Extremal Moment Methods and Stochastic Orders

Application in Actuarial Science

Chapters I, II and III.

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With 60 Tables

"Neglect of mathematics works injury to all knowledge, since he who is ignorant of it cannot know the other sciences or the things of this world."

Roger Bacon

CONTENTS

CHAPTER III. Best bounds for expected values by given range and known moments of higher order

CHAPTER IV. Stochastically ordered extremal random variables

CHAPTER V. Bounds for bivariate expected values

CHAPTER VI. Applications in Actuarial Science

PREFACE

This specialized monograph provides an account of some substantial classical and more recent results in the field of extremal moment problems and their relationship to stochastic orders. The presented topics are believed to be of primordial importance in several fields of Applied Probability and Statistics, especially in Actuarial Science and Finance. Probably the first and most classical prototype of an extremal moment problem consists of the classical Chebyshev-Markov distribution bounds, whose origin dates back to work by Chebyshev, Markov and Possé in the second half of the nineteenth century. Among the more recent developments, the construction of extremal stop-loss transforms turns out to be very similar and equally well important. Both subjects are treated in an unified way as optimization problems for expected values of random functions by given range and known moments up to a given order, which is the leading and guiding theme throughout the book. All in one, our work contributes to a general trend in the mathematical sciences, which puts more and more emphasis on robust, distribution-free and extremal methods.

The intended readership, which reflects the author's motivation for writing the book, are groups of applied mathematical researchers and actuaries working at the interface between academia and industry. The first group will benefit from the first five chapters, which makes 80% of the exposé. The second group will appreciate the final chapter, which culminates with a series of recent actuarial and related financial applications. This splitting into two parts mirrors also the historical genesis of the present subject, which has its origin in many mathematical statistical papers, some of which are quite old. Among the first group, we have especially in mind a subgroup of forthcoming twentyfirst century generation of applied mathematicians, which will have the task to implement in algorithmic language complex structures. Furthermore, the author hopes there is something to take home in several other fields involving related Applied and even Abstract Mathematics. For example, Chapter I develops a complete analytical-algebraic structure for sets of finite atomic random variables in low dimensions through the introduction of convenient involution mappings. This constitutes a clear invitation and challenge to algebraists for searching and developing a corresponding more general structure in arbitrary dimensions. Also, our results of this Chapter are seen as an application of the theory of orthogonal polynomials, which are known to be of great importance in many parts of Applied Mathematics. The interested actuary and finance specialist is advised to read first or in parallel Chapter VI, which provides motivation for most of the mathematical results presented in the first part.

The chosen mathematical language is adapted to experienced applied scientists, which not always need an utmost precise and rigorous form, yet a sufficiently general formulation. Besides introductory probability theory and statistics only classical mathematics is used. No prerequisites are made in functional analysis (Hilbert space theory for a modern treatment of orthogonal polynomials), measure theory (rigorous probability theory), and the theory of stochastic processes (modern applied probability modelling). However, to read the succintly written final actuarial chapter, background knowledge on insurance mathematics is assumed. For this purpose, many of the excellent textbooks mentioned in the notes of Chapter VI will suffice. To render the text reasonably short and fluent, most sources of results have been reported in notes at the end of each chapter, where one finds references to additional related material, which hopefully will be useful for future research in the present field. The given numerous cross-references constitute a representative but not at all an exhaustive list of the available material in the academic literature.

A great deal of non-mathematical motivation for a detailed analysis of the considered tools stems from Actuarial Science and Finance. For example, a main branch of Finance is devoted to the "modelling of financial returns", where one finds path-breaking works by Bachelier(1900), Mandelbrot(1963), Fama(1965) and others (see e.g. Taylor(1992)). More recent work includes Mittnik and Rachev(1993), as well as the current research in Chaos Theory along the books by Peters(1991/94) (see e.g. Thoma(1996) for a readable introduction into this fascinating and promising subject). Though models with infinite variance (and/or infinite kurtosis) can be considered, there presumably does not seem to exist a definitive answer for their universal applicability (see e.g. Granger and Orr(1972)). For this reason, moment methods still remain of general interest, also in this area, whatever the degree of development other methods have achieved. Furthermore, their importance can be justified from the sole purpose of useful comparative results. Let us underpin the need for distribution-free and extremal moment methods by a single concrete example. One often agrees that a satisfactory model for daily returns in financial markets should have a probability distribution, which is similar to the observed empirical distribution. Among the available models, symmetric distributions have often been considered adequate (e.g. Taylor(1992), Section 2.8). Since sample estimates of the kurtosis parameter take in a majority of situations values greater than 6 (normal distributions have a kurtosis equal to 3), there is an obvious need for statistical models allowing for variation of the kurtosis parameter. In our monograph, several Sections are especially formulated for the important special case of symmetric random variables, which often turns out to be mathematically more tractable.

To preserve the unity of the subject, a comparison with other methods is not provided. For example, parametric or semi-parametric statistical methods of estimation could and should be considered and put in relation to the various bounds. Though a real-life data study is not given, the importance and usefulness of the approach is illustrated through different applications from the field of Actuarial Science and related Finance. These applications emphasize the importance of the subject for the theoretically inclined reader, and hopefully acts as stimulus to investigate difficult open mathematical problems in the field.

An informal version of this monograph has been circulated among interested researchers since 1998. During the last decade many further advances have been reached in this area, some of which have been accounted for in the additional bibliography. In particular, a short account of some main results is found in the appendix of Hürlimann (2002a).

Finally, I wish to thank anonymous referees from Journal of Applied Probability as well as Statistics and Probability Letters for useful comments on Sections IV.1, IV.2 and some additional references about multivariate Chebyshev inequalities in Section V.7. A first review of the present work by Springer-Verlag has also been used for some minor adjustements of an earlier version of this monograph. My very warmest thanks go to Henryk Gzyl for inviting me to publish this monograph in the Boletín de la Asociación Matemática Venezolana

Zurich, October 2008, Werner Hürlimann

CHAPTER I

ORTHOGONAL POLYNOMIALS AND FINITE ATOMIC RANDOM VARIABLES

1. Orthogonal polynomials with respect to a given moment structure.

Consider a real random variable X with finite moments $\mu_k = E\left[X^k\right]$, k=0,1,2, ... If X takes an infinite number of values, then the moment determinants

(1.1)
$$\Delta_n = \begin{vmatrix} \mu_0 & . & . & . & \mu_n \\ . & & & & \\ . & & & & \\ . & & & & \\ \mu_n & . & . & . & \mu_{2n} \end{vmatrix}, \quad n = 0,1,2,\ldots,$$

are non-zero. Otherwise, only a finite number of them are non-zero (e.g. Cramèr(1946), Section 12.6). We will assume that all are non-zero. By convention one sets $p_0(x) = \mu_0 = 1$.

Definition 1.1. The *orthogonal polynomial* of degree $n \geq 1$ with respect to the finite moment structure $\{\mu_k\}_{k=0,\ldots,2n-1}$, also called orthogonal polynomial with respect to X, is the unique monic polynomial of degree n

(1.2)
$$p_n(x) = \sum_{j=0}^{n} (-1)^{n-j} t_j^n x^j, \quad t_n^n = 1,$$

which satisfies the n linear expected value equations

(1.3)
$$E\left[p_n(X) \cdot X^i\right] = 0, \quad i = 0,1,\ldots,n-1.$$

Note that the terminology "orthogonal" refers to the scalar product induced by the expectation operator $\langle X, Y \rangle = E[XY]$, where X, Y are random variables for which this quantity exists. These orthogonal polynomials coincide with the nowadays so-called classical Chebyshev polynomials.

