Intuitionistic Fuzzy θ-Closure Operator

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Abstract

The concepts of fuzzy θ-open (θ-closed)sets and fuzzy θ-closure operator are introduced and discussed in intuitionistic fuzzy topological spaces. As applications of these concepts, certain functions are characterized in terms of intuitionistic fuzzy θ-closure operator.

Keywords: Intuitionistic fuzzy $\theta-$closed set, intuitionistic fuzzy $\theta-$closure, intuitionistic fuzzy strongly $\theta-$continuous, intuitionistic fuzzy $\theta-$continuous, intuitionistic fuzzy weakly continuous, intuitionistic fuzzy $\lambda\theta-$continuous.

1. Introduction

Fuzzy sets were introduced by Zadeh[10] in 1965. A fuzzy set U in a universe X is a mapping from X to the unit interval [0, 1]. For any $x \in X$, the number $U(x)$ is called the membership degree of x in U. As a generalization of this notion, Atanassov[1] introduced the fundamental concept of intuitionistic fuzzy sets. While fuzzy sets give a degree of membership of an element in a given set, intuitionistic fuzzy sets give both a degree of membership and of non-membership. Both degrees belong to the interval [0, 1], and their sum shoud not exceed 1 (we will, formally, define the intuitionistic fuzzy set in section 2). Coker[2-4] and Hanafy[6] introduced the notion of intuitionistic fuzzy topological space, fuzzy continuity and some other related concepts. The concept of $\theta-$closure operator in a fuzzy topological spaces is introduced in [9]. In the present paper our aim is to introduce and study the concept of $\theta-$closure operator in intuitionistic fuzzy topological spaces. In section 3 of this paper we develop the concept of intuitionistic fuzzy $\theta-$closure operators. Intuitionistic fuzzy regular space is introduced and characterized in terms of intuitionistic fuzzy $\theta-$closure. The functions of fuzzy strongly $\theta-$continuous[8], fuzzy $\theta-$continuous[9], fuzzy weakly continuous[9] and fuzzy $\lambda\theta-$continuous[7] were introduced in fuzzy topological spaces. Section 4 is devoted to introduce these functions in intuitionistic fuzzy topological spaces and also includes the characterizations of these functions with the help of the notion of intuitionistic fuzzy

$\theta-$closures. For definitions and results not explained in this paper, we refer to the papers [1, 2, 4].

2. Preliminaries

Definition 2.1[1]. Let X be a nonempty fixed set. An intuitionistic fuzzy set (IFS, for short) U is an object having the form $U = \{\langle x, \mu_U(x), \gamma_U(x)\rangle : x \in X\}$ where the functions $\mu_U : X \to I$ and $\gamma_U : X \to I$ denote respectively the degree of membership (namely $\mu_U(x)$) and the degree of nonmembership (namely $\gamma_U(x)$) of each element $x \in X$ to the set U, and $0 \leq \mu_U(x) + \gamma_U(x) \leq 1$ for each $x \in X$.

For the sake of simplicity, we shall frequently use the symbol $U = \langle x, \mu_U(x), \gamma_U(x)\rangle$ for the IFS $U = \{\langle x, \mu_U(x), \gamma_U(x)\rangle : x \in X\}$. Every fuzzy set U on a nonempty set X is obviously an IFS having the form $U = \langle x, \mu_U(x), 1 - \mu_U(x)\rangle$.

Definition 2.2[1]. Let X be a nonempty set and let the IFS's U and V be in the form $U = \langle x, \mu_U(x), \gamma_U(x)\rangle$, $V = \langle x, \mu_V(x), \gamma_V(x)\rangle$ and let $\{U_j : j \in J\}$ be an arbitrary family of IFS's in X. Then
(i) $U \leq V$ iff $\mu_U(x) \leq \mu_V(x)$ and $\gamma_U(x) \geq \gamma_V(x), \forall x \in X$;
(ii) $\overline{U} = \{\langle x, \gamma_U(x), \mu_U(x)\rangle : x \in X\}$;
(iii) $\cap U_j = \{\langle x, \wedge\, \mu_{U_j}(x), \vee\, \gamma_{U_j}(x)\rangle : x \in X\}$;
(iv) $\cup U_j = \{\langle x, \vee\, \mu_{U_j}(x), \wedge\, \gamma_{U_j}(x)\rangle : x \in X\}$;
(v) $\tilde{1} = \{\langle x, 1, 0\rangle : x \in X\}$ and $\tilde{0} = \{\langle x, 0, 1\rangle : x \in X\}$;
(vi) $\overline{\overline{U}} = U, \overline{\tilde{0}} = \tilde{1}$ and $\overline{\tilde{1}} = \tilde{0}$.

Definition 2.3[2]. An intuitionistic fuzzy topology (IFT, for short) on a nonempty set X is a family Ψ of IFS's in X satisfying the following axioms:
(i) $\tilde{0}, \tilde{1} \in \Psi$;
(ii) $U_1 \cap U_2 \in \Psi$ for any $U_1, U_2 \in \Psi$;
(iii) $\cup U_j \in \Psi$ for any $\{ U_j : j \in J\} \subseteq \Psi$.
In this case the pair (X, Ψ) is called an intuitionistic fuzzy topological space (IFTS, for short) and each IFS in Ψ is known as an intuitionistic fuzzy open set (IFOS, for short) in X. The complement $\overline{U}$ of $IFOS$ U in $IFTS$ (X, Ψ) is called an intuitionistic fuzzy closed set (IFCS, for short).

Definition 2.4[2]. Let X and Y be two nonempty sets and $f : X \to Y$ a function.
(i) If $V = \{\langle y, \mu_V(y), \gamma_V(y)\rangle : y \in Y\}$ is an IFS in Y, then the preimage of V under f is denoted and defined by

$$f^{-1}(V) = \{\langle x, f^{-1}(\mu_V)(x), f^{-1}(\gamma_V)(x)\rangle \ : x \in X\}$$

where $f^{-1}(\mu_V)(x) = \mu_V(f(x))$ and $f^{-1}(\gamma_V)(x) = \gamma_V(f(x))$.

