overline{\zeta}}\\ &= \quad -\log(1-z\overline{\tilde{\zeta}}).\end{aligned}$$

Combining the two integral representations for w and ω leads to (13) as

$$\begin{aligned}-\frac{1}{\pi}\int\limits_{|\zeta|<1}\omega(\zeta)\frac{d\xi d\eta}{\overline{\zeta-z}} \quad &= \quad -\frac{1}{2\pi i}\int\limits_{|\tilde{\zeta}|=1}\gamma_1(\tilde{\zeta})\frac{1}{\pi}\int\limits_{|\zeta|<1}\frac{d\xi d\eta}{(\tilde{\zeta}-\zeta)(\overline{\zeta-z})}\ d\tilde{\zeta}\\ &\quad +\frac{1}{\pi}\int\limits_{|\tilde{\zeta}|<1}f(\tilde{\zeta})\frac{1}{\pi}\int\limits_{|\zeta|<1}\frac{d\xi d\eta}{(\tilde{\zeta}-\zeta)(\overline{\zeta-z})}\ d\tilde{\xi}d\tilde{\eta}\end{aligned}$$

where

$$-\frac{1}{\pi}\int\limits_{|\zeta|<1}\frac{d\xi d\eta}{(\overline{\zeta-z})(\zeta-\tilde{\zeta})} = \log|\tilde{\zeta}-z|^2-\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\log|\zeta-z|^2\frac{d\zeta}{\zeta-\tilde{\zeta}}$$

$$= \log|\tilde{\zeta}-z|^2-\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\log(1-\overline{z}\zeta)\frac{d\zeta}{\zeta-\tilde{\zeta}}$$

$$+\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\log(1-z\overline{\zeta})\frac{d\overline{\zeta}}{\overline{\zeta}(1-\tilde{\zeta}\overline{\zeta})}$$

$$= \log|\zeta-z|^2-\log(1-\overline{z}\tilde{\zeta})\ .$$

Remark In a similar way the problem

$$w_{z\overline{z}}=f \text{ in } \mathbb{D}\ ,\ w=\gamma_0\ ,\ w_{\overline{z}}=\gamma_1 \text{ on } \partial\mathbb{D}$$

with $f\in L_1(\mathbb{D};\mathbb{C}),\gamma_0,\gamma_1\in C(\partial\mathbb{D};\mathbb{C})$ can be solved.

That integral representations may not always be used to solve related boundary value problems as was done in the case of the Dirichlet problem with formula (3), can be seen from (6′). If w is a solution to the Poisson equation $w_{z\overline{z}}=f$ in $\mathbb{D}$ satisfying $\partial_\nu w=\gamma$ on $\partial\mathbb{D}$ and being normalized by

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1}w(\zeta)\frac{d\zeta}{\zeta}=c$$

for proper f and γ then on the basis of Theorem 16 it may be presented as

$$w(z) = c-\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\log|\zeta-z|^2\frac{d\zeta}{\zeta}$$
$$+\frac{1}{\pi}\int\limits_{|\zeta|<1}f(\zeta)\log|(\zeta-z)(1-z\overline{\zeta})|^2\,d\xi d\eta. \tag{15}$$

But this formula although providing always a solution to $w_{z\overline{z}}=f$ does not for all γ satisfy the respective boundary behaviour. Such a behaviour is also known from the Cauchy integral.

Theorem 8 *The Neumann problem for the Poisson equation in the unit disc*

$$w_{z\overline{z}}=f \textit{ in } \mathbb{D}\ ,\ \partial_\nu w=\gamma \textit{ on } \partial\mathbb{D}\ ,\ \frac{1}{2\pi i}\int\limits_{|\zeta|=1}w(\zeta)\frac{d\zeta}{\zeta}=c\ ,$$

for $f\in L_1(\mathbb{D};\mathbb{C}),\gamma\in C(\partial\mathbb{D};\mathbb{C}),c\in\mathbb{C}$ *is solvable if and only if*

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\frac{d\zeta}{\zeta}=\frac{2}{\pi}\int\limits_{|\zeta|<1}f(\zeta)d\xi d\eta\ . \tag{16}$$

The unique solution is then given by (15).

Proof As the Neumann function is a fundamental solution to the Laplace operator and the boundary integral is a harmonic function, (15) provides a solution to the Poisson equation. For checking the boundary behaviour the first order derivatives have to be considered. They are

$$w_z(z) = \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma(\zeta)\frac{d\zeta}{(\zeta - z)\zeta} - \frac{1}{\pi}\int\limits_{|\zeta|<1} f(\zeta)\Big(\frac{1}{\zeta - z} + \frac{\overline{\zeta}}{1 - z\overline{\zeta}}\Big)d\xi d\eta \ ,$$

$$w_{\overline{z}}(z) = \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma(\zeta)\frac{d\zeta}{\overline{(\zeta - z)}\zeta} - \frac{1}{\pi}\int\limits_{|\zeta|<1} f(\zeta)\Big(\frac{1}{\overline{\zeta - z}} + \frac{\zeta}{1 - \overline{z}\zeta}\Big)d\xi d\eta \ ,$$

so that

$$\partial_\nu w(z) = \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma(\zeta)\Big(\frac{\zeta}{\zeta - z} + \frac{\overline{z}}{\overline{\zeta - z}} - 1\Big)\frac{d\zeta}{\zeta}$$

$$-\frac{1}{\pi}\int\limits_{|\zeta|<1} f(\zeta)\Big[\frac{z}{\zeta - z} + \frac{\overline{z}}{\overline{\zeta - z}} + \frac{z\overline{\zeta}}{1 - z\overline{\zeta}} + \frac{\overline{z}\zeta}{1 - \overline{z}\zeta}\Big]d\xi d\eta$$

$$= \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma(\zeta)\Big(\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - 2\Big)\frac{d\zeta}{\zeta}$$

$$+\frac{1}{\pi}\int\limits_{|\zeta|=1} f(\zeta)\Big[2 - \frac{z}{\zeta - z} - \frac{\overline{z}}{\overline{\zeta - z}} - \frac{1}{1 - z\overline{\zeta}} - \frac{1}{1 - \overline{z}\zeta}\Big]d\xi d\eta \ .$$

For $|z|= 1$ this is using the property of the Poisson kernel

$$\partial_\nu w(z) = \gamma(z) - \frac{1}{2\pi i}\int \gamma(\zeta)\frac{d\zeta}{\zeta} + \frac{2}{\pi}\int\limits_{|\zeta|=1} f(\zeta)d\xi d\eta \ .$$

Therefore $\partial_\nu w = \gamma$ on $|z|= 1$ if and only if condition (16) holds. At last the normalization condition has to be verified. It follows from $|\zeta - z|=|1 - z\overline{\zeta}|$ for $|z|= 1$ and

$$\frac{1}{2\pi i}\int\limits_{|z|=1} \log|1 - z\overline{\zeta}|^2\frac{dz}{z} = \frac{1}{2\pi i}\int\limits_{|z|=1} \log(1 - z\overline{\zeta})\frac{dz}{z} - \frac{1}{2\pi i}\int\limits_{|z|=1} \log(1 - \overline{z}\zeta)\frac{d\overline{z}}{\overline{z}} = 0 \ .$$

Theorem 9 *The Dirichlet-Neumann problem for the inhomogeneous Bitsadze equation in the unit disc*

$$w_{\overline{z}\overline{z}} = f \ in\ \mathbb{D} \ , \ w = \gamma_0 \ , \ \partial_\nu w_{\overline{z}} = \gamma_1 \ on\ \partial\mathbb{D} \ , \ w_{\overline{z}}(0) = c \ ,$$

for $f \in L_1(\mathbb{D};\mathbb{C}) \cap C(\partial\mathbb{D};\mathbb{C}), \gamma_0, \gamma_1 \in C(\partial\mathbb{D};\mathbb{C}), c \in \mathbb{C}$ is solvable if and only if for $z \in \mathbb{D}$

$$c - \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma_0(\zeta)\frac{d\zeta}{1-\overline{z}\zeta} + \frac{1}{\pi}\int\limits_{|\zeta|<1} f(\zeta)\frac{1-|\zeta|^2}{\zeta(1-\overline{z}\zeta)}\, d\xi d\eta = 0 \tag{17}$$

and

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1} (\gamma_1(\zeta) - \overline{\zeta}f(\zeta))\frac{d\zeta}{\zeta(1-\overline{z}\zeta)} + \frac{1}{\pi}\int\limits_{|\zeta|<1} \frac{\overline{z}f(z)}{(1-\overline{z}\zeta)^2}\, d\xi d\eta = 0\ . \tag{18}$$

The solution then is

$$\begin{aligned} w(z) &= c\bar{z} + \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma_0(\zeta)\frac{d\zeta}{\zeta - z} \\ &\quad + \frac{1}{2\pi i}\int\limits_{|\zeta|=1} (\gamma_1(\zeta) - \overline{\zeta}f(\zeta))\frac{1-|z|^2}{z}\ \log(1-z\overline{\zeta})\frac{d\zeta}{\zeta} \\ &\quad + \frac{1}{\pi}\int\limits_{|\zeta|<1} f(\zeta)\frac{|\zeta|^2-|z|^2}{\zeta(\zeta-z)}\, d\xi d\eta\ . \end{aligned} \tag{19}$$

Proof The problem is equivalent to the system

$$w_{\overline{z}} = \omega \text{ in } \mathbb{D}\ ,\ w = \gamma_0 \text{ on } \partial\mathbb{D}\ ,$$

$$\omega_{\overline{z}} = f \text{ in } \mathbb{D}\ ,\ \partial_\nu \omega = \gamma_1 \text{ on } \partial\mathbb{D}\ ,\ \omega(0) = c\ .$$