Lemma 1.1. (*Chebyshev determinant representation of the orthogonal polynomials*) The orthogonal polynomial of degree n identifies with the determinant

(1.4) $$p_n(x)=\frac{1}{\Delta_{n-1}}\begin{vmatrix}\mu_0 & . & . & . & \mu_n\\ . & & & & .\\ . & & & & .\\ . & & & & .\\ \mu_{n-1} & & & & \mu_{2n-1}\\ 1 & . & . & . & x^n\end{vmatrix},\quad n=1,2,...$$

Proof. Let $0\le i<n$. Then the expected value

$$E\left[p_n(X)\cdot X^i\right]=\frac{1}{\Delta_{n-1}}\begin{vmatrix}\mu_0 & . & . & . & \mu_n\\ . & & & & .\\ . & & & & .\\ . & & & & .\\ \mu_{n-1} & & & & \mu_{2n-1}\\ E\left[X^i\right] & . & . & . & E\left[X^{n+i}\right]\end{vmatrix}=\frac{1}{\Delta_{n-1}}\begin{vmatrix}\mu_0 & . & . & . & \mu_n\\ . & & & & .\\ . & & & & .\\ . & & & & .\\ \mu_{n-1} & & & & \mu_{2n-1}\\ \mu_i & . & . & . & \mu_{n+i}\end{vmatrix}$$

vanishes because $0\le i<n$, and thus two rows of the determinant are equal. ◊

The orthogonal polynomials form an orthogonal system with respect to the scalar product $\langle X,Y\rangle=E[XY]$.

Lemma 1.2. (*Orthogonality relations*)

(O1) $$E\left[p_m(X)\cdot p_n(X)\right]=0 \quad \text{for } m\neq n$$

(O2) $$E\left[p_n(X)^2\right]=\frac{\Delta_n}{\Delta_{n-1}}\neq 0,\quad n=1,2,...$$

Proof. The relations (O1) follow by linearity of the expectation operator using the defining conditions (1.3). Since $p_n(X)$ is orthogonal to the powers of X of degree less than n, one has $E\left[p_n(X)^2\right]=E\left[p_n(X)\cdot X^n\right]$. The formula in the proof of Lemma 1.1, valid for i=n, shows (O2). ◊

The concrete computation of orthogonal polynomials depends recursively only on the "leading" coefficients t_{n-1}^n and on the moment determinants Δ_n.

Lemma 1.3. (*Three term recurrence relation for orthogonal polynomials*) The orthogonal polynomials satisfy the recurrence relation

(O3) $$\begin{aligned}&p_{n+1}(x)=(x+t_{n-1}^n-t_n^{n+1})p_n(x)-c_n p_{n-1}(x),\quad n=2,3,...,\\ &c_n=\frac{\Delta_{n-2}\cdot\Delta_n}{\Delta_{n-1}^2},\end{aligned}$$

where the starting values are given by

$$p_0(x)=1,\ p_1(x)=x-\mu_1,\ p_2(x)=x^2-\left(\frac{\mu_3-\mu_1\mu_2}{\sigma^2}\right)x+\left(\frac{\mu_1\mu_3-\mu_2^2}{\sigma^2}\right),\ \sigma^2=\mu_2-\mu_1^2.$$

Proof. Clearly one can set

(1.5) $$p_{n+1}(x)=(x+a_n)\cdot p_n(x)+\sum_{j=0}^{n-1}b_j p_j(x).$$

Multiply this with $p_i(x)$, $i<n-1$, and take expectations to see that $b_i\cdot E\left[p_i(X)^2\right]=0$, hence $b_i=0$ by (O2) of Lemma 1.2. Setting $b_{n-1}=-c_n$ one has obtained the three term recurrence relation

(1.6) $$p_{n+1}(x)=(x+a_n)\cdot p_n(x)-c_n p_{n-1}(x).$$

The remaining coefficients are determined in two steps. First, multiply this with X^{n-1} and take expectation to get using (O2)

(1.7) $$c_n=\frac{E\left[p_n(X)\cdot X^n\right]}{E\left[p_{n-1}(X)\cdot X^{n-1}\right]}=\frac{\Delta_{n-2}\cdot\Delta_n}{\Delta_{n-1}^2}.$$

Second, multiply (1.6) with $p_n(x)$ and take expectation to get using (1.2)

$$a_n\cdot E\left[p_n(X)^2\right]=-E\left[p_n(X)^2\cdot X\right]=t_{n-1}^n\cdot E\left[p_n(X)\cdot X^n\right]-E\left[p_n(X)\cdot X^{n+1}\right].$$

Using that $x^{n+1}=p_{n+1}(x)+t_n^{n+1}x^n\mp\ldots$, one gets further

(1.8) $$a_n\cdot E\left[p_n(X)^2\right]=t_{n-1}^n\cdot E\left[p_n(X)\cdot X^n\right]-t_n^{n+1}\cdot E\left[p_n(X)\cdot X^n\right].$$

Since $E\left[p_n(X)^2\right]=E\left[p_n(X)\cdot X^n\right]$, as shown in the proof of Lemma 1.2, the desired value for the coefficient a_n follows. ◊

To be effective an explicit expression for the "leading" coefficient t_{n-1}^n is required.

Lemma 1.4. (*Leading coefficient of orthogonal polynomials*) For $n\ge 2$ let $M^{(n)}=(M_{ij})_{i,j=0,\ldots,n-1}$ be the (nxn)-matrix with elements $M_{ij}=\mu_{i+j}$, which is assumed to be non-singular. Further let the column vectors $m^{(n)}=(\mu_n,\mu_{n+1},\ldots,\mu_{2n-1})^T$ and $t^{(n)}=(t_o^n,-t_1^n,t_2^n,\ldots,(-1)^{n-1}t_{n-1}^n)^T$. Then one has the linear algebraic relationship

(1.9) $$M^{(n)}\cdot t^{(n)}=(-1)^{n-1}m^{(n)}.$$

In particular the leading coefficient of the orthogonal polynomial of degree n is the (n-1)-th component of the vector $\left(M^{(n)}\right)^{-1}\cdot m^{(n)}$, and equals

(1.10) $$t_{n-1}^{n}=\frac{\sum_{i=0}^{n-1}\overline{M}_{in-1}^{(n)}\cdot\mu_{n+i}}{\Delta_{n-1}},$$

where $\Delta_{n-1}=\det M^{(n)}$ and $\overline{M}^{(n)}$ is the adjoint matrix of $M^{(n)}$ with elements $\overline{M}_{ij}^{(n)}=(-1)^{i+j}\det \breve{M}_{ij}^{(n)}$, i,j=0,...,n-1, $\breve{M}_{ij}^{(n)}$ the matrix obtained from $M^{(n)}$ through deletion of the i-th row and j-th column.

Proof. This is a standard exercise in Linear Algebra. By definition 1.1 the n linear expected value equations (1.3) are equivalent with the system of linear equations in the coefficients t_j^n:

(1.11) $$\sum_{j=0}^{n}(-1)^j\mu_{i+j}t_j^n=0,\quad i=0,\dots,n-1.$$

Since $t_n^n=1$ this is of the form (1.9). Inverting using the adjoint matrix shows (1.10). ◊

It is possible to reduce the amount of computation needed to evaluate t_{n-1}^n. Making the transformation of random variables $(X-\mu_1)/\sigma$, one can restrict the attention to standardized random variables X such that $\mu_1=0$, $\mu_2=\sigma^2=1$. The corresponding orthogonal polynomials will be called *standard orthogonal polynomials*. Clearly the standard orthogonal polynomial of degree two equals $p_2(x)=x^2-\gamma x-1$, where $t_1^2=\gamma=\mu_3$ is the skewness parameter.