(ii) If $U = \{\langle x, \lambda_U(x), \delta_U(x)\rangle : x \in X\}$ is an IFS in X, then the image of U under f is denoted and defined by

$$f(U) = \{\langle y, f(\lambda_U)(y), f_{\_}(\delta_U)(y)\rangle : y \in Y\}$$

where

$$f(\lambda_U)(y) = \begin{cases} \sup\limits_{x\in f^{-1}(\ y)} \lambda_U(x), & f^{-1}(y) \neq 0 \\ \\ 0 & otherwise. \end{cases}$$

and

$$f_{\_}(\delta_U)(y) = 1 - f(1-\delta_U)(y) = \begin{cases} \inf\limits_{x\in f^{-1}(\ y)} \delta_U(x), & f^{-1}(y) \neq 0 \\ \\ 1 & otherwise. \end{cases}$$

Corollary 2.5.[5]. Let $U, U_j (j \in J)$ IFS's in X, $V, V_j (j \in J)$ IFS's in Y and $f : X \to Y$ a function. Then:
(i) $U_1 \leq U_2 \Rightarrow f(U_1) \leq f(U_2)$.
(ii) $V_1 \leq V_2 \Rightarrow f^{-1}(V_1) \leq f^{-1}(V_2)$.
(iii) $U \leq f^{-1}f(U)$ (If f is injective, then $U = f^{-1}f(U)$).
(iv) $ff^{-1}(V) \leq V$ (If f is surjective, then $ff^{-1}(V) = V$).
(v) $f^{-1}(\cup V_j) = \cup f^{-1}(V_j)$ and $f^{-1}(\cap V_j) = \cap f^{-1}(V_j)$.
(vi) $f(\cup V_j) = \cup f(V_j)$.
(vii) $f(\cap U_j) \leq \cap f(U_j)$, (If f is injective, then $f(\cap U_j) = \cap f(U_j)$).
$(viii)$ $f^{-1}(\overline{V}) = \overline{f^{-1}(V)}$.

Definition 2.6[2]. Let (X, Ψ) be an IFTS and $U = \langle x, \mu_U(x), \gamma_U(x)\rangle$ an IFS in X. Then the fuzzy interior and the fuzzy closure of U are defined by:
$cl(U) = \cap\{K : K$ is an IFCS in X and $U \leq K\}$ and
$int(U) = \cup\{G : G$ is an IFOS in X and $G \leq U\}$.

Definition 2.7[5]. An IFS U of an IFTS X is called:
(i) an intuitionistic fuzzy regular open set (IFROS, for short) of X if $int(cl(U)) = U$;
(ii) an intuitionistic fuzzy $\lambda-$open set (IFλOS, for short) of X if $U \leq int(cl(\ int(U)))$.

The complement of IFROS (resp. IFλOS) is called intuitionistic fuzzy regular closed set (resp. λ-closed set)(IFRCS (resp. IFλCS), for short) of X.

Proposition 2.8[2]. For any IFS U in (X, Ψ) we have:
(i) $cl(\overline{U}) = \overline{int(U)}$, (ii) $int(\overline{U}) = \overline{cl(U)}$.

Definition 2.9[4]. Let X be a non empty set and $c \in X$ a fixed element in X. If $a \in (0, 1]$ and $b \in [0, 1)$ are two fixed real numbers such that $a + b \leq 1$, then the IFS $c(a, b) = \langle x, c_a, 1 - c_{1-b} \rangle$ is called an intuitionistic fuzzy point (IFP, for short) in X, where a denotes the degree of membership of $c(a, b)$, b the degree of nonmembership of $c(a, b)$, and $c \in X$ the support of $c(a, b)$.

Definition 2.10[4]. Let $c(a, b)$ be an IFP in X and $U = \langle x, \mu_U, \gamma_U \rangle$ be an IFS in X. Suppose further that $a, b \in (0, 1)$. $c(a, b)$ is said to be properly contained in U ($c(a, b) \in U$, for short) iff $a < \mu_U(c)$ and $b > \gamma_U(c)$.

Definition 2.11[4]. (i) An IFP $c(a, b)$ in X is said to be quasi-coincident with the IFS $U = \langle x, \mu_U, \gamma_U \rangle$, denoted by $c(a, b)qU$, iff $a > \gamma_U(c)$ or $b < \mu_U(c)$.

(ii) Let $U = \langle x, \mu_U, \gamma_U \rangle$ and $V = \langle x, \mu_V, \gamma_V \rangle$ are two $IFSs$ in X. Then, U and V are said to be quasi-coincident, denoted by UqV, iff there exists an element $x \in X$ such that $\mu_U(x) > \gamma_V(x)$ or $\gamma_U(x) < \mu_V(x)$.

The expression 'not quasi-coincident' will be abbreviated as $\widetilde{q}$.

Proposition 2.12[4]. Let U and V be two IFS's and $c(a, b)$ an IFP in X. Then:
(i) $U \, \widetilde{q} \, \overline{V}$ iff $U \leq V$, (ii) $U \, q \, V$ iff $U \nleq \overline{V}$,
(iii) $c(a, b) \leq U$ iff $c(a, b) \widetilde{q} \, \overline{U}$, (iv) $c(a, b) \, q \, U$ iff $c(a, b) \nleq \overline{U}$.

Definition 2.13[4]. Let $f : X \to Y$ be a function and $c(a, b)$ an IFP in X. Then the image of $c(a, b)$ under f, denoted by $f(c(a, b))$, is defined by

$$f(c(a, b)) = \langle \, y, \; f(c)_a \;\; , \;\; 1 - f(c)_{1-b} \rangle \; .$$

Proposition 2.14[6]. Let $f : X \to Y$ be a function and $c(a, b)$ an IFP in X.
(i) If for IFS V in Y we have $f(c(a, b)) \, q \, V$, then $c(a, b) \, q \, f^{-1}(V)$.
(ii) If for IFS U in X we have $c(a, b)) \, q \, U$, then $f(c(a, b)) \, q \, f(U)$.