The solvability conditions are

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma_0(\zeta)\frac{d\zeta}{1-\overline{z}\zeta} = \frac{1}{\pi}\int\limits_{|\zeta|<1} \omega(\zeta)\frac{d\xi d\eta}{1-\overline{z}\zeta}$$

and

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1} (\gamma_1(\zeta) - \overline{\zeta}f(\zeta))\frac{d\zeta}{1-\overline{z}\zeta)\zeta} + \frac{1}{\pi}\int\limits_{|\zeta|<1} \frac{\overline{z}f(\zeta)}{(1-\overline{z}\zeta)^2}\, d\xi d\eta = 0$$

and the unique solutions

$$w(z) = \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma_0(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int\limits_{|\zeta|<1} \omega(\zeta)\frac{d\xi d\eta}{\zeta - z}$$

and

$$\omega(z) = c - \frac{1}{2\pi i} \int\limits_{|\zeta|=1} (\gamma(\zeta) - \overline{\zeta} f(\zeta)) \log(1 - z\overline{\zeta}) \frac{d\zeta}{\zeta} - \frac{1}{\pi} \int\limits_{|\zeta|<1} \frac{z f(\zeta)}{\zeta(\zeta - z)} \, d\xi d\eta$$

according to [3], Theorems 10 and 11.
From

$$\frac{1}{\pi} \int\limits_{|\zeta|<1} \frac{d\xi d\eta}{1 - \overline{z}\zeta} = 1 \, , \ \frac{1}{\pi} \int\limits_{|\zeta|<1} \log(1 - \zeta\overline{\tilde{\zeta}}) \frac{d\xi d\eta}{1 - \overline{z}\zeta} = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \log(1 - \zeta\overline{\tilde{\zeta}}) \frac{d\zeta}{(1 - \overline{z}\zeta)\zeta} = 0 \, ,$$

and

$$\frac{1}{\pi} \int\limits_{|\zeta|<1} \frac{\zeta}{\tilde{\zeta} - \zeta} \frac{d\xi d\eta}{1 - \overline{z}\zeta} = \frac{|\tilde{\zeta}|^2 - 1}{\tilde{\zeta}(1 - \overline{z}\tilde{\zeta})}$$

condition (17) follows. Similarly (19) follows from

$$-\frac{1}{\pi} \int\limits_{|\zeta|<1} \frac{d\xi d\eta}{\zeta - z} = \overline{z} \, , \ \frac{1}{\pi} \int\limits_{|\zeta|<1} \log(1 - \zeta\overline{\tilde{\zeta}}) \frac{d\xi d\eta}{\zeta - z} = \frac{1 - |z|^2}{z} \ \log(1 - z\overline{\tilde{\zeta}})$$

and

$$\frac{1}{\pi} \int\limits_{|\zeta|<1} \frac{\zeta}{(\tilde{\zeta} - \zeta)} \frac{d\xi d\eta}{\zeta - z} = \frac{|\tilde{\zeta}|^2 - |z|^2}{(\tilde{\zeta} - z)} \, .$$

Theorem 10 *The boundary value problem for the inhomogeneous Bitsadze equation in the unit disc*

$$w_{\overline{z}\overline{z}} = f \ in \ \mathbb{D} \, , \ w = \gamma_0 \, , \ z w_{z\overline{z}} = \gamma_1 \ on \ \partial\mathbb{D} \, , \ w_{\overline{z}}(0) = c \, ,$$

is solvable for $f \in L_1(\mathbb{D};\mathbb{C}), \gamma_0, \gamma_1 \in C(\partial\mathbb{D};\mathbb{C}), c \in \mathbb{C}$ *if and only if for* $z \in \mathbb{D}$ *condition* (17) *together with*

$$\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma_1(\zeta) \frac{d\zeta}{\zeta(1 - \overline{z}\zeta)} + \frac{\overline{z}}{\pi} \int\limits_{|\zeta|<1} f(\zeta) \frac{d\xi d\eta}{(1 - \overline{z}\zeta)^2} = 0 \tag{20}$$

holds. The solution then is uniquely given by

$$\begin{aligned} w(z) \ = \ & c\overline{z} + \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma_0(\zeta) \frac{d\zeta}{\zeta - z} + \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma_1(\zeta) \frac{1 - |z|^2}{z} \ \log(1 - z\overline{\zeta}) \frac{d\zeta}{\zeta} \\ & + \frac{1}{\pi} \int\limits_{|\zeta|<1} f(\zeta) \frac{|\zeta|^2 - |z|^2}{\zeta(\zeta - z)} \ d\xi d\eta \, . \end{aligned} \tag{21}$$

The proof is as the last one but [3], Theorem 12 is involved rather than [3], Theorem 11.

Theorem 11 *The Neumann problem for the inhomogeneous Bitsadze equation in the unit disc*

$$w_{\overline{z}\overline{z}} = f \text{ in } \mathbb{D} \ , \ \partial_\nu w = \gamma_0 \ , \ \partial_\nu w_{\overline{z}} = \gamma_1 \text{ on } \partial\mathbb{D} \ , \ w(0) = c_0 \ , \ w_{\overline{z}}(0) = c_1$$

is uniquely solvable for $f \in C^\alpha(\mathbb{D};\mathbb{C}), 0 < \alpha < 1, \gamma_0, \gamma_1 \in C(\partial\mathbb{D};\mathbb{C}), c_0, c_1 \in \mathbb{C}$ *if and only if for* $z \in \partial\mathbb{D}$

$$c_1\overline{z} + \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma_0(\zeta)\frac{d\overline{\zeta}}{\overline{\zeta - z}} - \frac{1}{2\pi i}\int\limits_{|\zeta|=1}(\gamma_1(\zeta) - \overline{\zeta}f(\zeta))(1 - \overline{z}\zeta\log(1 - z\overline{\zeta}))d\overline{\zeta}$$
$$+\frac{1}{\pi}\int\limits_{|\zeta|<1}\frac{f(\zeta)}{\zeta}\Big(\frac{\overline{z}\zeta(\overline{\zeta - z})}{(1 - \overline{z}\zeta)^2} - \frac{1}{\zeta - z}\Big)d\xi d\eta = 0 \tag{22}$$

and

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1}(\gamma_1(\zeta) - \overline{\zeta}f(\zeta))\frac{d\overline{\zeta}}{\overline{\zeta - z}} - \frac{\overline{z}}{\pi}\int\limits_{|\zeta|<1}f(\zeta)\frac{d\xi d\eta}{(1 - \overline{z}\zeta)^2} = 0 \ . \tag{23}$$

The solution then is given as

$$w(z) = c_0 + c_1\overline{z} - \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma_0(\zeta)\log(1 - z\overline{\zeta})\frac{d\zeta}{\zeta}$$
$$+\frac{1}{2\pi i}\int\limits_{|\zeta|=1}(\gamma_1(\zeta) - \overline{\zeta}f(\zeta))(\overline{\zeta - z})\log(1 - z\overline{\zeta})\frac{d\overline{\zeta}}{\zeta} + \frac{z}{\pi}\int\limits_{|\zeta|<1}\frac{f(\zeta)}{\zeta}\frac{\overline{\zeta - z}}{\zeta - z}\,d\xi d\eta. \tag{24}$$

Proof From applying [3], Theorem 11

$$w(z) = c_0 - \frac{1}{2\pi i}\int\limits_{|\zeta|=1}(\gamma_0(\zeta) - \overline{\zeta}\omega(\zeta))\log(1 - z\overline{\zeta})\frac{d\zeta}{\zeta} - \frac{z}{\pi}\int\limits_{|\zeta|<1}\omega(\zeta)\frac{d\xi d\eta}{\zeta(\zeta - z)} \ ,$$

$$\omega(z) = c_1 - \frac{1}{2\pi i}\int\limits_{|\zeta|=1}(\gamma_1(\zeta) - \overline{\zeta}f(\zeta))\log(1 - z\overline{\zeta})\frac{d\zeta}{\zeta} - \frac{z}{\pi}\int\limits_{|\zeta|<1}f(\zeta)\frac{d\xi d\eta}{\zeta(\zeta - z)} \ ,$$

if and only if

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1}(\gamma_0(\zeta) - \overline{\zeta}\omega(\zeta))\frac{d\zeta}{\zeta(1 - \overline{z}\zeta)} + \frac{\overline{z}}{\pi}\int\limits_{|\zeta|<1}\frac{\omega(\zeta)}{(1 - \overline{z}\zeta)^2}\,d\xi d\eta = 0 \ ,$$

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1}(\gamma_1(\zeta)-\overline{\zeta}f(\zeta))\frac{d\zeta}{\zeta(1-\overline{z}\zeta)}+\frac{\overline{z}}{\pi}\int\limits_{|\zeta|<1}\frac{f(\zeta)}{(1-\overline{z}\zeta)^2}\,d\xi d\eta=0\;.$$

Inserting ω into the first condition leads to (22) on the basis of

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\omega(\zeta)\frac{d\zeta}{\zeta^2(1-\overline{z}\zeta)}$$
$$=c_1\overline{z}-\frac{1}{2\pi i}\int\limits_{|\tilde\zeta|=1}(\gamma_1(\tilde\zeta)-\overline{\tilde\zeta}f(\tilde\zeta))\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{\log(1-\zeta\overline{\tilde\zeta})}{1-\overline{z}\zeta}\frac{d\zeta}{\zeta^2}\frac{d\tilde\zeta}{\tilde\zeta}$$
$$-\frac{1}{\pi}\int\limits_{|\tilde\zeta|<1}\frac{f(\tilde\zeta)}{\tilde\zeta}\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{d\zeta}{(\tilde\zeta-\zeta)\zeta(1-\overline{z}\zeta)}\,d\tilde\xi d\tilde\eta$$
$$=c_1\overline{z}+\frac{1}{2\pi i}\int\limits_{|\zeta|=1}(\gamma_1(\zeta)-\overline{\zeta}f(\zeta))\overline{\zeta}\frac{d\zeta}{\zeta}+\frac{\overline{z}}{\pi}\int\limits_{|\zeta|<1}\frac{f(\zeta)}{\zeta(1-\overline{z}\zeta)}\,d\xi d\eta$$

and

$$\frac{\overline{z}}{\pi}\int\limits_{|\zeta|<1}\frac{\omega(\zeta)}{(1-\overline{z}\zeta)^2}\,d\xi d\eta=\frac{\overline{z}}{\pi}\int\limits_{|\zeta|<1}\partial_{\overline{\zeta}}\frac{(\overline{\zeta-z})\omega(\zeta)}{(1-\overline{z}\zeta)^2}\,d\xi d\eta-\frac{\overline{z}}{\pi}\int\limits_{|\zeta|<1}\frac{\overline{\zeta-z}}{(1-\overline{z}\zeta)^2}\,f(\zeta)d\xi d\eta$$