Lemma 1.5. (*Leading coefficient of standard orthogonal polynomials*)
For $n\ge 3$ let $R^{(n)}=(R_{ij})_{i,j=2,\dots,n-1}$ be the (n-2)x(n-2)-matrix with elements $R_{ij}=(-1)^j\cdot(\mu_{i+j}-\mu_i\mu_j-\mu_{i+1}\mu_{j+1})$, which is assumed to be non-singular. Further let the column vectors $r^{(n)}=(-R_{2n},\dots,-R_{n-1n})^T$ and $s^{(n)}=(t_2^n,\dots,t_{n-1}^n)^T$. Then one has

(1.12) $$R^{(n)}\cdot s^{(n)}=r^{(n)}.$$

In particular the leading coefficient of the standard orthogonal polynomial of degree n is the (n-1)-th component of the vector $\left(R^{(n)}\right)^{-1}\cdot r^{(n)}$, and equals

(1.13) $$t_{n-1}^n=-\frac{\sum_{i=2}^{n-1}\overline{R}_{in-1}^{(n)}\cdot R_{in}}{\sum_{i=2}^{n-1}\overline{R}_{in-1}^{(n)}\cdot R_{in-1}},$$

where $\overline{R}^{(n)}$ is the adjoint matrix of $R^{(n)}$ with elements $\overline{R}_{ij}^{(n)}=(-1)^{i+j}\det \grave{R}_{ij}^{(n)}$, i,j=2,...,n-1, $\grave{R}_{ij}^{(n)}$ the matrix obtained from $R^{(n)}$ through deletion of the i-th row and j-th column.

Proof. Solving for t_o^n, t_1^n in the first two equations indexed i=0,1 in (1.11), one obtains

(1.14) $$\sigma^2\cdot t_0^n=-\sum_{j=2}^{n}(-1)^j\cdot(\mu_2\mu_j-\mu_1\mu_{j+1})\cdot t_j^n,$$

(1.15) $$\sigma^2\cdot t_1^n=\sum_{j=2}^{n}(-1)^j\cdot(\mu_{j+1}-\mu_1\mu_j)\cdot t_j^n,$$

which specialize in the standardized case $\mu_1=0,\mu_2=1$ to

(1.16) $$t_0^n = -\sum_{j=2}^{n} (-1)^j \cdot \mu_j t_j^n,$$

(1.17) $$t_1^n = \sum_{j=2}^{n} (-1)^j \cdot \mu_{j+1} t_j^n.$$

Consider now the equations with index $i \geq 2$. Multiply them with σ^2 and use (1.14), (1.15) to get the linear system in $t_2^n, \ldots, t_{n-1}^n$ of order n-2 :

(1.18) $$\sum_{j=2}^{n} (-1)^j \cdot (\sigma^2 \mu_{i+j} - \mu_2 \mu_i \mu_j - \mu_{i+1} \mu_{j+1} + \mu_1 (\mu_i \mu_{j+1} + \mu_{i+1} \mu_j) \cdot t_j^n = 0,$$
$$i = 2, \ldots, n-1.$$

In the standardized case this system reduces to

(1.19) $$\sum_{j=2}^{n} (-1)^j \cdot (\mu_{i+j} - \mu_i \mu_j - \mu_{i+1} \mu_{j+1}) \cdot t_j^n = 0, \quad i = 2, \ldots, n-1.$$

Since $t_n^n = 1$ this is of the form (1.12). Inverting using the adjoint matrix shows (1.13). ◊

Examples 1.1.

For the lower degrees n=3,4 one obtains the leading coefficients

(1.20) $$t_2^3 = -\frac{M_{23}}{M_{22}} = \frac{\mu_5 - \mu_3 \mu_4 - \mu_3}{\mu_4 - \mu_3^2 - 1},$$

(1.21) $$t_3^4 = -\frac{M_{23} M_{24} + M_{22} M_{34}}{M_{23}^2 + M_{22} M_{33}}.$$

2. The algebraic moment problem.

Given the first 2n-1 moments of some real random variable X, the *algebraic moment problem* of order n asks for the existence and construction of a finite atomic random variable with ordered support $\{x_1, \ldots, x_n\}$ such that $x_1 < x_2 < \ldots < x_n$, and probabilities $\{p_1, \ldots, p_n\}$ such that the system of non-linear equations

AMP(n) $$\sum_{i=1}^{n} p_i x_i^k = \mu_k, \quad k = 0, \ldots, 2n-1$$

is solvable. For computational purposes it suffices to know that if a solution exists, then the atoms of the random variable solving AMP(n) must be identical with the distinct real zeros of the orthogonal polynomial of degree n, as shown by the following precise recipe.

Lemma 2.1. (*Solution of AMP(n)*) Given are positive numbers $p_1, \ldots, p_n$ and real distinct numbers $x_1 < x_2 < \ldots < x_n$ such that the system AMP(n) is solvable. Then the x_i's are the distinct real zeros of the orthogonal polynomial of degree n, that is $p_n(x_i) = 0$, i=1,...,n, and

(2.1) $$p_i = \frac{E\left[\prod_{j\neq i}(Z-x_j)\right]}{\prod_{j\neq i}(x_i-x_j)}, i=1,...,n,$$

where Z denotes the discrete random variable with support $\{x_1,...,x_n\}$ and probabilities $\{p_1,...,p_n\}$ defined by AMP(n).

Proof. Consider the matrix factorization

$$\begin{pmatrix} \mu_0 & \mu_1 & . & . & . & \mu_n \\ \mu_1 & \mu_2 & . & . & . & \mu_{n+1} \\ . & & & & & . \\ . & & & & & . \\ \mu_{n-1} & \mu_n & . & . & . & \mu_{2n-1} \\ 1 & x & . & . & . & x^n \end{pmatrix} = \begin{pmatrix} p_1 & p_2 & . & . & p_n & 0 \\ p_1x_1 & p_2x_2 & . & . & p_nx_n & 0 \\ . & & & & & . \\ . & & & & & . \\ p_1x_1^{n-1} & & & & & 0 \\ 0 & 0 & . & . & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & x_1 & . & . & . & x_1^n \\ 1 & x_2 & . & . & . & x_2^n \\ . & & & & & . \\ . & & & & & . \\ 1 & x_n & . & . & . & x_n^n \\ 1 & x & . & . & . & x^n \end{pmatrix}$$

By assumption the p_i's are positive and the x_i's are distinct. Therefore by Lemma 1.1 and this factorization, one sees that

(2.2) $$\Delta_{n-1}\cdot p_n(x) = \det\begin{pmatrix} \mu_0 & \mu_1 & . & . & . & \mu_n \\ \mu_1 & \mu_2 & . & . & . & \mu_{n+1} \\ . & & & & & . \\ . & & & & & . \\ \mu_{n-1} & \mu_n & . & . & . & \mu_{2n-1} \\ 1 & x & . & . & . & x^n \end{pmatrix} = 0$$

holds if and only if x is one of the x_i's. ◊

Remark 2.1. Multiplying each column in the determinant (2.2) by x and substracting it from the next column one obtains that

(2.3) $$\Delta_{n-1}\cdot p_n(x) = \det(M_1 - xM_0),$$

with M_0, M_1 the moment matrices

(2.4) $$M_0 = \begin{pmatrix} \mu_0 & . & . & . & \mu_{n-1} \\ . & & & & \\ . & & & & \\ . & & & & \\ \mu_{n-1} & . & . & . & \mu_{2n-2} \end{pmatrix}, M_1 = \begin{pmatrix} \mu_1 & . & . & . & \mu_n \\ . & & & & \\ . & & & & \\ . & & & & \\ \mu_n & . & . & . & \mu_{2n-1} \end{pmatrix}.$$

It follows that the atoms $x_1, \dots, x_n$ are the *eigenvalues* of the symmetric matrix $M_0^{-\frac{1}{2}} M_1 M_0^{\frac{1}{2}}$.