Definition 2.15. Let (X, Ψ) be an IFTS on X and $c(a, b)$ an IFP in X. An IFS A is called $\varepsilon q - nbd(\varepsilon\lambda q - nbd)$ of $c(a, b)$, if there exists an $IFOS$ $(IF\lambda OS)U$ in X such that $c(a, b)qU$ and $U \leq A$.

The family of all $\varepsilon q{-}nbd(\varepsilon\lambda q{-}nbd)$ of $c(a,b)$ will be denoted by $N_\varepsilon^q(N_\varepsilon^{\lambda q})(c(a,b))$.

Definition 2.16[6]. An IFTS (X,Ψ) is said to be intuitionistic fuzzy extremely disconnected (IFEDS, for short) iff the intuitionistic fuzzy closure of each IFOS in X is IFOS.

3. θ-closure operator in IFTS's

Definition 3.1. An IFP $c(a,b)$ is said to be intuitionistic fuzzy θ-*cluster* point (IFθ-*cluster* point, for short) of an $IFS\ U$ iff for each $A \in N_\varepsilon^q(c(a,b))$, $cl(A)\ q\ U$.

The set of all $IF\theta$-*cluster* points of U is called the intuitionistic fuzzy θ $-$closure of U and denoted by $cl_\theta(U)$. An IFS U will be called $IF\theta$-*closed*(IFθCS, for short) iff $U = cl_\theta(U)$. The complement of an $IF\theta$-*closed* set is $IF\theta$-*open* (IFθOS, for short). The θ-*interior* of U is denoted and defined by

$int_\theta(U) = \tilde{1} - cl_\theta(\tilde{1} - U)$.

Definition 3.2. An IFS U of an IFTS X is said to be $\varepsilon\theta q - nbd$ of an IFP $c(a,b)$ if there exists an $\varepsilon q - nbd\ \ A$ of $c(a,b)$ such that $cl(A)\ \tilde{q}\ \overline{U}$.

The family of all $\varepsilon\theta q - nbd$ of $c(a,b)$ will be denoted by $N_\varepsilon^{\theta q}(c(a,b))$.

Remark 3.3. It is clear that :

(i) IFθOS⊆IFOS and IFROS⊆IFOS

(ii) For any IFS U in an IFTS X, $cl(U) \le cl_\theta(U)$.

Example 3.4. Let $X = \{a,b\}$ and $U = \left\langle x, (\frac{a}{0.4}, \frac{b}{0.5}), (\frac{a}{0.4}, \frac{b}{0.4})\right\rangle$. Then the family $\Psi = \{\tilde{0}, \tilde{1}, U\}$ of IFS's in X is an IFT on X. Clearly U is an IFOS in X but not IFROS (Indeed $int(cl(U)) = \tilde{1} \neq U$).

Example 3.5. Let $X = \{a,b,c\}$ and

$U = \left\langle x, (\frac{a}{0.0}, \frac{b}{0.0}, \frac{c}{0.5}), (\frac{a}{1.0}, \frac{b}{1.0}, \frac{c}{0.0})\right\rangle$, $V = \left\langle x, (\frac{a}{1.0}, \frac{b}{0.5}, \frac{c}{0.0}), (\frac{a}{0.0}, \frac{b}{0.5}, \frac{c}{1.0})\right\rangle$.

Then the family $\Psi = \{\tilde{0}, \tilde{1}, U, V, U \cup V\}$ of IFS's in X is an IFT on X. Clearly U is an IFROS but not IFθOS.

Theorem 3.6. If U is an IFOS in an IFTS X, then $cl(U) = cl_\theta(U)$.

Proof. It is enough to prove $cl_\theta(U) \le cl(U)$.

Let $c(a,b)$ be an IFP in X such that $c(a,b) \notin cl(U)$, then there exists $V \in N_\varepsilon^q(c(a,b))$ such that $V\tilde{q}U$ and hence $V \le \overline{U}$. Then $cl(V) \le \overline{int(U)} \le \overline{U}$, since U is an IFOS in X. Hence $cl(V)\tilde{q}U$ which implies $c(a,b) \notin cl_\theta(U)$. Then $cl_\theta(U) \le cl(U)$. Thus $cl(U) = cl_\theta(U)$. ■

Theorem 3.7. In an IFTS (X, Ψ), the following hold :
(i) Finite union and arbitrary intersection of IFθCS's in X is an IFθCS.
(ii) For two IFS's U and V in X , if $U \leq V$, then $cl_\theta(U) \leq cl_\theta(V)$.
(iii) The IFS's $\tilde{0}$ and $\tilde{1}$ are IFθ-closed.

Proof. The straightforward proofs are omitted. ■

Remark 3.8. The set of all IFθOS's in an IFTS (X, Ψ) induce an IFTS (X, Ψ_θ) (say) which is coarser than the IFTS (X, Ψ).

Remark 3.9. For an IFS U in an IFTS X, $cl_\theta(U)$ is evidently IFCS but not necessarily IFθCS as is seen in the following example.

Example 3.10. Let $X = \{a, b, c\}$ and $U = \left\langle x, (\frac{a}{0.5}, \frac{b}{0.6}, \frac{c}{0.2}), (\frac{a}{0.4}, \frac{b}{0.4}, \frac{c}{0.1}) \right\rangle$, $V = \left\langle x, (\frac{a}{0.4}, \frac{b}{0.3}, \frac{c}{0.2}), (\frac{a}{0.5}, \frac{b}{0.6}, \frac{c}{0.3}) \right\rangle$. Then the family $\Psi = \{\tilde{0}, \tilde{1}, U, V\}$ of IFS's in X is an IFT on X.