$$=\frac{\overline{z}}{2\pi i}\int\limits_{|\zeta|=1}\frac{\overline{\zeta-z}}{(1-\overline{z}\zeta)^2}\,\omega(\zeta)d\zeta-\frac{\overline{z}}{\pi}\int\limits_{|\zeta|<1}\frac{\overline{\zeta-z}}{(1-\overline{z}\zeta)^2}\,f(\zeta)d\xi d\eta,$$

where for $|z|=1$

$$\frac{\overline{z}}{2\pi i}\int\limits_{|\zeta|=1}\frac{\overline{\zeta-z}}{(1-\overline{z}\zeta)^2}\,\omega(\zeta)d\zeta=\frac{z}{2\pi i}\int\limits_{|\zeta|=1}\frac{\overline{\zeta-z}}{(\zeta-z)^2}\,\omega(\zeta)d\zeta$$
$$=-\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{\omega(\zeta)}{\zeta-z}\frac{d\zeta}{\zeta}$$
$$=\frac{1}{2\pi i}\int\limits_{|\tilde\zeta|=1}(\gamma_1(\tilde\zeta)-\overline{\tilde\zeta}f(\tilde\zeta))\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{\log(1-\zeta\overline{\tilde\zeta})}{\zeta(\zeta-z)}\,d\zeta\frac{d\tilde\zeta}{\tilde\zeta}$$
$$-\frac{1}{\pi}\int\limits_{|\tilde\zeta|<1}\frac{f(\tilde\zeta)}{\tilde\zeta}\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{d\zeta}{(\tilde\zeta-\zeta)(\zeta-z)}\,d\tilde\xi d\tilde\eta$$

$$= \frac{1}{2\pi i} \int\limits_{|\zeta|=1} (\gamma_1(\zeta) - \overline{\zeta} f(\zeta)) \frac{1}{z} \; \log(1 - z\overline{\zeta}) \frac{d\zeta}{\zeta}$$
$$= \frac{1}{2\pi i} \int\limits_{|\zeta|=1} (\gamma_1(\zeta) - \overline{\zeta} f(\zeta)) \overline{z} \log(1 - z\overline{\zeta}) \frac{d\zeta}{\zeta} \; .$$

From $\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \log(1 - z\overline{\zeta}) d\overline{\zeta} = -\overline{\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \log(1 - \overline{z}\zeta) d\zeta} = 0$,

$$\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \log(1 - \zeta\overline{\tilde{\zeta}}) \log(1 - z\overline{\zeta}) d\overline{\zeta} = -\overline{\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \log(1 - \overline{\zeta}\tilde{\zeta}) \log(1 - \overline{z}\zeta) d\zeta}$$
$$= \sum_{k=1}^{+\infty} \frac{\overline{\tilde{\zeta}}^k}{k} \overline{\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \log(1 - \overline{z}\zeta) \frac{d\zeta}{\zeta^2}} = \sum_{k=2}^{+\infty} \frac{\overline{\tilde{\zeta}}^k}{k!} \overline{\partial_\zeta^{k-1} \log(1 - \overline{z}\zeta)} \, |_{\zeta=0}$$
$$= -\sum_{k=2}^{+\infty} \frac{\overline{\tilde{\zeta}}^k z^{k-1}}{(k-1)k}$$
$$= \overline{\tilde{\zeta}} \log(1 - z\overline{\tilde{\zeta}}) - \frac{1}{z} \; (\log(1 - z\overline{\tilde{\zeta}}) + z\overline{\tilde{\zeta}}) = -\frac{1 - z\overline{\tilde{\zeta}}}{z} \; \log(1 - z\overline{\tilde{\zeta}}) - \overline{\tilde{\zeta}} \; ,$$

and

$$\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \frac{\zeta \log(1 - z\overline{\zeta})}{\tilde{\zeta} - \zeta} \; d\overline{\zeta} \quad = \quad -\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \frac{\log(1 - z\overline{\zeta})}{1 - \tilde{\zeta}\overline{\zeta}} \; d\overline{\zeta}$$
$$= \quad \overline{\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \frac{\log 1 - \overline{z}\zeta}{1 - \overline{\tilde{\zeta}}\zeta} \; d\zeta} = 0$$

the relation

$$\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \omega(\zeta) \log(1 - z\overline{\zeta}) d\overline{\zeta}$$
$$= \frac{1}{2\pi i} \int\limits_{|\zeta|=1} (\gamma_1(\zeta) - \overline{\zeta} f(\zeta)) \Big(\frac{1 - z\overline{\zeta}}{z} \; \log(1 - z\overline{\zeta}) + \overline{\zeta} \Big) \frac{d\zeta}{\zeta} \qquad (25)$$

follows. Similarly from

$$\frac{z}{\pi}\int\limits_{|\zeta|<1}\frac{d\xi d\eta}{\zeta(\zeta-z)}=\frac{1}{\pi}\int\limits_{|\zeta|<1}\frac{d\xi d\eta}{\zeta-z}-\frac{1}{\pi}\int\limits_{|\zeta|<1}\frac{d\xi d\eta}{\zeta}=-\overline{z}\ ,$$

$$\frac{1}{\pi}\int\limits_{|\zeta|<1}\frac{\log(1-\zeta\overline{\tilde{\zeta}})}{\zeta-z}\,d\xi d\eta$$

$$=(\overline{\tilde{\zeta}-z})\log(1-z\overline{\tilde{\zeta}})-\frac{1}{2\pi i}\int\limits_{|\zeta|=1}(\overline{\tilde{\zeta}-\zeta})\log(1-\zeta\overline{\tilde{\zeta}})\frac{d\zeta}{\zeta-z}$$

$$=(\overline{\tilde{\zeta}-z})\log(1-z\overline{\tilde{\zeta}})+\frac{1}{2\pi i}\int\limits_{|\zeta|=1}(1-\zeta\overline{\tilde{\zeta}})\log(1-\zeta\overline{\tilde{\zeta}})\frac{d\zeta}{\zeta(\zeta-z)}$$

$$=(\overline{\tilde{\zeta}-z})\log(1-z\overline{\tilde{\zeta}})+\frac{1-z\overline{\tilde{\zeta}}}{z}\ \log(1-z\overline{\tilde{\zeta}})=\frac{1-|z|^2}{z}\ \log(1-z\overline{\tilde{\zeta}})\ ,$$

$$\frac{1}{\pi}\int\limits_{|\zeta|<1}\log(1-\zeta\overline{\tilde{\zeta}})\frac{d\xi d\eta}{\zeta}=\frac{1}{\pi}\int\limits_{|\zeta|<1}\partial_{\overline{\zeta}}\frac{\overline{\zeta}}{\zeta}\ \log(1-\zeta\overline{\tilde{\zeta}})d\xi d\eta$$

$$=\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\log(1-\zeta\overline{\tilde{\zeta}})\frac{d\zeta}{\zeta^2}=-\overline{\tilde{\zeta}}\ ,$$

and

$$\frac{1}{\pi}\int\limits_{|\zeta|<1}\frac{d\xi d\eta}{(\zeta-\tilde{\zeta})(\zeta-z)}=\frac{1}{\pi(\tilde{\zeta}-z)}\int\limits_{|\zeta|<1}\Big(\frac{1}{\zeta-\tilde{\zeta}}-\frac{1}{\zeta-z}\Big)d\xi d\eta=-\frac{\overline{\tilde{\zeta}-z}}{\tilde{\zeta}-z}$$

it follows

$$\frac{z}{\pi}\int\limits_{|\zeta|<1}\omega(\zeta)\frac{d\xi d\eta}{\zeta(\zeta-z)}$$

$$=-c_1\overline{z}+\frac{1}{2\pi i}\int\limits_{|\zeta|=1}(\gamma_1(\zeta)-\overline{\zeta}f(\zeta))\Big(\frac{1-|z|^2}{z}\ \log(1-z\overline{\zeta})+\overline{\zeta}\Big)\frac{d\zeta}{\zeta}$$

$$-\frac{z}{\pi}\int\limits_{|\zeta|<1}\frac{f(\zeta)}{\zeta}\frac{\overline{\zeta-z}}{\zeta-z}\ d\xi d\eta\ . \tag{26}$$

From (25) and (26) the representation (24) follows.

Theorem 12 *The boundary value problem for the inhomogeneous Bitsadze equation in the unit disc*

$$w_{\overline{z}\overline{z}} = f \textit{ in } \mathbb{D} \ , \ zw_z = \gamma_0 \ , \ zw_{z\overline{z}} = \gamma_1 \textit{ on } \partial\mathbb{D} \ , \ w(0) = c_0 \ , \ w_{\overline{z}}(0) = c_1 \ ,$$

for $f \in L_1(\mathbb{D};\mathbb{C}), \gamma_0, \gamma_1 \in C(\partial\mathbb{D};\mathbb{C}), c_0, c_1 \in \mathbb{C}$ *is uniquely solvable if and only if for* $|z| = 1$

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma_0(\zeta)\frac{d\zeta}{(1-z\overline{\zeta})\zeta} + \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma_1(\zeta)\frac{1}{z}\log(1-z\overline{\zeta})\frac{d\zeta}{\zeta}$$
$$= \frac{\overline{z}}{\pi}\int\limits_{|\zeta|<1} f(\zeta)\frac{\overline{\zeta - z}}{(1-\overline{z}\zeta)^2}d\xi d\eta \ , \tag{27}$$

and

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma_1(\zeta)\frac{d\zeta}{(1-\overline{z}\zeta)\zeta} + \frac{\overline{z}}{\pi}\int\limits_{|\zeta|<1} f(\zeta)\frac{d\xi d\eta}{(1-\overline{z}\zeta)^2} = 0 \ . \tag{28}$$