As a consequence, computation simplifies in the case of symmetric random variables.

Corollary 2.1. The standard orthogonal polynomials of a symmetric random variable satisfy the simplified three term recurrence relation

$$(2.5) \qquad p_{n+1}(x) = x \cdot p_n(x) - c_n p_{n-1}(x), \quad n = 2,3,\dots, \quad c_n = \frac{\Delta_{n-2} \cdot \Delta_n}{\Delta_{n-1}^2},$$

where the starting values are given by $p_0(x) = 1, \quad p_1(x) = x, \quad p_2(x) = x^2 - 1.$

Proof. A finite atomic symmetric random variable, which solves AMP(n) in case $\mu_{2k+1} = 0$, k=0,1,..., must have a symmetric support and symmetrically distributed probabilities. Two cases are possible. If n=2r is even it has support $\{-x_1,\dots,-x_r, x_r,\dots, x_1\}$ and probabilities $\{p_1,\dots, p_r, p_r,\dots, p_1\}$, and if n=2r+1 it has support $\{-x_1,\dots,-x_r, 0, x_r,\dots, x_1\}$ and probabilities $\{p_1,\dots, p_r, p_0, p_r,\dots, p_1\}$. By Lemma 2.1 and the Fundamental Theorem of Algebra, the corresponding orthogonal polynomials are $p_n(x) = \prod_{i=1}^{r}(x^2 - x_i^2)$ if n=2r, and $p_n(x) = x \cdot \prod_{i=1}^{r}(x^2 - x_i^2)$ if n=2r+1. In both cases the leading coefficient t_{n-1}^n vanishes, and the result follows by Lemma 1.3. ◊

A main application of the algebraic moment problem, and thus also of orthogonal polynomials, will be the complete determination in Sections 4, 5 (respectively Section 6) of the sets of finite atomic random variables (respectively symmetric random variables) by given range and known moments up to the fourth order.

3. The classical orthogonal polynomials.

The developed results are illustrated at several fairly classical examples, which are known to be of great importance in the mathematical theory of special functions, and have applications in Physics, Engineering, and Computational Statistics.

3.1. Hermite polynomials.

The Hermite polynomials, defined by the recursion

$$(3.1) \qquad \begin{aligned} &H_{n+1}(x) = xH_n(x) - nH_{n-1}(x), \quad n = 1,2,\dots, \\ &H_0(x) = 1, \quad H_1(x) = x, \end{aligned}$$

are the standard orthogonal polynomials with respect to the standard normal random variable X with moments $\mu_{2n+1} = 0, \quad \mu_{2n} = \prod_{k=0}^{n-1}(2k+1)$. The orthogonality relation (O2) from Lemma 1.2 equals $E\left[H_n(X)^2\right] = \Delta_n / \Delta_{n-1} = n!$, which one finds in any book discussing orthogonal polynomials. It follows that $c_n = n$ and (3.1) follows by Corollary 2.1.

3.2. Chebyshev polynomials of the first kind.

The most famous polynomials of Chebyshev type are the $T_n(x) = \cos(n \cdot \arccos(x))$ and satisfy the recursion

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n = 1,2,\ldots, \qquad (3.2)$$
$$T_0(x) = 1, \quad T_1(x) = x.$$

They are orthogonal with respect to the symmetric weight function $f_X(x) = \dfrac{1}{\pi\sqrt{1-x^2}}$ defined on (-1,1), which has moments $\mu_{2n+1} = 0, \mu_{2n} = \dfrac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})n!} = \binom{n}{\frac{1}{2}}$, in particular $\mu_2 = \frac{1}{2}$. To get the standard orthogonal Chebyshev polynomials one must rescale the weight function to the standard probability density

$$f_S(x) = \frac{1}{\sqrt{2}} f_X\left(\frac{x}{\sqrt{2}}\right) = \frac{1}{\pi\sqrt{2-x^2}}, \quad x \in (-\sqrt{2}, \sqrt{2}),$$

with moments $\mu_{2n+1}^S = 0, \mu_{2n}^S = 2^n \cdot \binom{n}{\frac{1}{2}}$. This suggests the rescaling

$$T_n^S(x) = 2^{-(\frac{n}{2}-1)} \cdot T_n\left(\frac{x}{\sqrt{2}}\right), \quad n = 1,2,\ldots, \qquad (3.3)$$

where the inner rescaling of the argument yields $\mu_2 = 1$ and the outer one ensures a monic polynomial for all n=1,2,... . From the orthogonality relation (O2) in Lemma 1.2 one gets

$$E\left[T_n^S(S)^2\right] = 2^{-(n-2)} \cdot E\left[T_n(X)^2\right] = 2^{-(n-1)}. \qquad (3.4)$$

It follows that $c_n = \frac{1}{2}$, and by Corollary 2.1 the rescaled recursion

$$T_{n+1}^S(x) = xT_n^S(x) - \tfrac{1}{2}T_{n-1}^S(x), \quad n = 2,3,\ldots, \qquad (3.5)$$
$$T_0^S(x) = 1, \quad T_1^S(x) = x, \quad T_2^S(x) = x^2 - 1,$$

generates the standard orthogonal Chebyshev polynomials with respect to the symmetric random variable S with probability distribution

$$F_S(x) = \frac{1}{2} + \frac{1}{\pi}\arcsin\left(\frac{x}{\sqrt{\pi}}\right), \quad x \in \left(-\sqrt{2}, \sqrt{2}\right), \quad \arcsin(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

In this situation the support $\{x_1, \ldots, x_n\}$, which solves AMP(n), is given explicitely by the analytical formulas (zeros of the Chebyshev polynomials) :

$$x_i = \sqrt{2} \cdot \cos\left\{\left(\frac{2i-1}{n}\right) \cdot \frac{\pi}{2}\right\}, \quad i = 1,\ldots,n. \qquad (3.6)$$

3.3. Legendre polynomials.

The Legendre polynomials, defined by the recursion

(3.7)
$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x), \quad n = 1,2,\ldots,$$
$$P_0(x) = 1, \quad P_1(x) = x,$$

are orthogonal with respect to the symmetric uniform density $f_X(x) = \frac{1}{2}$ if $x \in (-1,1)$, $f_X(x) = 0$ if $|x| > 1$, which has moments $\mu_{2n+1} = 0$, $\mu_{2n} = \frac{1}{2n+1}$, in particular $\mu_2 = \frac{1}{3}$. The standard orthogonal Legendre polynomials with respect to

$$f_S(x) = \sqrt{3} \cdot f_X(\frac{x}{\sqrt{3}}), \quad x \in [-\sqrt{3}, \sqrt{3}],$$

with moments $\mu^S_{2n+1} = 0$, $\mu^S_{2n} = \frac{3^n}{2n+1}$, are obtained through rescaling by setting