Let $A = \left\langle x, (\frac{a}{0.3}, \frac{b}{0.3}, \frac{c}{0.0}), (\frac{a}{0.6}, \frac{b}{0.7}, \frac{c}{0.1}) \right\rangle$ be an IFS in X. Then an IFP $a(0.6, 0.3) \in cl_\theta(A)$ (because $a(0.6, 0.3)qU \leq U$, $cl(U) = \tilde{1}\ qA$), also $a(0.8, 0.1) \notin cl_\theta(A)$ (because $a(0.8, 0.1)qV$, $cl(V) = \overline{V}\tilde{q}A$). But $a(0.8, 0.1) \in cl_\theta(a(0.6, 0.3)) \leq cl_\theta(cl_\theta(A))$. Hence, $cl_\theta(A)$ is not IFθCS.

Lemma 3.11. If U, V are IFOS's in an IFEDS X, then $cl(V)\tilde{q}U \Rightarrow cl(V)\tilde{q}cl_\theta(U)$.

Proof. Let $cl(V)\tilde{q}U \Rightarrow U \leq \overline{cl(V)} \Rightarrow cl(U) \leq \overline{cl(V)}$ since X is an IFEDS. Hence $cl(V)\tilde{q}U \Rightarrow cl(V)\tilde{q}cl(U) \Rightarrow cl(V)\tilde{q}cl_\theta(U)$, by Remark 3.3. ■

Theorem 3.12. If U is an IFOS in an IFEDS (X, Ψ), then $cl_\theta(U)$ is an IFθCS in X.

Proof. Let $c(a, b)$ be an IFP in X and let $c(a, b) \notin cl_\theta(U)$. Then there is $V \in N^q_\varepsilon(c(a, b))$ such that $cl(V)\tilde{q}U$. By Lemma 3.11, $cl(V)\tilde{q}cl_\theta(U)$ and hence $c(a, b) > cl_\theta(U)$ implies $c(a, b) \notin cl_\theta(cl_\theta(U))$. Then $cl_\theta(cl_\theta(U)) \leq cl_\theta(U)$. But $cl_\theta(U) \leq cl_\theta(cl_\theta(U))$, then $cl_\theta(U) = cl_\theta(cl_\theta(U))$. Thus $cl_\theta(U)$ is an IFθCS. ■

Theorem 3.13. In an IFEDS (X, Ψ), every IFROS in X is an IFθOS.

Proof. Let U be an IFROS in an IFEDS (X, Ψ). Then $U = int(cl(U)) = cl(U) = int(U)$. Since U is an IFCS, $\overline{U}$ is an IFOS and by Theorem 3.6 , $cl(\overline{U}) = cl_\theta(\overline{U})$. Now $\overline{cl(\overline{U})} = \overline{cl_\theta(\overline{U})}$, i.e. $int(U) = int_\theta(U)$ (by Proposition 2.8). Thus $U = int(U) = int_\theta(U)$, and hence U is an IFθOS in (X, Ψ). ■

Theorem 3.14. An IFS U in an IFTS X is IFθO iff for each IFP $c(a, b)$ in X with $c(a, b)qU$, U is an $\varepsilon\theta q - nbd$ of $c(a, b)$.

Proof. Let U be an $IF\theta OS$ and $c(a,b)$ be an IFP in X with $c(a,b)qU$. Then by Proposition 2.12, $c(a,b) \nleq \overline{U}$. Since $\overline{U}$ is an IFθCS, $c(a,b) \nleq \overline{U} = cl_\theta(\overline{U})$. Then there exists $\varepsilon q - nbd$ A of $c(a,b)$ such that $cl(A)\widetilde{q}\overline{U}$. Hence U is an $\varepsilon\theta q - nbd$ of $c(a,b)$.

Conversely, if $c(a,b) \nleq \overline{U}$, then by by Proposition2.12, $c(a,b)qU$. Since U is an $\varepsilon\theta q - nbd$ of $c(a,b)$, then there exists $\varepsilon q - nbd$ A of $c(a,b)$ such that $cl(A)$ $\widetilde{q}\,\overline{U}$ and so $c(a,b) \nleq cl_\theta(\overline{U})$. Hence $\overline{U}$ is an IFθCS and then U is an IFθOS. ■

Theorem 3.15. For any IFS U in an IFTS (X,Ψ), $cl_\theta(U) = \cap\{cl_\theta(A) : A \in \Psi \text{ and } U \leq A\}$.

Proof. Obviously that $cl_\theta(U) \leq \cap\{cl_\theta(A) : A \in \Psi \text{ and } U \leq A\}$.

Now, let $c(a,b) \in \cap\{cl_\theta(A) : A \in \Psi \text{ and } U \leq A\}$, but $c(a,b) \notin cl_\theta(U)$. Then there exists an $\varepsilon q - nbd$ G of $c(a,b)$ such that $cl(G)\,\widetilde{q}\,U$ and hence by Proposition 2.12, $U \leq \overline{cl(G)}$. Then $c(a,b) \in cl_\theta(\overline{cl(G)})$ and consequently, $cl(G)q\overline{cl(G)}$ which is not true. Hence the result. ■

Definition 3.16. An IFTS X is said to be intuitionistic fuzzy regular (IFRS, for short) iff for each IFP $c(a,b)$ in X and each $\varepsilon q - nbd$ U of $c(a,b)$, there exists $\varepsilon q - nbd$ V of $c(a,b)$ such that $cl(V) \leq U$.

Theorem 3.17. An IFTS X is IFRS iff for any IFS U in X, $cl(U) = cl_\theta(U)$.

Proof. Let X be an IFRS. It is always true that $cl(U) \leq cl_\theta(U)$ for any IFS U. Now, let $c(a,b)$ be an IFP in X with $c(a,b) \in cl_\theta(U)$ and let A be an $\varepsilon q - nbd$ of $c(a,b)$. Then by IFRS X, there exists $\varepsilon q - nbd$ V of $c(a,b)$ such that $cl(V) \leq A$. Now, $c(a,b) \in cl_\theta(U)$ implies $cl(V)qU$ implies AqU implies $c(a,b) \in cl(U)$. Hence $cl_\theta(U) \leq cl(U)$. Thus $cl_\theta(U) = cl(U)$.