The solution then is

$$\begin{aligned} w(z) &= c_0 + c_1\overline{z} - \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma_0(\zeta)\log(1-z\overline{\zeta})\frac{d\zeta}{\zeta} \\ &+ \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma_1(\zeta)\Big(\frac{1-|z|^2}{z}\ \log(1-z\overline{\zeta}) + \overline{\zeta}\Big)\frac{d\zeta}{\zeta} \\ &+ \frac{z}{\pi}\int\limits_{|\zeta|<1} \frac{f(\zeta)}{\zeta}\frac{\overline{\zeta - z}}{\zeta - z}\ d\xi d\eta \ . \end{aligned} \tag{29}$$

Proof The system

$$w_{\overline{z}} = \omega \text{ in } \mathbb{D} \ , \ zw_z = \gamma_0 \text{ on } \partial\mathbb{D} \ , \ w(0) = c_0 \ ,$$

$$\omega_{\overline{z}} = f \text{ in } \mathbb{D} \ , \ z\omega_z = \gamma_1 \text{ on } \partial\mathbb{D} \ , \ \omega(0) = c_1 \ ,$$

is uniquely solvable if and only if

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma_0(\zeta)\frac{d\zeta}{(1-\overline{z}\zeta)\zeta} + \frac{\overline{z}}{\pi}\int\limits_{|\zeta|<1} \omega(\zeta)\frac{d\xi d\eta}{(1-\overline{z}\zeta)^2} = 0$$

and

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma_1(\zeta)\frac{d\zeta}{(1-\overline{z}\zeta)\zeta} + \frac{\overline{z}}{\pi}\int\limits_{|\zeta|=1} \omega(\zeta)\frac{d\xi d\eta}{(1-\overline{z}\zeta)^2} = 0 \ .$$

The solution then is

$$w(z) = c_0 - \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma_0(\zeta)\log(1 - z\overline{\zeta})\frac{d\zeta}{\zeta} - \frac{z}{\pi}\int\limits_{|\zeta|<1} \omega(\zeta)\frac{d\xi d\eta}{\zeta(\zeta - z)} \ ,$$

$$\omega(z) = c_1 - \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma_1(\zeta)\log(1 - z\overline{\zeta})\frac{d\zeta}{\zeta} - \frac{z}{\pi}\int\limits_{|\zeta|<1} f(\zeta)\frac{d\xi d\eta}{\zeta(\zeta - z)} \ .$$

Inserting the expression for ω into the first condition gives (27) because as in the preceding proof on $|z| = 1$

$$\begin{aligned}&\frac{\overline{z}}{\pi}\int\limits_{|\zeta|<1} \frac{\omega(\zeta)}{(1-\overline{z}\zeta)^2}\, d\xi d\eta \\ &= \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma_1(\zeta)\frac{1}{z}\ \log(1 - z\overline{\zeta})\frac{d\zeta}{\zeta} - \frac{\overline{z}}{\pi}\int \frac{\overline{\zeta - z}}{(1-\overline{z}\zeta)^2}\, f(\zeta)d\xi d\eta \ .\end{aligned}$$

Also from

$$\begin{aligned}\frac{z}{\pi}\int\limits_{|\zeta|<1} \omega(\zeta)\frac{d\xi d\eta}{\zeta(\zeta - z)} \quad = \quad & -c_1\overline{z} \\ & + \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma_1(\zeta)\Big(\frac{1-|z|^2}{z}\ \log(1 - z\overline{\zeta}) + \overline{\zeta}\Big)\frac{d\zeta}{\zeta} \\ & - \frac{z}{\pi}\int\limits_{|\zeta|<1} \frac{f(\zeta)}{\zeta}\frac{\overline{\zeta - z}}{\zeta - z}\, d\xi d\eta\end{aligned}$$

formula (29) follows.

Boundary value problems as in Theorem 12 can also be solved for the Poisson equation. One case is considered in the next theorem.

Theorem 13 *The boundary value problem for the Poisson equation in the unit disc*

$$w_{z\overline{z}} = f \ in\ \mathbb{D}\ ,\ \overline{z}w_{\overline{z}} = \gamma_0\ ,\ zw_{zz} = \gamma_1\ on\ \partial\mathbb{D}\ ,\ w(0) = c_0\ ,\ w_z(0) = c_1$$

are uniquely solvable for $f \in L_1(\mathbb{D}; \mathbb{C}), \gamma_0, \gamma_1 \in C(\partial\mathbb{D}; \mathbb{C}), c_0, c_1 \in \mathbb{C}$ *if and only if*

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma_0(\zeta)\frac{d\zeta}{(1 - z\overline{\zeta})\zeta} = \frac{1}{\pi}\int\limits_{|\zeta|<1} f(\zeta)\Big(\frac{1}{1 - z\overline{\zeta}} + \frac{1}{1 - \overline{z}\zeta} - 1\Big)d\xi d\eta \qquad (30)$$

and

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma_1(\zeta)\frac{d\zeta}{(1-\overline{z}\zeta)\zeta}+\frac{\overline{z}}{\pi}\int\limits_{|\zeta|<1}f(\zeta)\frac{d\xi d\eta}{(1-\overline{z}\zeta)^2}=0\ . \tag{31}$$

The solution then is

$$\begin{aligned} w(z) \quad = \quad & c_0+c_1z-\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma_0(\zeta)\log(1-\overline{z}\zeta)\frac{d\zeta}{\zeta} \\ & +\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma_1(\zeta)\Big(\frac{1-z\overline{\zeta}}{\overline{\zeta}}\ \log(1-z\overline{\zeta})+z\Big)\frac{d\zeta}{\zeta} \\ & +\frac{1}{\pi}\int\limits_{|\zeta|<1}f(\zeta)\Big(\frac{z}{\zeta}+\log|\zeta-z|^2-\log(1-\overline{z}\zeta)-\log|\zeta|^2\Big)d\xi d\eta\ . \end{aligned} \tag{32}$$

Proof The problem is equivalent to the system

$$w_z=\omega \text{ in } \mathbb{D}\ ,\ \overline{z}w_{\overline{z}}=\gamma_0 \text{ on } \partial\mathbb{D}\ ,\ w(0)=c_0\ ,$$

$$\omega_{\overline{z}}=f \text{ in } \mathbb{D}\ ,\ z\omega=\gamma_1 \text{ on } \partial\mathbb{D}\ ,\ \omega(0)=c_1\ .$$

It is solvable if and only if

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma_0(\zeta)\frac{d\zeta}{(1-z\overline{\zeta})\zeta}+\frac{z}{\pi}\int\limits_{|\zeta|<1}\omega(\zeta)\frac{d\xi d\eta}{(1-z\overline{\zeta})^2}=0$$

and

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma_1(\zeta)\frac{d\zeta}{(1-\overline{z}\zeta)\zeta}+\frac{\overline{z}}{\pi}\int\limits_{|\zeta|<1}f(\zeta)\frac{d\zeta d\eta}{(1-\overline{z}\zeta)^2}=0$$

and the solutions are according to [3], Theorem 12

$$w(z)=c_0-\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma_0(\zeta)\log(1-\overline{z}\zeta)\frac{d\zeta}{\zeta}-\frac{\overline{z}}{\pi}\int\limits_{|\zeta|<1}\omega(\zeta)\frac{d\xi d\eta}{\overline{\zeta(\zeta-z)}}\ ,$$

$$\omega(z)=c_1-\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma_1(\zeta)\log(1-z\overline{\zeta})\frac{d\zeta}{\zeta}-\frac{z}{\pi}\int\limits_{|\zeta|<1}f(\zeta)\frac{d\xi d\eta}{\zeta(\zeta-z)}\ .$$

For the first problem [3], Theorem 12 is applied to $\overline{w}$ and the formulas then complex conjugated. For (30)

$$\frac{z}{\pi}\int\limits_{|\zeta|<1}\omega(\zeta)\frac{d\xi d\eta}{(1-z\overline{\zeta})^2} = \frac{1}{\pi}\int\limits_{|\zeta|<1}\Big[\partial_{\overline{\zeta}}\Big[\frac{\omega(\zeta)}{1-z\overline{\zeta}}\Big) - \frac{f(\zeta)}{1-z\overline{\zeta}}\Big]d\xi d\eta$$
$$= \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{\omega(\zeta)}{1-z\overline{\zeta}}\,d\zeta - \frac{1}{\pi}\int\limits_{|\zeta|<1}\frac{f(\zeta)}{1-z\overline{\zeta}}\,d\xi d\eta$$

has to be evaluated. For $|z|=1$

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{\omega(\zeta)}{1-z\overline{\zeta}}\,d\zeta = -\frac{\overline{z}}{2\pi i}\int\limits_{|\zeta|=1}\frac{\zeta\omega(\zeta)}{1-\overline{z}\zeta}\,d\zeta$$
$$= \frac{1}{2\pi i}\int\limits_{|\tilde{\zeta}|=1}\gamma_1(\tilde{\zeta})\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\log(1-\zeta\overline{\tilde{\zeta}})\frac{\overline{z}\zeta}{1-\overline{z}\zeta}\,d\zeta\frac{d\tilde{\zeta}}{\tilde{\zeta}}$$
$$+\frac{1}{\pi}\int\limits_{|\tilde{\zeta}|<1}\frac{f(\tilde{\zeta})}{\tilde{\zeta}}\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{\zeta}{\tilde{\zeta}-\zeta}\frac{\overline{z}\zeta}{1-\overline{z}\zeta}\,d\zeta d\tilde{\xi}d\tilde{\eta}$$
$$= -\frac{1}{\pi}\int\limits_{|\zeta|<1}f(\zeta)\frac{\overline{z}\zeta}{1-\overline{z}\zeta}\,d\xi d\eta$$

so that

$$\frac{z}{\pi}\int\limits_{|\zeta|<1}\omega(\zeta)\frac{d\xi d\eta}{(1-z\overline{\zeta})^2} = -\frac{1}{\pi}\int\limits_{|\zeta|<1}f(\zeta)\Big(\frac{\overline{z}\zeta}{1-\overline{z}\zeta}+\frac{1}{1-z\overline{\zeta}}\Big)d\xi d\eta\,.$$