(3.8)
$$\alpha_n P^S_n(x) = P_n(\frac{x}{\sqrt{3}}), \quad n = 1,2,3,\ldots .$$

Inserting this in the standardized recursion

(3.9)
$$P^S_{n+1}(x) = xP^S_n(x) - c_n P^S_{n-1}(x), \quad n = 2,3,\ldots,$$
$$P^S_0(x) = 1, \quad P^S_1(x) = x, \quad P^S_2(x) = x^2 - 1,$$

and comparing with (3.7) one obtains the relations

(3.10)
$$\sqrt{3} \cdot (n+1) \cdot \alpha_{n+1} = (2n+1) \cdot \alpha_n, \quad c_n = \frac{n\alpha_{n-1}}{(n+1)\alpha_{n+1}},$$

from which one deduces that

(3.11)
$$\alpha_n = \frac{\prod_{j=1}^{n-1}(2j+1)}{n! 3^{\frac{n}{2}}}, \quad n = 1,2,\ldots, \quad c_n = \frac{3n^2}{(2n-1)(2n+1)}, \quad n = 2,3,\ldots .$$

3.4. Other orthogonal polynomials.

The other classical orthogonal polynomials are the *Laguerre* and *generalized Laguerre* polynomials, known to be orthogonal with respect to a Gamma random variable, and the *Jacobi* polynomials. In these cases extensions of classical results have been obtained by Morton and Krall(1978). The *Bessel* polynomials, introduced by Krall and Frink(1949), are also known to be orthogonal with respect to a measure of bounded variation, wich however has failed all attempts to identify it completely (see Morton and Krall(1978) and Krall(1993)). It is interesting to note that the Chebyshev polynomials (1.4) are very generally orthogonal with respect to any *moment generating linear functional* w defined on polynomials such that

$\langle w, x^i \rangle = \mu_i$, $i = 0,1,...,$ as claimed by Krall(1978) (see also Maroni(1991)). Examples include a Cauchy representation and the explicit distributional linear functional

$$w(x) = \sum_{n=0}^{\infty} (-1)^n \mu_n \frac{\delta^{(n)}(x)}{n!}, \tag{3.12}$$

where $\delta^{(n)}$ denotes the Dirac function and its derivatives. A major problem consists to extend w to appropriate spaces for which it is a continuous linear functional. Another important subject is the connection with ordinary differential equations, for which we recommend the survey paper by Krall(1993).

4. Moment inequalities.

In Section 5 the complete algebraic-analytical structure of the sets of di- and triatomic standard random variables by given range and known moments to the fourth order will be given. As a preliminary step, it is important to state the conditions under which these sets are non-empty. This is done in Theorem 4.1 below.

Let $[a,b]$ be a given real interval, where the limiting cases $a=-\infty$ and/or $b=\infty$ are allowed. Let X be a random variable taking values in $[a,b]$, with finite mean μ and variance σ^2. Making use of the standard location-scale transformation $(X-\mu)/\sigma$, it suffices to consider standard random variables X with mean zero and variance one. Under this assumption, let $\gamma = \mu_3$ be the skewness and $\gamma_2 = \delta - 3$ the kurtosis, where $\delta = \mu_4$ is the fourth order central moment. It will be useful to set $\Delta = \delta - (\gamma^2 + 1)$.

The whole set of standard random variables taking values in $[a,b]$ is denoted by D(a,b). For fixed $n, k \geq 2$, one considers the subsets of standard n-atomic random variables with known moments up to the order k, which are defined and denoted by

$$D_k^{(n)}(a,b) = \{ X \in D(a,b) \text{ has a finite n-atomic ordered support } \{x_1,...,x_n\}, \; x_i \in [a,b], \text{ such that } E[X^j] = \mu_j, \; j=1,...,k \}. \tag{4.1}$$

In case the moment values are to be distinguished, the set $D_k^{(n)}(a,b)$ may be alternatively denoted by $D^{(n)}(a,b;\mu_1,...,\mu_k)$. The probabilities at the atoms x_i of a representative element $X \in D_k^{(n)}(a,b)$ will be denoted by $p_i^{(n)}$, or simply p_i in case the context is clear.

4.1. Structure of standard di- and triatomic random variables.

Since they appear as extremal random variables of the moment inequalities required in the existence results of Subsection 4.2, it is necessary to determine partially the structure of the standard di- and triatomic random variables $D_2^{(2)}(a,b)$ and $D_2^{(3)}(a,b)$.

Lemma 4.1. (*Representation of diatomic random variables*) A standard random variable $X \in D_2^{(2)}(a,b)$ is uniquely determined by its ordered support $\{x, y\}$, $a \leq x < y \leq b$, such that $xy=-1$. Moreover the probabilities at the atoms x, y take the values

(4.2) $$p_x^{(2)} = \frac{y}{y-x}, \quad p_y^{(2)} = \frac{-x}{y-x}.$$

Proof. By Lemma 2.1 (solution of AMP(2)) the atoms x, y are the distinct real zeros of the standard quadratic orthogonal polynomial

(4.3) $$p_2(x) = x^2 - \gamma x - 1,$$

where γ is a variable skewness parameter. The Vietà formulas imply the relations

(4.4) $$x + y = \gamma, \quad xy = -1.$$

This shows the first part of the affirmation. The formulas (4.2) for the probabilities follow from (2.1) of Lemma 2.1, that is from

(4.5) $$p_x^{(2)} = E\left[\frac{X-y}{x-y}\right], \quad p_y^{(2)} = E\left[\frac{X-x}{y-x}\right].$$

Since $x<y$ and $xy=-1$, one must have $x<0<y$, hence the probabilities are positive. ◊

Definition 4.1. For mathematical convenience it is useful to consider a real map j on $(-\infty,0)\cup(0,\infty)$, which maps a non-zero element x to its negative inverse $j(x) = -1/x$. Since $j^2(x) = x$ is the identity, the map j is called an *involution*, which for simplicity one denotes with a bar, that is one sets $\bar{x} = j(x)$. This notation will be used throughout.

Lemma 4.2. (*Representation of triatomic random variables*) A standard random variable $X \in D_2^{(3)}(a,b)$ is uniquely determined by its ordered support $\{x, y, z\}, a \le x < y < z \le b$, such that the following inequalities hold :

(4.6) $$x \le \bar{z} < 0, \quad \bar{z} \le y \le \bar{x}.$$

Moreover the probabilities at the atoms x, y, z take the values

(4.7) $$p_x^{(3)} = \frac{1+yz}{(y-x)(z-x)}, \quad p_y^{(3)} = \frac{-(1+xz)}{(y-x)(z-y)}, \quad p_z^{(3)} = \frac{1+xy}{(z-x)(z-y)}.$$

Proof. By Lemma 1.3, the standard cubic orthogonal polynomial equals

$$p_3(x) = (x + t_1^2 - t_2^3)p_2(x) - c_2 p_1(x).$$

Since $p_2(x) = x^2 - \gamma x - 1$, $t_1^2 = \gamma$, $c_2 = \frac{\Delta_0\Delta_2}{\Delta_1^2} = \Delta = \delta - (\gamma^2 + 1)$, one finds further

(4.8) $$p_3(x) = x^3 - t_2^3 x^2 + (\gamma t_2^3 - \delta)x - (\gamma - t_2^3).$$

By Lemma 2.1 (solution of AMP(3)), the atoms x, y, z of $X \in D_2^{(3)}(a,b)$ are the distinct real zeros of $p_3(x)=0$, where γ, δ, t_2^3 are viewed as variables. The Vietà formulas imply that

$$(4.9)\qquad \begin{aligned} &x+y+z=t_2^3, \\ &xy+xz+yz=\gamma t_2^3-\delta, \\ &xyz=\gamma-t_2^3. \end{aligned}$$