Conversely, let $c(a,b)$ be an IFP in X and U an $\varepsilon q - nbd$ of $c(a,b)$. Then $c(a,b) \notin \overline{U} = cl(\overline{U}) = cl_\theta(\overline{U})$. Thus there exists an $\varepsilon q - nbd$ V of $c(a,b)$ such that $cl(V)\widetilde{q}\overline{U}$ and then $cl(V) \leq U$. Hence X is IFRS. ■

Corollary 3.18 In an IFRS (X,Ψ) , an IFCS is an IFθCS and hence for any IFS U in X , $cl_\theta(U)$ is an IFθCS.

4. Characterizations for some types of functions in terms of IFθ-closure

Definition 4.1. A function $f : (X,\Psi) \to (Y,\Phi)$ is said to be intuitionistic fuzzy strongly $\theta-$continuous ($IFStr$-θ continuous, for short), if for each IFP $c(a,b)$ in X and $V \in N_\varepsilon^q\,(f(c(a,b)))$, there exists $U \in N_\varepsilon^q(c(a,b))$ such that $f(cl(U)) \leq V$.

Theorem 4.2. For a function $f : (X, \Psi) \to (Y, \Phi)$, the following are equivalent:
(*i*) f is $IFStr - \theta$ continuous.
(*ii*) $f(cl_\theta(U)) \leq cl(f(U))$ for each $IFS\ U$ in X.
(*iii*) $cl_\theta(f^{-1}(V)) \leq f^{-1}(cl(V))$ for each $IFS\ V$ in Y.
(*iv*) $f^{-1}(V)$ is an $IF\theta CS$ in X for each $IFCS\ V$ in Y.
(*v*) $f^{-1}(V)$ is an $IF\theta OS$ in X for each $IFOS\ V$ in Y.

Proof. $(i) \Longrightarrow (ii)$: Let $c(a,b) \in cl_\theta(U)$ and $B \in N_\varepsilon^q\ (f(c(a,b)))$. By (i), there exists $A \in N_\varepsilon^q(c(a,b))$ such that $f(cl(A)) \leq B$. Now, using Definition 3.1 and Proposition 2.14, we have $c(a,b) \in cl_\theta(U) \Rightarrow cl(A)qU \Rightarrow f(cl(A))qf(U) \Rightarrow Bqf(U) \Rightarrow f(c(a,b)) \in cl(f(U)) \Rightarrow c(a,b) \in f^{-1}(cl(f(U)))$. Hence $cl_\theta(U) \leq f^{-1}(cl(f(U)))$ and so $f(cl_\theta(U) \leq cl(f(U))$.

$(ii) \Longrightarrow (iii)$: Obvious by putting $U = f^{-1}(V)$.

$(iii) \Longrightarrow (iv)$: Let V be an $IFCS$ in Y. By (iii), we have $cl_\theta(f^{-1}(V)) \leq f^{-1}(cl(V)) = f^{-1}(V)$ which implies that $f^{-1}(V) = cl_\theta(V)$. Hence $f^{-1}(V)$ is an IFθCS in X.

$(iv) \Longrightarrow (v)$: By taking the complement.

$(v) \Longrightarrow (i)$: Let $c(a,b)$ be an IFP and $B \in N_\varepsilon^q\ (f(c(a,b)))$. By (v), $f^{-1}(B)$ is an $IF\theta OS$ in X. Now, using Proposition 2.14, we have $f(c(a,b))qB \Rightarrow c(a,b)qf^{-1}(B) \Rightarrow c(a,b) \notin \overline{f^{-1}(B)}$. Hence $\overline{f^{-1}(B)}$ is an IFθCS such that $c(a,b) \notin \overline{f^{-1}(B)}$. Then there exists $A \in N_\varepsilon^q(c(a,b))$ such that $cl(A)\ \widetilde{q}\overline{f^{-1}(B)}$ which implies that $f(cl(A)) \leq B$. Hence f is an $IFStr - \theta$ continuous. ■

Definition 4.3. A function $f : (X, \Psi) \to (Y, \Phi)$ is said to be intuitionistic fuzzy weakly continuous (IFw continuous, for short) , iff for each $IFOS\ V$ in Y, $f^{-1}(V) \leq int(f^{-1}(cl(V)))$.

Lemma 4.4. Let $f : (X, \Psi) \to (Y, \Phi)$ be a function. Then for an IFS B in Y, $f(\overline{f^{-1}(B)}) \leq \overline{B}$, where equality holds if f is onto.

Proof. Let $B = \{\langle y, \mu_B(y), \gamma_B(y)\rangle : y \in Y\}$ be an IFS in Y. From Definition 2.4, if $f^{-1}(y) = \tilde{0}$, then $(f(\overline{f^{-1}(B)}))(y) = 0 \leq \overline{B}(y)$. But if $f^{-1}(y) \neq \tilde{0}$ and since $f^{-1}(B) = \langle x,\ f^{-1}(\mu_B)(x),\ f^{-1}(\gamma_B)(x)\rangle$ implies $\overline{f^{-1}(B)} = \langle x,\ f^{-1}(\gamma_B)(x)\ ,\ f^{-1}(\mu_B)(x)\rangle$. Then, we have:

$$f(\overline{f^{-1}(B)})(y) = \langle y, f(f^{-1}(\gamma_B))(y),\ (1 - f(1 - f^{-1}(\mu_B)))(y)\rangle$$

where

$$f(f^{-1}(\gamma_B))(y) = \sup_{x \in f^{-1}(y)} f^{-1}(\gamma_B)(x) = \sup_{x \in f^{-1}(y)} \gamma_B f(x) = \gamma_B(y)$$

and

$$(1-f(1-\ f^{-1}(\mu_B)))(y)=\inf_{x\in f^{-1}(\ y)} f^{-1}(\mu_B)(y)=\inf_{x\in f^{-1}(\ y)}\mu_B f(x)=\mu_B(y)$$

i.e.

$$f(\overline{f^{-1}(B)})(y)=\langle y,\gamma_B(y),\ \mu_B(y)\rangle=\overline{B}(y).$$

If f is onto, then for each $y\in Y$, $f^{-1}(y)\neq\tilde{0}$ and hence we have $f(\overline{f^{-1}(B)})=\overline{B}$. ■

Lemma 4.5. Let U be an IFS and $c(a,b)$ be IFP in an IFTS (X,Ψ). Then for a function $f:(X,\Psi)\to(Y,\Phi)$ if $c(a,b)\in U$ then $f(c(a,b))\in f(U)$.