For (32)

$$-\frac{1}{\pi}\int\limits_{|\zeta|<1}\omega(\zeta)\frac{d\xi d\eta}{\overline{\zeta-z}} = c_1 z$$
$$+\frac{1}{2\pi i}\int\limits_{|\tilde{\zeta}|=1}\frac{\gamma_1(\tilde{\zeta})}{\tilde{\zeta}}\frac{1}{\pi}\int\limits_{|\zeta|<1}\log(1-\zeta\overline{\tilde{\zeta}})\frac{d\xi d\eta}{\overline{\zeta-z}}\,d\tilde{\zeta}$$
$$+\frac{1}{\pi}\int\limits_{|\tilde{\zeta}|<1}\frac{f(\tilde{\zeta})}{\tilde{\zeta}}\frac{1}{\pi}\int\limits_{|\zeta|<1}\frac{\zeta}{\tilde{\zeta}-\zeta}\frac{d\xi d\eta}{\overline{\zeta-z}}\,d\tilde{\xi}d\tilde{\eta}$$

needs some modification. From

$$\begin{aligned}
\frac{1-z\overline{\tilde{\zeta}}}{\overline{\tilde{\zeta}}}\,(\log(1-z\overline{\tilde{\zeta}})-1) &= -\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{1-\zeta\overline{\tilde{\zeta}}}{\overline{\tilde{\zeta}}}\,(\log(1-\zeta\overline{\tilde{\zeta}})-1)\frac{d\overline{\zeta}}{\overline{\zeta-z}} \\
&\quad +\frac{1}{\pi}\int\limits_{|\zeta|<1}\log(1-\zeta\overline{\tilde{\zeta}})\frac{d\xi d\eta}{\overline{\zeta-z}} \\
&= \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{1-\zeta\overline{\tilde{\zeta}}}{\overline{\tilde{\zeta}}}\,(\log(1-\zeta\overline{\tilde{\zeta}})-1)\frac{d\zeta}{\zeta(1-\overline{z}\zeta)} \\
&\quad +\frac{1}{\pi}\int\limits_{|\zeta|<1}\log(1-\zeta\overline{\tilde{\zeta}})\frac{d\xi d\eta}{\overline{\zeta-z}} \\
&= -\frac{1}{\overline{\tilde{\zeta}}}+\frac{1}{\pi}\int\limits_{|\zeta|<1}\log(1-\zeta\overline{\tilde{\zeta}})\frac{d\xi d\eta}{\overline{\zeta-z}}
\end{aligned}$$

from which also

$$\frac{1}{\pi}\int\limits_{|\zeta|<1}\log(1-\zeta\overline{\tilde{\zeta}})\frac{d\xi d\eta}{\overline{\zeta}}=0$$

follows, and

$$\begin{aligned}
-\frac{1}{\pi}\int\limits_{|\zeta|<1}\frac{1}{\zeta-\tilde{\zeta}}\frac{d\xi d\eta}{\overline{\zeta-z}} &= \log|\tilde{\zeta}-z|^2-\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\log|\zeta-z|^2\,\frac{d\zeta}{\zeta-\tilde{\zeta}} \\
&= \log|\tilde{\zeta}-z|^2+\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\log(1-z\overline{\zeta})\frac{d\overline{\zeta}}{\overline{\zeta}(1-\tilde{\zeta}\overline{\zeta})} \\
&\quad -\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\log(1-\overline{z}\zeta)\frac{d\zeta}{\zeta-\tilde{\zeta}} \\
&= -\log|\tilde{\zeta}-z|^2-\log(1-\overline{z}\tilde{\zeta})
\end{aligned}$$

from what

$$\begin{aligned}
&-\frac{1}{\pi}\int\limits_{|\zeta|<1}\frac{\zeta}{\zeta-\tilde{\zeta}}\frac{d\xi d\eta}{\overline{\zeta-z}}=-\frac{1}{\pi}\int\limits_{|\zeta|<1}\Big(1+\frac{\tilde{\zeta}}{\zeta-\tilde{\zeta}}\Big)\frac{d\xi d\eta}{\overline{\zeta-z}} \\
&= z+\tilde{\zeta}(\log|\tilde{\zeta}-z|^2-\log(1-\overline{z}\tilde{\zeta}))
\end{aligned}$$

and

$$-\frac{1}{\pi}\int\limits_{|\zeta|<1}\frac{\zeta}{\zeta-\tilde{\zeta}}\frac{d\xi d\eta}{\overline{\zeta}}=\tilde{\zeta}\log|\tilde{\zeta}|^2$$

is seen,

$$-\frac{\overline{z}}{\pi}\int\limits_{|\tilde{\zeta}|<1}\omega(\zeta)\frac{d\xi d\eta}{\overline{\zeta}(\overline{\zeta-z})}=c_1 z+\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma_1(\zeta)\Big(\frac{1-z\overline{\zeta}}{\overline{\zeta}}\ \log(1-z\overline{\zeta})+z\Big)\frac{d\zeta}{\zeta}$$

$$\frac{1}{\pi}\int\limits_{|\zeta|<1}f(\zeta)\Big(\frac{z}{\zeta}+\log|\zeta-z|^2-\log(1-\overline{z}\zeta)-\log|\zeta|^2\Big)d\xi d\eta$$

follows.

Remark Instead of this constructive way the proof can be given by verification. From (32)

$$\overline{z}w_{\overline{z}}(z)=\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma_0(\zeta)\frac{\overline{z}\zeta}{1-\overline{z}\zeta}\frac{d\zeta}{\zeta}-\frac{\overline{z}}{\pi}\int\limits_{|\zeta|<1}f(\zeta)\Big(\frac{1}{\overline{\zeta-z}}-\frac{\zeta}{1-\overline{z}\zeta}\Big)d\xi d\eta\ .$$

This obviously coincides with γ_0 on $\partial\mathbb{D}$ if and only if (31) is satisfied. Similarly from

$$w_z(z)=c_1-\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma_1(\zeta)\log(1-z\overline{\zeta})\frac{d\zeta}{\zeta}-\frac{1}{\pi}\int\limits_{|\zeta|<1}f(\zeta)\Big(\frac{1}{\zeta-z}-\frac{1}{\zeta}\Big)d\xi d\eta\ ,$$

$$zw_{zz}(z)=\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma_1(\zeta)\frac{z\overline{\zeta}}{1-z\overline{\zeta}}\frac{d\zeta}{\zeta}-\frac{1}{\pi}\int\limits_{|\zeta|<1}f(\zeta)\frac{z}{(\zeta-z)^2}\ d\xi d\eta$$

it is seen that zw_{zz} coinsides with γ_1 on $\partial\mathbb{D}$ if and only if (31) holds. The other two conditions are obviously satisfied.

2 The inhomogeneous polyanalytic equation

As second order equations of special type were treated in the preceding section model equations of third, forth, fifth etc. order can be investigated. From the material presented it is clear how to proceed and what kind of boundary conditions can be posed. However, there is a variety of boundary conditions possible. All kind of combinations of the three kinds, Schwarz, Dirichlet, Neumann conditions can be posed. And there are even others e.g. boundary conditions of mixed type which are not investigated here.

As simple examples the Schwarz problem will be studied for the inhomogeneous polyanalytic equation. Another possibility is the Neumann problem for the inhomogeneous polyharmonic equation, see [4, 5], and the Dirichlet problem, see [2].

Lemma 3 *For* $|z|<1, |\tilde{\zeta}|<1$ *and* $k \in \mathbb{N}_0$

$$\frac{1}{k+1}\,(\tilde{\zeta}-z+\overline{\tilde{\zeta}-z})^{k+1} = \frac{(-1)^{k+1}}{k+1}\,(z+\overline{z})^{k+1}$$
$$-\frac{1}{2\pi}\int\limits_{|\zeta|<1}\Big(\frac{1}{\zeta}\,\frac{\zeta+\tilde{\zeta}}{\zeta-\tilde{\zeta}}-\frac{1}{\overline{\overline{\zeta}}}\,\frac{1+\tilde{\zeta}\overline{\zeta}}{1-\tilde{\zeta}\overline{\zeta}}\Big)(\zeta-z+\overline{\zeta-z})^k d\xi d\eta\;. \tag{33}$$

Proof The function $w(\tilde{\zeta}) = i(\tilde{\zeta}-z+\overline{\tilde{\zeta}-z})^{k+1}/(k+1)$ satisfies the Schwarz condition

$$w_{\overline{\tilde{\zeta}}}(\tilde{\zeta}) = i(\tilde{\zeta}-z+\overline{\tilde{\zeta}-z})^k \text{ in } \mathbb{D}\;,\ \mathrm{Re}\, w(\tilde{\zeta}) = 0 \text{ on } \partial\mathbb{D}\;,\ \mathrm{Im}\, w(0) = \frac{(-1)^{k+1}}{k+1}\,(z+\overline{z})^{k+1}\;,$$

so that according to [3], (33)

$$w(\tilde{\zeta}) = i\frac{(-1)^{k+1}}{k+1}\,(z+\overline{z})^{k+1} - \frac{i}{2\pi}\int\limits_{|\zeta|<1}\Big(\frac{1}{\zeta}\,\frac{\zeta+\tilde{\zeta}}{\zeta-\tilde{\zeta}}-\frac{1}{\overline{\overline{\zeta}}}\,\frac{1+\tilde{\zeta}\overline{\zeta}}{1-\tilde{\zeta}\overline{\zeta}}\Big)(\zeta-z+\overline{\zeta-z})^k d\xi d\eta\;.$$

This is (33).

Corollary 1 *For* $|z|<1$ *and* $k \in \mathbb{N}_0$

$$\frac{1}{2\pi}\int\limits_{|\zeta|<1}\Big(\frac{1}{\zeta}-\frac{1}{\overline{\overline{\zeta}}}\Big)(\zeta-z+\overline{\zeta-z})^k d\xi d\eta = 0 \tag{34}$$

and

$$\frac{1}{2\pi}\int\limits_{|\zeta|<1}\Big(\frac{1}{\zeta}\,\frac{\zeta+z}{\zeta-z}-\frac{1}{\overline{\overline{\zeta}}}\,\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}\Big)(\zeta-z+\overline{\zeta-z})^k d\xi d\eta = \frac{(-1)^{k+1}}{k+1}\,(z+\overline{z})^{k+1}\;. \tag{35}$$

Proof (34) and (35) are particular cases of (33) for $\tilde{\zeta}=0$ and $\tilde{\zeta}=z$, respectively.