Using that $\Delta t_2^3=\mu_5-\gamma(\delta+1)$ by (1.20), the variable parameters, or equivalently the variable moments of order 3, 4, and 5 are thus determined by the atoms as follows :

$$(4.10)\qquad \begin{aligned} &\gamma=x+y+z+xyz \ \text{ (sum of first and third equation)}, \\ &\delta=\gamma(x+y+z)+xy+xz+yz \ \text{ (second and first equation)}, \\ &\mu_5=\gamma(\delta+1)+\Delta(x+y+z) \ \text{ (first equation)}. \end{aligned}$$

The expressions (4.7) for the probabilities follow from (2.1) of Lemma 2.1, that is from

$$(4.11)\qquad p_x^{(3)}=E\left[\frac{(X-y)(X-z)}{(x-y)(x-z)}\right],\ p_y^{(3)}=E\left[\frac{(X-x)(X-z)}{(y-x)(y-z)}\right],\ p_z^{(3)}=E\left[\frac{(X-x)(X-y)}{(z-x)(z-y)}\right].$$

By (4.10) the only restrictions about the atoms is that their probabilities (4.7) must be positive. Since $x<y<z$ one must have $1+yz\geq 0, 1+xz\leq 0, 1+xy\geq 0$. Since $xz\leq -1$ one must have $x<0<z$, hence also $\bar{z}<0<\bar{x}$. It follows that

$$(4.12)\qquad \begin{aligned} &\bar{z}\cdot(1+yz)=\bar{z}-y\leq 0 \quad \Leftrightarrow \quad \bar{z}\leq y \\ &\bar{z}\cdot(1+xz)=\bar{z}-x\geq 0 \quad \Leftrightarrow \quad x\leq \bar{z} \\ &\bar{x}\cdot(1+xy)=\bar{x}-y\geq 0 \quad \Leftrightarrow \quad y\leq \bar{x} \end{aligned}$$

Therefore the atoms satisfy the inequalities (4.6). ◊

Remark 4.1. Each of the boundary conditions $1+xy=0$ (z arbitrary), $1+xz=0$ (y arbitrary), $1+yz=0$ (x arbitrary), identifies the set $D_2^{(2)}(a,b)$ as a subset of $D_2^{(3)}(a,b)$.

4.2. Moment conditions for the existence of random variables.

It is now possible to state the conditions under which there exist standard random variables defined on a given range with known moments up to the fourth order.

Theorem 4.1. (*Moment inequalities*) There exist non-degenerate standard random variables on $[a,b]$ with given moments to order four if and only if the following moment inequalities hold :

$$(4.13)\qquad a<0<b \ \text{ (inequalities on the mean } \mu=0)$$

$$(4.14)\qquad 1+ab\leq 0 \ \text{ (inequality on the variance } \sigma^2=1)$$

(4.15) $\gamma_{\min} = a + \bar{a} \le \gamma \le \gamma_{\max} = b + \bar{b}$ (inequalities on the skewness)

If 1+ab<0 then one has

(4.16) $0 \le \Delta \le \left(\frac{ab}{1+ab}\right)(\gamma - \gamma_{\min})(\gamma_{\max} - \gamma)$ (inequalities between skewness and kurtosis)

The above inequalities are sharp and attained as follows :

(4.17) 1+ab=0 for a diatomic random variable with support $\{a, \bar{a}\}$ provided $b = \bar{a}$

(4.18) $\gamma = \gamma_{\min}$ for a diatomic random variable with support $\{a, \bar{a}\}$

(4.19) $\gamma = \gamma_{\max}$ for a diatomic random variable with support $\{\bar{b}, b\}$

(4.20) $\Delta = \delta - (\gamma^2 + 1) = 0$ for a diatomic random variable with support $\{c, \bar{c}\}$, with $c = \frac{1}{2}(\gamma - \sqrt{4+\gamma^2})$

(4.21) $\Delta = \left(\frac{ab}{1+ab}\right)(\gamma - \gamma_{\min})(\gamma_{\max} - \gamma)$ for a triatomic random variable with support $\{a, \varphi(a,b), b\}$, with $\varphi(a,b) = \frac{\gamma - (a+b)}{1+ab}$

Proof. The real inequalities follow by taking expectations in the following random inequalities, which are valid with probability one for all $X \in D(a,b)$:

(4.13') $a \le X \le b$

(4.14') $(b - X)(X - a) \ge 0$

(4.15') $(X - \bar{a})^2 \cdot (X - a) \ge 0, \quad (X - \bar{b})^2 \cdot (b - X) \ge 0$

(4.16') $(X - c)^2 \cdot (X - \bar{c})^2 \ge 0, \quad (b - X)(X - a)(X - \varphi(a,b))^2 \ge 0$

For a non-degenerate random variable, the inequalities in (4.13) must be strict. Sharpness of the inequalities is verified without difficulty. For example (4.18), (4.19) follow directly from the relations (4.4) in the proof of Lemma 4.1. To show (4.21) one must use the first relation in (4.10) for the skewness of a triatomic random variable, which yields the formula for the middle atom $\varphi(a,b) = \frac{\gamma - (a+b)}{1+ab}$. ◊

The inequalities in (4.15) yield the minimal and maximal values of the skewness for standard random variables on $[a,b]$. The corresponding extremal values of the kurtosis can be determined from the inequalities in (4.16).

Corollary 4.1. The extremal values of the kurtosis for standard random variables on $[a,b]$ are given by

(4.22) $$\gamma_{2,\min} = -2, \text{ attained when } \gamma = 0$$

(4.23) $$\gamma_{2,\max} = \frac{1}{4}\left(\frac{ab}{1+ab}\right)(\gamma_{\min} - \gamma_{\max})^2 - \frac{1}{4}ab(\gamma_{\min} + \gamma_{\max})^2 - 2,$$
attained when $\gamma = -\frac{1}{2}ab(\gamma_{\min} + \gamma_{\max})$.

Proof. Let $\delta^*_{\min} = \delta_{\min}(\gamma^*)$ be the minimal value of δ for which the lower bound $\Delta = \delta - (\gamma^2 + 1) = 0$ is attained. It is immediate that $\gamma^* = 0$, hence $\gamma_{2,\min} = \delta^*_{\min} - 3 = -2$. Similarly let $\delta^*_{\max} = \delta_{\max}(\gamma^*)$ be the maximal value of δ for which the upper bound in (4.16) is attained. It suffices to maximize the univariate function of the skewness

(4.24) $$\delta(\gamma) = \gamma^2 + 1 + \left(\frac{ab}{1+ab}\right)\left(-\gamma^2 + (\gamma_{\min} + \gamma_{\max})\gamma - \gamma_{\min}\gamma_{\max}\right).$$

Its first and second derivatives are

(4.25) $$\delta'(\gamma) = 2\gamma + \left(\frac{ab}{1+ab}\right)(\gamma_{\min} + \gamma_{\max} - 2\gamma)$$

(4.26) $$\delta''(\gamma) = \frac{2}{1+ab} \le 0$$

It follows that $\delta(\gamma)$ is local maximal at $\delta'(\gamma^*) = 0$, hence $\gamma^* = -\frac{1}{2}ab(\gamma_{\min} + \gamma_{\max})$. ◊

Examples 4.1.

(i) Suppose that $[a,b] = [-1,b]$ with $b \ge 1$. Then the minimal kurtosis $\gamma_{2,\min} = -2$ is attained for the diatomic random variable with support $\{-1,1\}$ and probabilities $\{\frac{1}{2}, \frac{1}{2}\}$.

(ii) Suppose that $[a,b] = [-b,b]$ with $b \ge 1$. Then the maximal kurtosis $\gamma_{2,\max} = b^2 - 3 \ge -2$ is attained for the triatomic random variable with support $\{-b, 0, b\}$ and probabilities $\left\{\frac{1}{2b^2}, \frac{b^2-1}{b^2}, \frac{1}{2b^2}\right\}$.