Proof. Let $c(a,b)\in U=\langle\ x,\mu_U\ ,\gamma_U\ \rangle$. Using Definitions 2.10 and 2.13, we have $a<\mu_U(c)$ and $b>\gamma_U(c)$
$\Rightarrow f(c)_a(y)<f(\mu_U(U))(y)$ and $1-b<1-\ \gamma_U(c)$
$\Rightarrow f(c)_a(y)<f(\mu_U(U))(y)$ and $f(c)_{1-b}(y)<f(1-\ \gamma_U(c))(y)$
$\Rightarrow f(c)_a(y)<f(\mu_U(U))(y)$ and $(1-f(c)_{1-b})(y)>(1-f(1-\ \gamma_U(c)))(y)$
$\Rightarrow f(c(a,b))\in f(U)$. ■

Theorem 4.6. For a function $f:(X,\Psi)\to(Y,\Phi)$, the following are equivalent:
(i) f is an IFw continuous.
(ii) $f(cl(U))\leq cl_\theta(f(U))$ for each $IFS\ U$ in X.
(iii) $cl(f^{-1}(V))\leq f^{-1}(cl_\theta(V))$ for each $IFS\ V$ in Y.
(iv) $cl(f^{-1}(V))\leq f^{-1}(cl(V))$ for each $IFOS\ V$ in Y.

Proof. $(i)\Rightarrow(ii)$: Let f be an IFw continuous and U any IFS in X. Suppose $c(a,b)\in cl(U)$, then by Lemma 4.5 $f(c(a,b))\in f(cl(U))$. It is enough to show that $f(c(a,b))\in cl_\theta(f(U))$. Let $G\in N^q_\varepsilon(f(c(a,b)))$. Then by Proposition 2.14, we have $f^{-1}(G)qc(a,b)$. By IFw continuous of f, $f^{-1}(G)\leq int(f^{-1}(cl(G)))$ and $int(f^{-1}(cl(G)))\in N^q_\varepsilon(c(a,b))$. Since $c(a,b)\in cl(U)$, we have $int(f^{-1}(cl(G)))\ qU$ and hence $cl(G)qf(U)$. Thus $f(c(a,b))\in cl_\theta(f(U))$.

$(ii)\Rightarrow(iii)$: Let V be an IFS in Y, then $f^{-1}(V)$ is an IFS in X. By (ii) we have $f(cl(f^{-1}(V)))\leq cl_\theta(f(f^{-1}(V)))\leq cl_\theta(V)$. Hence $cl(f^{-1}(V))\leq f^{-1}(cl_\theta(V))$.

$(iii)\Rightarrow(iv)$: Let V be an IFOS in Y. By Theorem 3.6, $cl(V)=cl_\theta(V)$ and by (iii), we have $cl(f^{-1}(V))\leq f^{-1}(cl(V))$.

$(iv)\Rightarrow(i)$: Let V be an IFOS in Y, and $cl(f^{-1}(V))\leq f^{-1}(cl(V))$. Then from $f^{-1}(V)\leq cl(f^{-1}(V))$ and the fact that V be an IFOS it follows that

$f^{-1}(V) = int(f^{-1}(V)) \leq int(cl(f^{-1}(V))) \leq int(f^{-1}(cl(V)))$. Hence f is an IFw continuous. ■

Theorem 4.7. Let $f : (X, \Psi) \to (Y, \Phi)$ be an IFw continuous function, then:
(i) $f^{-1}(V)$ is an $IFCS$ in X, for each $IF\theta CS$ V in Y.
(ii) $f^{-1}(V)$ is an $IFOS$ in X, for each $IF\theta OS$ V in Y.

Proof. (i) Let V be an IFθCS in Y, then $V = cl_\theta(V)$. By Theorem 4.6(iii), we have $cl(f^{-1}(V)) \leq f^{-1}(cl_\theta(V)) = f^{-1}(V)$. Hence $f^{-1}(V)$ is an IFCS in X.

$(i) \Leftrightarrow (ii)$: Obvious. ■

Definition 4.8. A function $f : (X, \Psi) \to (Y, \Phi)$ is called intuitionistic fuzzy $\theta-$continuous ($IF\theta$- continuous, for short), iff for each IFP $c(a, b)$ in X and each $V \in N^q_\varepsilon$ $(f(c(a, b)))$, there exists $U \in N^q_\varepsilon(c(a, b))$ such that $f(cl(U)) \leq cl(V)$.

Theorem 4.9. For a function $f : (X, \Psi) \to (Y, \Phi)$, the following are equivalent:
(i) f is an $IF\theta$-continuous.
(ii) $f(cl_\theta(U)) \leq cl_\theta(f(U))$, for each IFS U in X.
(iii) $cl_\theta(f^{-1}(V)) \leq f^{-1}(cl_\theta(V))$, for each IFS V in Y.
(iv) $cl_\theta(f^{-1}(V)) \leq f^{-1}(cl(V))$, for each IFOS V in Y.