Theorem 14 *The Schwarz problem for the inhomogeneous polyanalytic equation in the unit disc*

$$\partial_{\overline{z}}^n w = f \textit{ in } \mathbb{D}\;,\ \mathrm{Re}\,\partial_{\overline{z}}^\nu w = \gamma_\nu \textit{ on } \partial\mathbb{D}\;,\ \mathrm{Im}\,\partial_{\overline{z}}^\nu w(0) = 0\;,\ 0 \le \nu \le n-1\;,$$

is uniquely solvable for $f \in L_1(\mathbb{D};\mathbb{C}), \gamma_\nu \in C(\partial\mathbb{D};\mathbb{R}), c_\nu \in \mathbb{R}, 0 \le \nu \le n-1$.

The solution is

$$w(z) = i\sum_{\nu=0}^{n-1} \frac{c_\nu}{\nu!}\ (z+\overline{z})^\nu + \sum_{\nu=0}^{n-1} \frac{(-1)^\nu}{2\pi i\nu!} \int\limits_{|\zeta|=1} \gamma_\nu(\zeta)\frac{\zeta+z}{\zeta-z}\ (\zeta - z + \overline{\zeta - z})^\nu \frac{d\zeta}{\zeta}$$

$$+\frac{(-1)^n}{2\pi(n-1)!} \int\limits_{|\zeta|<1} \Big(\frac{f(\zeta)}{\zeta}\frac{\zeta+z}{\zeta-z} + \frac{\overline{f(\zeta)}}{\overline{\zeta}}\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}\Big)(\zeta - z + \overline{\zeta-z})^{n-1} d\xi d\eta\ . \quad (36)$$

Proof For $n = 1$ formula (36) is just [3], (33). Assuming it holds for $n-1$ rather than for n the Schwarz problem is rewritten as the system

$$\partial_{\overline{z}}^{n-1} w = \omega \text{ in } \mathbb{D}\ ,\ \text{Re}\ \partial_{\overline{z}}^{\nu} w = \gamma_\nu \text{ on } \partial\mathbb{D}\ ,\ \text{Im}\ \partial_{\overline{z}}^{\nu} w(0) = c_\nu\ ,\ 0 \le \nu \le n-2\ ,$$

$$\omega_{\overline{z}} = f \text{ in } \mathbb{D}\ ,\ \text{Re}\ \omega = \gamma_{n-1} \text{ on } \partial\mathbb{D}\ ,\ \text{Im}\ \omega(0) = c_{n-1}\ ,$$

having the solution

$$w(z) = i\sum_{\nu=0}^{n-2} \frac{c_\nu}{\nu!}\ (z+\overline{z})^\nu + \sum_{\nu=0}^{n-2} \frac{(-1)^\nu}{2\pi i\nu!} \int\limits_{|\zeta|=1} \gamma_\nu(\zeta)\frac{\zeta+z}{\zeta-z}\ (z - z + \overline{\zeta - z})^\nu \frac{d\zeta}{\zeta}$$

$$+\frac{(-1)^{n-1}}{2\pi(n-2)!} \int\limits_{|\zeta|<1} \Big(\frac{\omega(\zeta)}{\zeta}\frac{\zeta+z}{\zeta-z} + \frac{\overline{\omega(\zeta)}}{\overline{\zeta}}\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}\Big)(\zeta - z + \overline{\zeta-z})^{n-2} d\xi d\eta\ ,$$

$$\omega(z) = ic_{n-2} + \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma_{n-1}(\zeta)\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta}$$

$$-\frac{1}{2\pi} \int\limits_{|\zeta|<1} \Big(\frac{f(\zeta)}{\zeta}\frac{\zeta+z}{\zeta-z} + \frac{\overline{f(\zeta)}}{\overline{\zeta}}\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}\Big) d\xi d\eta\ .$$

Using (35)

$$\frac{(-1)^{n-1}}{2\pi(n-2)!} \int\limits_{|\zeta|<1} \Big(\frac{\omega(\zeta)}{\zeta}\frac{\zeta+z}{\zeta-z} + \frac{\overline{\omega(\zeta)}}{\overline{\zeta}}\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}\Big)(\zeta - z + \overline{\zeta-z})^{n-2} d\xi d\eta$$

$$= i\frac{c_{n-1}}{(n-1)!}\ (z+\overline{z})^{n-1} + \frac{(-1)^{n-1}}{2\pi i(n-2)!} \int\limits_{|\tilde{\zeta}|=1} \gamma_{n-1}(\tilde{\zeta})$$

$$\times\frac{1}{2\pi} \int\limits_{|\zeta|<1} \Big(\frac{\tilde{\zeta}+\zeta}{\tilde{\zeta}-\zeta}\frac{1}{\zeta}\frac{\zeta+z}{\zeta-z} + \overline{\frac{\tilde{\zeta}+\zeta}{\tilde{\zeta}-\zeta}}\frac{1}{\overline{\zeta}}\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}\Big)(\zeta - z + \overline{\zeta-z})^{n-2} d\xi d\eta \frac{d\tilde{\zeta}}{\tilde{\zeta}}$$

$$+\frac{(-1)^n}{2\pi(n-2)!}\int\limits_{|\tilde{\zeta}|<1}\frac{f(\tilde{\zeta})}{\tilde{\zeta}}$$

$$\times\frac{1}{2\pi}\int\limits_{|\zeta|<1}\Big(\frac{\tilde{\zeta}+\zeta}{\tilde{\zeta}-\zeta}\frac{1}{\zeta}\frac{\zeta+z}{\zeta-z}+\frac{1+\overline{\zeta}\tilde{\zeta}}{1-\overline{\zeta}\tilde{\zeta}}\frac{1}{\overline{\zeta}}\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}\Big)(\zeta-z+\overline{\zeta-z})^{n-2}d\xi d\eta d\tilde{\xi}d\tilde{\eta}$$

$$+\frac{(-1)^n}{2\pi(n-2)!}\int\limits_{|\tilde{\zeta}|<1}\frac{\overline{f(\tilde{\zeta})}}{\overline{\tilde{\zeta}}}$$

$$\times\frac{1}{2\pi}\int\limits_{|\zeta|<1}\Big(\frac{1+\zeta\overline{\tilde{\zeta}}}{1-\zeta\overline{\tilde{\zeta}}}\frac{1}{\zeta}\frac{\zeta+z}{\zeta-z}+\frac{\overline{\tilde{\zeta}+\zeta}}{\overline{\tilde{\zeta}-\zeta}}\frac{1}{\overline{\zeta}}\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}\Big)(\zeta-z+\overline{\zeta-z})^{n-2}d\xi d\eta d\tilde{\xi}d\tilde{\eta}$$

follows. Because

$$\begin{aligned}
&\frac{\tilde{\zeta}+\zeta}{\tilde{\zeta}-\zeta}\frac{1}{\zeta}\frac{\zeta+z}{\zeta-z}+\frac{1+\overline{\zeta}\tilde{\zeta}}{1-\overline{\zeta}\tilde{\zeta}}\frac{1}{\overline{\zeta}}\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}\\
&=-\Big(\frac{2\tilde{\zeta}}{\zeta-\tilde{\zeta}}+1\Big)\Big(\frac{2}{\zeta-z}-\frac{1}{\zeta}\Big)+\Big(\frac{2}{1-\overline{\zeta}\tilde{\zeta}}-1\Big)\Big(\frac{2z}{1-z\overline{\zeta}}+\frac{1}{\overline{\zeta}}\Big)\\
&=-\frac{4\tilde{\zeta}}{\tilde{\zeta}-z}\ \Big(\frac{1}{\zeta-\tilde{\zeta}}-\frac{1}{\zeta-z}\Big)+\frac{2}{\zeta-\tilde{\zeta}}-\frac{2}{\zeta}-\frac{2}{\zeta-z}+\frac{1}{\zeta}\\
&+\frac{4z}{\tilde{\zeta}-z}\ \Big(\frac{\tilde{\zeta}}{1-\tilde{\zeta}\overline{\zeta}}-\frac{z}{1-z\overline{\zeta}}\Big)+\frac{2\tilde{\zeta}}{1-\tilde{\zeta}\overline{\zeta}}+\frac{2}{\overline{\zeta}}-\frac{2z}{1-z\overline{\zeta}}-\frac{1}{\overline{\zeta}}\\
&=-2\frac{\tilde{\zeta}+z}{\tilde{\zeta}-z}\ \Big(\frac{1}{\zeta-\tilde{\zeta}}-\frac{1}{\zeta-z}\Big)+2\frac{\tilde{\zeta}+z}{\tilde{\zeta}-z}\ \Big(\frac{\tilde{\zeta}}{1-\tilde{\zeta}\overline{\zeta}}-\frac{z}{1-z\overline{\zeta}}\Big)-\frac{1}{\zeta}+\frac{1}{\overline{\zeta}}\\
&=-2\frac{\tilde{\zeta}+z}{\tilde{\zeta}-z}\ \Big(\frac{1}{\zeta-\tilde{\zeta}}-\frac{\tilde{\zeta}}{1-\tilde{\zeta}\overline{\zeta}}-\frac{1}{\zeta-z}+\frac{z}{1-z\overline{\zeta}}\Big)-\frac{1}{\zeta}+\frac{1}{\overline{\zeta}}
\end{aligned}$$