4.3. Other moment inequalities.

Clearly Theorem 4.1 is only one, although a most important one, among the numerous results in the vast subject of "moment inequalities". To stimulate further research in this area, let us illustrate with a survey of various possibilities, which are closely related to the results of Section 4.2, however without giving any proof.

Example 4.2.

A classical *problem of Chebyshev* consists to find the extremal bounds of the expected value

(4.27) $$E[g(X)] = \int_a^b g(x)dF(x),$$

where g(x) is a positive function, and X is a random variable defined on $[a,b]$ with given properties, say the first k moments known. A general solution to this problem is given by the Markov-Krein Theorem (see e.g. Karlin and Studden(1966)). For example, if the k=2n-1 first moments are known and $g(x) = x^{2n}$, then the moment of order 2n satisfies the best lower bound (in the notation of Lemma 1.4) :

(4.28) $$\mu_{2n} \geq \mu_{2n,\min} = \frac{1}{\Delta_{n-1}} \cdot m^{(n)^T} \cdot \overline{M}^{(n)} \cdot m^{(n)},$$

and the minimum is attained at the unique n-atomic random variable, which solves AMP(n) and whose atoms are the zeros of the orthogonal polynomial $p_n(x) = 0$. In the special case n=2 one obtains with $\mu_1 = 0, \mu_2 = 1, \mu_3 = \gamma \in [a + \overline{a}, b + \overline{b}]$, the minimum value

(4.29) $$\mu_{4,\min} = (1 \quad \gamma)^T \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \gamma \end{pmatrix} = 1 + \gamma^2,$$

which is attained at the unique $X \in D_3^{(2)}(a,b)$ with support $\{c, \overline{c}\}$. Clearly one recovers (4.20) and the fact $\Delta \geq 0$ in (4.16). Using the main result (4.28), the recent paper by Barnes(1995) relates the range $[x_1, x_n]$ of an arbitrary standard n-atomic random variable $X \in D_{2k}^{(n)}(-\infty, \infty)$ with ordered support $\{x_1, ..., x_n\}$ to certain determinants of moment matrices. He obtains an interesting general result, which turns out to be a refinement of the (variance) inequality $1 + x_1 x_n \leq 0$, which can be viewed as a restatement of the moment condition (4.14). For example, by known skewness and kurtosis, that is k=2, one obtains the sharper (variance) inequality

(4.30) $$1 + x_1 x_n \leq -\left(\frac{2\Delta}{4 + \gamma^2}\right).$$

Example 4.3.

It is also possible to refine the main moment inequalities of Theorem 4.1 by imposing additional geometric restrictions, for example symmetry, unimodality, etc. For unimodal distributions the best lower bound for the kurtosis in dependence on the skewness has been given by Johnson and Rogers(1951). The corresponding best upper bound has only been found recently by Teuscher and Guiard(1995). Sharp inequalities between skewness and kurtosis for symmetric unimodal random variables are described in Rohatgi and Székely(1989).

Example 4.4.

An important area for future research is the extension of the various existing *univariate* moment inequalities to the *multivariate* context. For example, the maximal *sample* skewness and kurtosis have been determined by Wilkins(1944) and Picard(1951) (see also Dalén(1987)).

Let $\mathbf{X}=(X_1,...,X_n)$ be a *random sample* of size n consisting of independent and identically distributed random variables, and let $\overline{X}=\frac{1}{n}\sum_{i=1}^{n}X_i$, $S=\frac{1}{n}\sum_{i=1}^{n}(X_i-\overline{X})^2$ be the random sample mean and variance. Making the standard transformation $\frac{X_i-\overline{X}}{\sqrt{S}}$ one may, without loss of generality, assume that $\overline{X}=0$, S=1. A *sample*, which is a realization of a random sample, is simply denoted with small letters by $\mathbf{x}=(x_1,...,x_n)$. Denote by SS_n the set of all standardized samples such that $\overline{x}=0$, s=1, and by $\gamma(\mathbf{x})$, $\gamma_2(\mathbf{x})=\delta(\mathbf{x})-3$ the sample skewness and kurtosis of the sample $\mathbf{x}$. Then the maximal sample skewness equals

(4.31) $$\max_{x\in SS_n}\{\gamma(x)\}=\frac{n-2}{\sqrt{n-1}},$$

and is attained at the "extremal" sample

(4.32) $$\mathbf{x}^*=(\sqrt{n-1},-\frac{1}{\sqrt{n-1}},-\frac{1}{\sqrt{n-1}},...,-\frac{1}{\sqrt{n-1}}).$$

Similarly the maximal sample kurtosis is determined by

(4.33) $$\max_{x\in SS_n}\{\delta(x)\}=\frac{n^2-3n+3}{n-1},$$

and is attained at the same "extremal" sample $\mathbf{x}^*$. Moreover the relation $\delta(\mathbf{x}^*)=\gamma(\mathbf{x}^*)^2+1$ shows that the equality (4.20) holds in this extreme situation. The maximal values of the sample moments of higher order have also been determined by Picard.

5. Structure of finite atomic random variables by known moments to order four.

As a probabilistic application of the theory of orthogonal polynomials, the complete algebraic-analytical structure of the sets of finite atomic random variables $D_2^{(2)}(a,b)$, $D_3^{(3)}(a,b)$ and $D_4^{(3)}(a,b)$ is derived. The structure of a special subset of $D_4^{(4)}(a,b)$ needed later in Section III.2 is also displayed.

A minimal number of notions taken from the field of Abstract Algebra is required in our presentation. Under an *algebraic set* we mean a set together with one or more operations acting on the elements of this set. Two sets are called *isomorphic* if there exists a one-to-one correspondence between the elements of these sets. The *symbol* $\cong$ is used when there is an isomorphism between two sets.

Theorem 5.1. (*Characterization of standard diatomic random variables on* $[a,b]$) Suppose that a<0<b and 1+ab≤0. Let $S_2(a,b)=\{[a,\overline{b}],\bar{\cdot}\}$ be the algebraic set consisting of the real interval $[a,\overline{b}]$ and the strictly increasing involution mapping $\bar{\cdot}$ from $[a,\overline{b}]$ to $[\overline{a},b]$,

which maps x to $\bar{x} = -1/x$. Then the set $D_2^{(2)}(a,b)$ of all standardized diatomic random variables defined on $[a,b]$ is isomorphic to the algebraic set $S_2(a,b)$, that is there exists a one-to-one correspondence between these sets. More precisely each $X \in D_2^{(2)}(a,b)$ is uniquely determined by its support $\{x, \bar{x}\}, x \in [a, \bar{b}]$, and the probabilities are given by

$$(5.1) \qquad p_x^{(2)} = \frac{\bar{x}}{\bar{x}-x} = \frac{1}{1+x^2}, \quad p_{\bar{x}}^{(2)} = \frac{-x}{\bar{x}-x} = \frac{x^2}{1+x^2}.$$

Proof. By Theorem 4.1 the conditions (4.13) and (4.14) are required. Let $X \in D_2^{(2)}(a,b)$ has support $\{x,y\}, a \le x < y \le b$. By Lemma 4.1 and its proof one must have $y = \bar{x}$ and $x<0<y$. If $x \in (\bar{b},0)$ then $y = \bar{x} \in (b,\infty)$ and the support $\{x,\bar{x}\}$ is not feasible. Therefore one must have $x \in [a,\bar{b}]$. Since the involution $\bar{\cdot}$ is strictly increasing, each X is uniquely determined by an atom $x \in [a,\bar{b}]$, which means that the sets $D_2^{(2)}(a,b)$ and $S_2(a,b)$ are isomorphic. The formulas (5.1) are a restatement of (4.2). ◊