Proof. $(i) \Longrightarrow (ii)$: Let $c(a, b) \in cl_\theta(U)$ and $B \in N^q_\varepsilon$ $(f(c(a, b)))$. By (i), there is $A \in N^q_\varepsilon(c(a, b))$ such that $f(cl(A)) \leq cl(B)$. Now, if $c(a, b) \in cl_\theta(U)$ then $cl(A)qU$ so that $f(cl(A))qf(U)$ and hence $cl(B)qf(U)$. Therefore $f(c(a, b)) \in cl_\theta(f(U))$ and it follows that $c(a, b) \in f^{-1}(cl_\theta(f(U)))$. Thus $cl_\theta(U) \leq f^{-1}(cl_\theta(f(U)))$ and hence $f(cl_\theta(U) \leq cl_\theta(f(U))$.

$(ii) \Rightarrow (iii)$: By (ii), if $f(cl_\theta(f^{-1}(V))) \leq cl_\theta(f(f^{-1}(V))) \leq cl_\theta(V)$, then it follows that $cl_\theta(f^{-1}(V)) \leq f^{-1}(cl_\theta(V))$.

$(iii) \Rightarrow (iv)$: Clear by Theorem 3.6.

$(iv) \Rightarrow (i)$: Let $c(a, b)$ be an IFP in X and $\overline{V \in N^q_\varepsilon}$ $(f(c(a, b)))$. Then $f(c(a, b)) \notin \underline{cl(\overline{cl(V)})}$, and hence $c(a, b) \notin f^{-1}(cl(\overline{cl(V)}))$. By (iv), we have $c(a, b) \notin \underline{cl_\theta(f^{-1}(\overline{cl(V)}))}$ and hence there exists $U \in N^q_\varepsilon(c(a, b))$ such that $cl(U)$ $\widetilde{q}$ $f^{-1}(cl(V)) = \overline{f^{-1}(cl(V))}$ which implies $f(cl(U)) \leq cl(V)$. Thus f is an $IF\theta$-continuous. ■

Theorem 4.10. Let $f : (X, \Psi) \to (Y, \Phi)$ be a function. If (X, Ψ) is an IFEDS, then the following are equivalent:

(i) f is an $IF\theta$- continuous.
(ii) $f^{-1}(V)$ is an $IF\theta CS$ in X for each $IF\theta CS$ V in Y.
(iii) $f^{-1}(V)$ is an $IF\theta OS$ in X for each $IF\theta OS$ V in Y.

Proof. $(i) \Longrightarrow (ii)$: Let V be an $IF\theta CS$ in Y. Since f is an $IF\theta$-continuous, then by (iii) in Theorem 4.9 we have $cl_\theta(f^{-1}(V)) \leq f^{-1}(cl_\theta(V)) = f^{-1}(V)$ which implies that $f^{-1}(V) = cl_\theta(f^{-1}(V))$. Hence $f^{-1}(V)$ is an $IF\theta CS$ in X.

$(ii) \Leftrightarrow (iii)$: Obvious.

$(ii) \Longrightarrow (i)$: Let V be an $IFOS$ in Y. Then by Theorem 3.6 $cl(V) = cl_\theta(V)$ which is an $IF\theta CS$ by Theorem 3.12. From (ii), $f^{-1}(cl(V)) = f^{-1}(cl_\theta(V))$ is an $IF\theta CS$ in X. Since $f^{-1}(V) \leq f^{-1}(cl(V))$, then $cl_\theta(f^{-1}(V)) \leq f^{-1}(cl(V))$. Hence f is an $IF\theta$-continuous. ■

Definition 4.11. A function $f : (X, \Psi) \to (Y, \Phi)$ is said to be intuitionistic fuzzy $\lambda\theta-$continuous($IF\lambda\theta$- continuous, for short), if for each IFP $c(a, b)$ in X and $V \in N_\varepsilon^{\lambda q}$ $(f(c(a, b)))$, there exists $U \in N_\varepsilon^{\theta q}(c(a, b))$ such that $f(U) \leq V$.

Definition 4.12. Let U be an IFS of an IFTS X Then:
(i) The $\lambda-$closure of U is denoted and defined by:

$$cl_\lambda(U) = \wedge\{K : K \text{ is } IF\lambda CS \text{ in } X \text{ and } U \leq K\}.$$

(iii) The $\lambda-$interior of U is denoted and defined by:

$$int_\lambda(U) = \vee\{G : G \text{ is } IF\lambda OS \text{ in } X \text{ and } G \leq U\}.$$

Theorem 4.13. Let $f : (X, \Psi) \to (Y, \Phi)$ be a function. then the following are equivalent:
(i) f is an $IF\lambda\theta$-continuous.
(ii) $f^{-1}(V)$ is an $IF\theta OS$ in X, for each $IF\lambda OS$ V in Y.
(iii) $f^{-1}(H)$ is an $IF\theta CS$ in X, for each $IF\lambda CS$ H in Y.
(iv) $cl_\theta(f^{-1}(V)) \leq f^{-1}(cl_\lambda(V))$, for each IFS V in Y.
(v) $f^{-1}(int_\lambda(G)) \leq int_\theta(f^{-1}(G))$, for each IFS G in Y.

Proof. $(i) \Longrightarrow (ii)$ Let V be an $IF\lambda OS$ in Y and $c(a, b)$ be IFP in X such that $c(a, b)qf^{-1}(V)$.Since f is $IF\lambda\theta$ continuous, there exists an $U \in N_\varepsilon^{\theta q}(c(a, b))$ such that $f(U) \leq V$. Then $c(a, b)qU \leq f^{-1}f(U) \leq f^{-1}(V)$ which shows that $f^{-1}(V) \in N_\varepsilon^{\theta q}(c(a, b))$ and then is an $IF\theta OS$ of X.

$(ii) \Longrightarrow (iii)$ by taking the complement.