$$\begin{aligned}
&=-\frac{\tilde{\zeta}+z}{\tilde{\zeta}-z}\ \Big(\frac{2}{\zeta-\tilde{\zeta}}-\frac{1}{\zeta}-\frac{2\tilde{\zeta}}{1-\tilde{\zeta}\overline{\zeta}}-\frac{1}{\overline{\zeta}}-\frac{2}{\zeta-z}+\frac{1}{\zeta}+\frac{2z}{1-z\overline{\zeta}}+\frac{1}{\overline{\zeta}}\Big)-\frac{1}{\zeta}+\frac{1}{\overline{\zeta}}\\
&=-\frac{\tilde{\zeta}+z}{\tilde{\zeta}-z}\ \Big(\frac{1}{\zeta}\frac{\zeta+\tilde{\zeta}}{\zeta-\tilde{\zeta}}-\frac{1}{\overline{\zeta}}\frac{1+\tilde{\zeta}\overline{\zeta}}{1-\tilde{\zeta}\overline{\zeta}}-\frac{1}{\zeta}\frac{\zeta+z}{\zeta-z}+\frac{1}{\overline{\zeta}}\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}\Big)-\frac{1}{\zeta}+\frac{1}{\overline{\zeta}}
\end{aligned}$$

and similarly

$$\frac{1+\zeta\overline{\tilde\zeta}}{1-\zeta\overline{\tilde\zeta}}\frac{1}{\zeta}\frac{\zeta+z}{\zeta-z}+\frac{\overline{\tilde\zeta+\zeta}}{\overline{\tilde\zeta-\zeta}}\frac{1}{\overline{\zeta}}\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}$$
$$=\Big(\frac{2}{1-\zeta\overline{\tilde\zeta}}-1\Big)\Big(\frac{2}{\zeta-z}-\frac{1}{\zeta}\Big)-\Big(\frac{2}{\overline{\zeta-\tilde\zeta}}-\frac{1}{\overline{\zeta}}\Big)\Big(\frac{2}{1-z\overline{\zeta}}-1\Big)$$
$$=\frac{4}{1-z\overline{\tilde\zeta}}\ \Big(\frac{1}{\zeta-z}+\frac{\overline{\tilde\zeta}}{1-\zeta\overline{\tilde\zeta}}\Big)-\Big(\frac{2\overline{\tilde\zeta}}{1-\zeta\overline{\tilde\zeta}}+\frac{2}{\zeta}\Big)$$
$$-\frac{2}{\zeta-z}+\frac{1}{\zeta}-\frac{4}{1-z\overline{\tilde\zeta}}\ \Big(\frac{1}{\overline{\zeta-\tilde\zeta}}+\frac{z}{1-z\overline{\zeta}}\Big)+\frac{2z}{1-z\overline{\zeta}}+\frac{2}{\overline{\zeta}}+\frac{2}{\overline{\zeta-\tilde\zeta}}-\frac{1}{\overline{\zeta}}$$
$$=2\frac{1+z\overline{\tilde\zeta}}{1-z\overline{\tilde\zeta}}\ \Big(\frac{1}{\zeta-z}+\frac{\overline{\tilde\zeta}}{1-\zeta\overline{\tilde\zeta}}-\frac{1}{\overline{\zeta-\tilde\zeta}}-\frac{z}{1-z\overline{\zeta}}\Big)-\frac{1}{\zeta}+\frac{1}{\overline{\zeta}}$$
$$=-\frac{1+z\overline{\tilde\zeta}}{1-z\overline{\tilde\zeta}}\ \Big(\frac{2}{\overline{\zeta-\tilde\zeta}}-\frac{1}{\overline{\zeta}}-\frac{2\overline{\tilde\zeta}}{1-\zeta\overline{\tilde\zeta}}-\frac{1}{\zeta}-\frac{2}{\zeta-z}+\frac{1}{\zeta}+\frac{2z}{1-z\overline{\zeta}}+\frac{1}{\overline{\zeta}}\Big)-\frac{1}{\zeta}+\frac{1}{\overline{\zeta}}$$
$$=-\frac{1+z\overline{\tilde\zeta}}{1-z\overline{\tilde\zeta}}\ \Big(\frac{1}{\overline{\zeta}}\frac{\overline{\zeta+\tilde\zeta}}{\overline{\zeta-\tilde\zeta}}-\frac{1}{\zeta}\frac{1+\zeta\overline{\tilde\zeta}}{1-\zeta\overline{\tilde\zeta}}-\frac{1}{\zeta}\frac{\zeta+z}{\zeta-z}+\frac{1}{\overline{\zeta}}\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}\Big)-\frac{1}{\zeta}+\frac{1}{\overline{\zeta}}$$

and applying (33), (34), and (35)

$$\frac{1}{2\pi}\int\limits_{|\zeta|<1}\ \Big(\frac{\tilde\zeta+\zeta}{\tilde\zeta-\zeta}\frac{1}{\zeta}\frac{\zeta+z}{\zeta-z}+\frac{1+\overline{\zeta}\tilde\zeta}{1-\overline{\zeta}\tilde\zeta}\frac{1}{\overline{\zeta}}\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}\Big)(\zeta-z+\overline{\zeta-z})^{n-2}d\xi d\eta$$

$$=-\frac{\tilde\zeta+z}{\tilde\zeta-z}\frac{1}{2\pi}\int\limits_{|\zeta|<1}\ \Big(\frac{1}{\zeta}\frac{\zeta+\tilde\zeta}{\zeta-\tilde\zeta}-\frac{1}{\overline{\zeta}}\frac{1+\tilde\zeta\overline{\zeta}}{1-\tilde\zeta\overline{\zeta}}-\frac{1}{\zeta}\frac{\zeta+z}{\zeta-z}+\frac{1}{\overline{\zeta}}\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}\Big)(\zeta-z+\overline{\zeta-z})^{n-2}d\xi d\eta$$

$$=\frac{\tilde\zeta+z}{\tilde\zeta-z}\ \Big[\frac{1}{n-1}\ (\tilde\zeta-z+\overline{\tilde\zeta-z})^{n-1}-\frac{(-1)^{n-1}}{n-1}\ (z+\overline{z})^{n-1}+\frac{(-1)^{n-1}}{n-1}\ (z+\overline{z})^{n-1}\Big]$$

$$=\frac{\tilde\zeta+z}{\tilde\zeta-z}\frac{1}{n-1}\ (\tilde\zeta-z+\tilde\zeta-z)^{n-1}\ ,$$

$$\frac{1}{2\pi}\int\limits_{|\zeta|<1}\ \Big(\frac{\tilde\zeta+\zeta}{\tilde\zeta-\zeta}\frac{1}{\zeta}\frac{\zeta+z}{\zeta-z}+\frac{1+\overline{\zeta}\tilde\zeta}{1-\overline{\zeta}\tilde\zeta}\frac{1}{\overline{\zeta}}\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}\Big)(\zeta-z+\overline{\zeta-z})^{n-2}d\xi d\eta$$

$$=-\frac{1+z\overline{\tilde\zeta}}{1-z\overline{\tilde\zeta}}\frac{1}{2\pi}\int\limits_{|\zeta|<1}\ \Big(\frac{1}{\overline{\zeta}}\frac{\overline{\zeta+\tilde\zeta}}{\overline{\zeta-\tilde\zeta}}-\frac{1}{\zeta}\frac{1-\overline{\tilde\zeta}\zeta}{1-\overline{\tilde\zeta}\zeta}-\frac{1}{\zeta}\frac{\zeta+z}{\zeta-z}+\frac{1}{\overline{\zeta}}\frac{1-z\overline{\zeta}}{1-z\overline{\zeta}}\Big)(\zeta-z+\overline{\zeta-z})^{n-2}d\xi d\eta$$

$$= \frac{1+z\overline{\tilde{\zeta}}}{1-z\overline{\tilde{\zeta}}}\Big[\frac{1}{n-1}\ (\tilde{\zeta}-z+\overline{\tilde{\zeta}-z})^{n-1} - \frac{(-1)^{n-1}}{n-1}\ (z+\overline{z})^{n-1} + \frac{(-1)^{n-1}}{n-1}\ (z+\overline{z})^{n-1}\Big]$$

$$= \frac{1+z\overline{\tilde{\zeta}}}{1-z\overline{\tilde{\zeta}}}\frac{1}{n-1}\ (\tilde{\zeta}-z+\overline{\tilde{\zeta}-z})^{n-1}\ ,$$

then

$$\frac{(-1)^{n-1}}{2\pi(n-2)!}\int\limits_{|\zeta|<1}\Big(\frac{\omega(\zeta)}{\zeta}\frac{\zeta+z}{\zeta-z}+\frac{\overline{\omega(\zeta)}}{\overline{\zeta}}\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}\Big)(\zeta-z+\overline{\zeta-z})^{n-2}d\xi d\eta$$

$$= i\frac{c_{n-1}}{(n-1)!}\ (z+\overline{z})^{n-1}+\frac{(-1)^{n-1}}{2\pi i(n-1)!}\int\limits_{|\zeta|=1}\gamma_{n-1}(\zeta)\frac{\zeta+z}{\zeta-z}\ (\zeta-z+\overline{\zeta-z})^{n-1}\frac{d\zeta}{\zeta}$$

$$+\frac{(-1)^{n}}{2\pi(n-1)!}\int\limits_{|\zeta|<1}\Big(\frac{f(\zeta)}{\zeta}\frac{\zeta+z}{\zeta-z}+\frac{\overline{f(\zeta)}}{\overline{\zeta}}\frac{1+z\overline{\tilde{\zeta}}}{1-z\overline{\tilde{\zeta}}}\Big)(\zeta-z+\overline{\zeta-z})^{n-1}d\xi d\eta\ .$$

This proves formula (36).

Theorem 15 *The Dirichlet problem for the inhomogeneous polyanalytic equation in the unit disc*

$$\partial_{\overline{z}}^{n}w=f\ \textit{in}\ \mathbb{D}\ ,\ \partial_{\overline{z}}^{\nu}w=\gamma_\nu\ \textit{on}\ \partial\mathbb{D}\ ,\ 0\le\nu\le n-1\ ,$$

is uniquely solvable for $f\in L_1(\mathbb{D};\mathbb{C}),\gamma_\nu\in C(\partial\mathbb{D};\mathbb{C}),0\le\nu\le n-1$*, if and only if for* $0\le\nu\le n-1$

$$\sum_{\lambda=\nu}^{n-1}\frac{\overline{z}}{2\pi i}\int\limits_{|\zeta|=1}(-1)^{\lambda-\nu}\frac{\gamma_\lambda(\zeta)}{1-\overline{z}\zeta}\frac{(\overline{\zeta-z})^{\lambda-\nu}}{(\lambda-\nu)!}\ d\zeta$$
$$+\frac{(-1)^{n-\nu}\overline{z}}{\pi}\int\limits_{|\zeta|<1}\frac{f(\zeta)}{1-\overline{z}\zeta}\frac{(\overline{\zeta-z})^{n-1-\nu}}{(n-1-\nu)!}\ d\xi d\eta=0\ . \tag{37}$$

The solution then is

$$w(z)\ =\ \sum_{\nu=0}^{n-1}\frac{(-1)^{\nu}}{2\pi i}\int\limits_{|\zeta|=1}\frac{\gamma_\nu(\zeta)}{\nu!}\frac{(\overline{\zeta-z})^{\nu}}{\zeta-z}\ d\zeta$$
$$+\frac{(-1)^{n}}{\pi}\int\limits_{|\zeta|<1}\frac{f(\zeta)}{(n-1)!}\frac{(\overline{\zeta-z})^{n-1}}{\zeta-z}\ d\xi d\eta\ . \tag{38}$$

Proof For $n = 1$ condition (37) coinsides with [3], (34) and (38) is [3], (35). Assuming Theorem 27 is proved for $n-1$ rather than for n the problem is decomposed into the system

$$\begin{aligned} \partial_{\overline{z}}^{n-1} w &= \omega \text{ in } \mathbb{D} \ , \ \partial_{\overline{z}}^{\nu} w = \gamma_\nu \text{ on } \partial\mathbb{D} \ , \ 0 \le \nu \le n-2 \ , \\ \partial_{\overline{z}}\omega &= f \text{ in } \mathbb{D} \ , \ \partial_{\overline{z}}\omega = \gamma_{n-1} \text{ on } \partial\mathbb{D} \ , \end{aligned}$$

with the solvability conditions (37) for $0 \le \nu \le n-2$ and ω instead of f together with

$$\frac{\overline{z}}{2\pi i} \int\limits_{|\zeta|=1} \gamma_{n-1}(\zeta) \frac{d\zeta}{1-\overline{z}\zeta} - \frac{\overline{z}}{\pi} \int\limits_{|\zeta|<1} f(\zeta) \frac{d\xi d\eta}{1-\overline{z}\zeta} = 0$$

and the solutions (38) for $n-1$ instead of n and ω instead of f where

$$\omega(z) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma_{n-1}(\zeta) \frac{d\zeta}{\zeta - z} - \frac{1}{\pi} \int\limits_{|\zeta|<1} f(\zeta) \frac{d\xi d\eta}{\zeta - z} \ .$$

Then for $0 \le \nu \le n-2$

$$\begin{aligned} &\frac{1}{\pi} \int\limits_{|\zeta|<1} \frac{\omega(\zeta)}{1-\overline{z}\zeta} \frac{(\overline{\zeta - z})^{n-2-\nu}}{(n-2-\nu)!} \ d\xi d\eta \\ &= \frac{1}{2\pi i} \int\limits_{|\tilde{\zeta}|=1} \gamma_{n-1}(\tilde{\zeta}) \psi_\nu(\tilde{\zeta}, z) d\tilde{\zeta} - \frac{1}{\pi} \int\limits_{|\tilde{\zeta}|<1} f(\tilde{\zeta}) \psi_\nu(\tilde{\zeta}, z) d\tilde{\xi} d\tilde{\eta} \ , \end{aligned}$$

where

$$\begin{aligned} \psi_\nu(\tilde{\zeta}, z) &= -\frac{1}{\pi} \int\limits_{|\zeta|<1} \frac{(\overline{\zeta - z})^{n-2-\nu}}{(n-2-\nu)!(1-\overline{z}\zeta)} \frac{d\xi d\eta}{\zeta - \tilde{\zeta}} \\ &= \frac{(\overline{\tilde{\zeta} - z})^{n-1-\nu}}{(n-1-\nu)!(1-\overline{z}\tilde{\zeta})} - \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \frac{(\overline{\zeta - z})^{n-1-\nu}}{(n-1-\nu!(1-\overline{z}\zeta)} \frac{d\zeta}{\zeta - \tilde{\zeta}} \\ &= \frac{(\overline{\tilde{\zeta} - z})^{n-1-\nu}}{(n-1-\nu)!(1-\overline{z}\tilde{\zeta})} \ . \end{aligned}$$

The last equality holds because

$$\begin{aligned} &\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \frac{(\overline{\zeta - z})^{n-1-\nu}}{(1-\overline{z}\zeta)(\zeta - \tilde{\zeta})} \ d\zeta = -\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \frac{(\overline{\zeta - z})^{n-1-\nu} d\overline{\zeta}}{(\overline{\zeta} - \overline{z})(1-\tilde{\zeta}\overline{\zeta})} \\ &= -\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \frac{(\overline{\zeta - z})^{n-2-\nu}}{1-\tilde{\zeta}\overline{\zeta}} \ d\overline{\zeta} = 0 \ . \end{aligned}$$

Thus for $0 \le \nu \le n-2$

$$\sum_{\lambda=\nu}^{n-2} \frac{\overline{z}}{2\pi i} \int\limits_{|\zeta|=1} (-1)^{\lambda-\nu} \frac{\gamma_\lambda(\zeta)}{1-\overline{z}\zeta} \frac{(\overline{\zeta - z})^{\lambda-\nu}}{(\lambda-\nu)!}\, d\zeta$$
$$+\frac{(-1)^{n-1-\nu}\overline{z}}{\pi} \int\limits_{|\zeta|<1} \frac{\omega(\zeta)}{1-\overline{z}\zeta} \frac{(\overline{\zeta - z})^{n-2-\nu}}{(n-2-\nu)!}\, d\xi d\eta$$
$$= \sum_{\lambda=\nu}^{n-1} \frac{\overline{z}}{2\pi i} \int\limits_{|\zeta|=1} (-1)^{\lambda-\nu} \frac{\gamma_\lambda(\zeta)}{1-\overline{z}\zeta} \frac{(\overline{\zeta - z})^{\lambda-\nu}}{(\lambda-\nu)!}\, d\zeta$$
$$+\frac{(-1)^{n-\nu}\overline{z}}{\pi} \int\limits_{|\zeta|<1} \frac{f(\zeta)}{1-\overline{z}\zeta} \frac{(\overline{\zeta - z})^{n-1-\nu}}{(n-1-\nu)!}\, d\xi d\eta = 0\ .$$

This is (37). For showing (38) similarly

$$\frac{1}{\pi} \int\limits_{|\zeta|<1} \omega(\zeta) \frac{(\overline{\zeta - z})^{n-2}}{(n-2)!(\zeta - z)}\, d\xi d\eta$$
$$= \frac{1}{2\pi i} \int\limits_{|\tilde\zeta|=1} \gamma_{n-1}(\tilde\zeta)\psi_{n-1}(\tilde\zeta, z) d\tilde\zeta - \frac{1}{\pi} \int\limits_{|\tilde\zeta|<1} f(\tilde\zeta)\psi_{n-1}(\tilde\zeta, z) d\tilde\xi d\tilde\eta$$

with

$$\psi_{n-1}(\tilde\zeta, z) = -\frac{1}{\pi} \int\limits_{|\zeta|<1} \frac{(\overline{\zeta - z})^{n-2}}{(n-2)!(\zeta - z)} \frac{d\xi d\eta}{\zeta - \tilde\zeta}$$
$$= -\frac{1}{\pi} \int\limits_{|\zeta|<1} \frac{(\overline{\zeta - z})^{n-2}}{(n-2)!(\tilde\zeta - z)} \Big(\frac{1}{\zeta - \tilde\zeta} - \frac{1}{\zeta - z}\Big) d\xi d\eta = \frac{(\overline{\tilde\zeta - z})^{n-1}}{(n-1)!(\tilde\zeta - z)}$$
$$-\frac{1}{2\pi i(n-1)!(\tilde\zeta - z)} \int\limits_{|\zeta|=1} \Big(\frac{(\overline{\zeta - z})^{n-1}}{(\zeta - \tilde\zeta)} - \frac{(\overline{\zeta - z})^{n-1}}{\zeta - z}\Big) d\zeta = \frac{(\overline{\tilde\zeta - z})^{n-1}}{(n-1)!(\tilde\zeta - z)}$$
$$+\frac{1}{2\pi i(n-1)!(\tilde\zeta - z)} \int\limits_{|\zeta|=1} (\overline{\zeta - z})^{n-1} \Big(\frac{1}{1-\tilde\zeta\overline{\zeta}} - \frac{1}{1-z\overline{\zeta}}\Big) \frac{d\overline{\zeta}}{\overline{\zeta}} = \frac{(\overline{\tilde\zeta - z})^{n-1}}{(n-1)!(\tilde\zeta - z)}\ .$$

Hence, $w(z)$ is equal to

$$\sum_{\nu=0}^{n-2} \frac{(-1)^\nu}{2\pi i} \int\limits_{|\zeta|=1} \frac{\gamma_\nu(\zeta)}{\nu!} \frac{(\overline{\zeta - z})^\nu}{\zeta - z}\, d\zeta \ + \ \frac{(-1)^{n-1}}{\pi} \int\limits_{|\zeta|<1} \frac{\omega(\zeta)}{(n-2)!} \frac{(\overline{\zeta - z})^{n-2}}{\zeta - z}\, d\xi d\eta$$

$$= \sum_{\nu=0}^{n-1} \frac{(-1)^\nu}{2\pi i} \int_{|\zeta|=1} \frac{\gamma_\nu(\zeta)}{\nu!} \frac{\overline{(\zeta - z)}^\nu}{\zeta - z} d\zeta + \frac{(-1)^n}{\pi} \int_{|\zeta|<1} \frac{f(\zeta)}{(n-1)!} \frac{\overline{(\zeta - z)}^{n-1}}{\zeta - z} d\xi d\eta \; ,$$

i.e. (38) is valid.

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Heinrich Begehr
I. Math. Inst., FU Berlin
Arnimallee 3
14195 Berlin, Germany
email: begehr@math.fu-berlin.de