$(iii) \Longrightarrow (iv)$ Let V be an IFS in Y Since $V \leq cl_\lambda(V)$,then $f^{-1}(V) \leq f^{-1}(cl_\lambda(V))$. Using (iii), $f^{-1}(cl_\lambda(V))$ is an $IF\theta CS$ in X .Thus $cl_\theta(f^{-1}(V)) \leq cl_\theta(f^{-1}(cl_\lambda(V))) = f^{-1}(cl_\lambda(V))$.

$(iv) \Longrightarrow (v)$ Using (iv) $cl_\theta(f^{-1}(V)) \leq f^{-1}(cl_\lambda(V))$, then
$\overline{cl_\theta(f^{-1}(V))} \geq \overline{f^{-1}(cl_\lambda(V))}$.
Hence $int_\theta(\overline{f^{-1}(V)}) \geq f^{-1}(\overline{cl_\lambda(V)})$.
Thus $f^{-1}(int_\lambda(\overline{V})) \leq int_\theta(f^{-1}(\overline{V}))$.
Put $G = \overline{V}$, then $f^{-1}(int_\lambda(G)) \leq int_\theta(f^{-1}(G))$

$(v) \Longrightarrow (i)$ Let V be an $IF\lambda OS$ in Y. Then $int_\lambda(V) = V$. Using (v), $f^{-1}(V) \leq int_\theta(f^{-1}(V))$. Hence $f^{-1}(V) = int_\theta(f^{-1}(V))$ i.e. $f^{-1}(V)$ is an $IF\theta OS$ in X. Let $c(a,b)$ be any IFP in $f^{-1}(V)$. Then $c(a,b)qf^{-1}(V)$, hence $f(c(a,b))qff^{-1}(V) \leq V$. Thus for any IFP $c(a,b)$ and each $V \in N_\varepsilon^{\lambda q}(f(c(a,b)))$, there exists $U = f^{-1}(V) \in N_\varepsilon^{\theta q}(c(a,b))$ such that $f(U) \leq V$. Thus f is $IF\lambda\theta$ continuous function. ■

Theorem 4.14. Let f be a bijective function from an $IFTS(X,\Psi)$ into an $IFTS(Y,\Phi)$. Then f is an $IF\lambda\theta$ continuous iff $int_\lambda(f(U)) \leq f(int_\theta(U))$, for each IFS U of X.

Proof. $(\Longrightarrow)$: Let f be an $IF\lambda\theta$ continuous function and U be an IFS in X. Hence $f^{-1}(int_\lambda(f(U)))$ is an $IF\theta OS$ in X. Since f is injective and using Theorem 4.13(v),we have: $f^{-1}(int_\lambda(f(U))) \leq int_\theta(f^{-1}(f(U))) = int_\theta(U)$.
Since f is surjective,
$ff^{-1}(int_\lambda(f(U))) \leq f(int_\theta(U))$.i.e. $int_\lambda(f(U)) \leq f(int_\theta(U))$.

$(\Longleftarrow)$: Let V be an $IF\lambda OS$ in Y. Then $V = int_\lambda(V)$. Using the hypothesis, we have: $V = int_\lambda(V) = int_\lambda(ff^{-1}(V)) \leq f(int_\theta(f^{-1}(V)))$, which implies that $f^{-1}(V) \leq f^{-1}f(int_\theta(f^{-1}(V)))$. From the fact that f is injective, we have: $f^{-1}(V) \leq int_\theta(f^{-1}(V))$. Hence $f^{-1}(V) = int_\theta(f^{-1}(V))$ i.e. $f^{-1}(V)$ is an $IF\theta OS$ in X. Thus f is $IF\lambda\theta$ continuous. ■

Theorem 4.15. Let $f : (X,\Psi) \to (Y,\Phi)$ be a bijective function. Then f is an $IF\lambda\theta$ continuous iff $f(cl_\theta(U)) \leq cl_\lambda(f(U))$, for each IFS U of X.

Proof. Similar to the proof of Theorem 4.14. ■

Remark 4.16. From the above definitions, one can illustrate the following implications:

$IF\lambda\theta$-continuous $\Rightarrow$ $IFstr\theta$-continuous $\Rightarrow$ IF-continuous $\Rightarrow$ IFw-continuous
$\Downarrow$
$IF\theta$-continuous

Example 4.17. Let $X = \{1, 2, 3\}$, $Y = \{a, b, c\}$ and
$A = \langle x, (\frac{1}{0.4}, \frac{2}{0.5}, \frac{3}{0.5}), (\frac{1}{0.3}, \frac{2}{0.4}, \frac{3}{0.4})\rangle$, $B = \langle x, (\frac{1}{0.5}, \frac{2}{0.5}, \frac{3}{0.5}), (\frac{1}{0.2}, \frac{2}{0.3}, \frac{3}{0.1})\rangle$,
$U = \langle y, (\frac{a}{0.5}, \frac{b}{0.4}, \frac{c}{0.5}), (\frac{a}{0.4}, \frac{b}{0.4}, \frac{c}{0.3})\rangle$, $V = \langle y, (\frac{a}{0.4}, \frac{b}{0.2}, \frac{c}{0.4}), (\frac{a}{0.5}, \frac{b}{0.4}, \frac{c}{0.5})\rangle$.
Then the family $\Psi = \{\tilde{0}, \tilde{1}, A, B\}$ of IFS's in X is an IFT on X and the family $\Phi = \{\tilde{0}, \tilde{1}, U, V\}$ of IFS's in Y is an IFT on Y. Let $f : (X, \Psi) \to (Y, \Phi)$ be a function defined as follows: $f(a) = 2$, $f(b) = 3$ and $f(c) = 1$. Then $f^{-1}(U) \subseteq int(f^{-1}(cl(U))) = \tilde{1}$ and $f^{-1}(V) \subseteq int(f^{-1}(cl(V))) = A$. Thus f is an IFw continuous but not IF-continuous.

Remark 4.18. From the above example, one can show that IFw continuous does not implies each of the concepts IF$\lambda\theta$-continuous, IFstr θ-continuous and IFθ-continuous.

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