Remark 5.1. In applications the following limiting cases of one- and double-sided infinite intervals are often encountered :

$$(5.2) \qquad \begin{aligned} D_2^{(2)}(a,\infty) &= \{\, X \in D([a,\infty)) : X \text{ has support } \{x,\bar{x}\}, x \in [a,0) \,\} \\ &\cong S_2(a,\infty) = \{[a,0); \bar{\cdot}\} \end{aligned}$$

$$(5.3) \qquad \begin{aligned} D_2^{(2)}(-\infty,\infty) &= \{\, X \in D((-\infty,\infty)) : X \text{ has support } \{x,\bar{x}\}, x \in (-\infty,0) \,\} \\ &\cong S_2(-\infty,\infty) = \{(-\infty,0); \bar{\cdot}\} \end{aligned}$$

Theorem 5.2. (*Characterization of standard triatomic random variables on* $[a,b]$ *with skewness* γ) Suppose the inequalities (4.13), (4.14) and (4.15) are fulfilled. Let $S_3(a,b) = \{[a,c]x[\bar{c},b]; \varphi\}$ be the algebraic set consisting of the product of real intervals $[a,c]x[\bar{c},b]$, $c = \frac{1}{2}(\gamma - \sqrt{4+\gamma^2})$, and the mapping φ from $[a,c]x[\bar{c},b]$ to $[c,\bar{c}]$, which maps (x,z) to $\varphi(x,z) = \frac{\gamma-(x+z)}{1+xz}$. Then the set $D_3^{(3)}(a,b)$ is isomorphic to the algebraic set $S_3(a,b)$. More precisely each $X \in D_3^{(3)}(a,b)$ is uniquely determined by its support $\{x, \varphi(x,z), z\}, (x,z) \in [a,c]x[\bar{c},b]$, and the probabilities are given by the formulas

$$(5.4) \qquad \begin{aligned} p_x^{(3)} &= \frac{1+\varphi(x,z)z}{(\varphi(x,z)-x)(z-x)}, \quad p_{\varphi(x,z)}^{(3)} = \frac{-(1+xz)}{(\varphi(x,z)-x)(z-\varphi(x,z))}, \\ p_z^{(3)} &= \frac{1+x\varphi(x,z)}{(z-x)(z-\varphi(x,z))} \end{aligned}$$

Proof. By Theorem 4.1 the conditions (4.13) to (4.15) are required. Let $X \in D_3^{(3)}(a,b)$ has support $\{x,y,z\}, a \le x < y < z \le b$. The proof of Lemma 4.2 (with γ now fixed instead of variable) shows the relations (4.10). The first one, which is equivalent with the condition $E[p_3(X)] = E[(X-x)(X-y)(X-z)] = \gamma - (x+y+z) - xyz = 0$ in (1.3), implies that $y = \varphi(x,z) = \frac{\gamma-(x+z)}{1+xz}$, while the other two determine δ and μ_5 in terms of γ and the atoms x, z. The formulas (5.4) are a restatement of (4.7). The inequalities (4.6) must also hold. Using that $y=\frac{\gamma-(x+z)}{1+xz}$, one sees that the inequalities $\bar{z}\le y\le \bar{x}$ are equivalent with

(5.5) $$z^2-\gamma z-1\ge 0,\quad x^2-\gamma x-1\ge 0.$$

Let $c=\frac{1}{2}(\gamma-\sqrt{4+\gamma^2})$, $\bar{c}=\frac{1}{2}(\gamma+\sqrt{4+\gamma^2})$ be the zeros of the quadratic equations. Since x<0<z one must have $(x,z)\in[a,c]x[\bar{c},b]$. Since $\bar{z}\le y\le \bar{x}$ one gets $y=\varphi(x,z)\in[c,\bar{c}]$. ◊

Remarks 5.2.

(i) One shows that for fixed $x\in[a,c]$ the function $\varphi(x,z)$ is strictly increasing in $z\in[\bar{c},b]$, and for fixed $z\in[\bar{c},b]$ the function $\varphi(x,z)$ is strictly increasing in $x\in[a,c]$.

(ii) One has the following limiting cases :

(5.6) $D_3^{(3)}(a,\infty)=\{\, X\in D([a,\infty)) :$ X has support $\{x,\varphi(x,z),z\},(x,z)\in[a,c]x[\bar{c},\infty)$, and $E[X^3]=\gamma\ \}$
$\cong S_3(a,\infty)=\{[a,c]x[\bar{c},\infty);\varphi\}$

(5.7) $D_3^{(3)}(-\infty,\infty)=\{\, X\in D((-\infty,\infty)) :$ X has support $\{x,\varphi(x,z),z\}$, $(x,z)\in(-\infty,c]x[\bar{c},\infty)$, and $E[X^3]=\gamma\ \}$
$\cong S_3(-\infty,\infty)=\{(-\infty,c]x[\bar{c},\infty);\varphi\}$

(iii) Under the assumption that a<0<b and 1+ab≤0, it becomes now clear that Lemma 4.2 classifies the set $D_2^{(3)}(a,b)$ of all standard triatomic random variables on $[a,b]$.

Theorem 5.3. (*Characterization of standard triatomic random variables on* $[a,b]$ *with skewness* γ *and kurtosis* $\gamma_2=\delta-3$) Suppose the moment inequalities of Theorem 4.1 are fulfilled. Let $S_4(a,b)=\{[a,b^*];\varphi,^*\}$ be the algebraic set consisting of the real interval $[a,b^*]$, the map $\varphi(x,z)=\frac{\gamma-(x+z)}{1+xz}$, and the strictly increasing involution mapping * from $[a,b^*]$ to $[a^*,b]$, which maps x to

(5.8) $$x^*=\frac{1}{2}\left\{\frac{C(x)-\sqrt{C(x)^2+4q(x)D(x)}}{q(x)}\right\},\text{ where}$$

(5.9) $$q(x)=1+\gamma x-x^2,\quad C(x)=\gamma q(x)+\Delta x,$$
$$D(x)=\Delta+q(x),\quad \Delta=\delta-(\gamma^2+1).$$

Then the set $D_4^{(3)}(a,b)$ is isomorphic to the algebraic set $S_4(a,b)$. More precisely each $X \in D_4^{(3)}(a,b)$ is uniquely determined by its support $\{x,y,z\} := \{x,\varphi(x,x^*),x^*\}, x \in [a,b^*]$, and the probabilities are given by the formulas

$$(5.10) \quad p_x^{(3)} = \frac{\Delta}{q(x)^2 + \Delta(1+x^2)}, \quad p_y^{(3)} = \frac{\Delta}{q(y)^2 + \Delta(1+y^2)}, \quad p_z^{(3)} = \frac{\Delta}{q(z)^2 + \Delta(1+z^2)}.$$

Proof. Clearly the moment inequalities of Theorem 4.1 are required. Let $X \in D_4^{(3)}(a,b)$ has support $\{x,y,z\}, a \le x < y < z \le b$. The proof of Lemma 4.2 (with γ, δ now fixed) shows the relations (4.10). The first one, which is equivalent with the condition $E[p_3(X)] = E[(X-x)(X-y)(X-z)] = \gamma - (x+y+z) - xyz = 0$ in (1.3), implies that $y = \varphi(x,z) = \frac{\gamma-(x+z)}{1+xz}$. Inserted into the second one, which is equivalent with the condition $E[Xp_3(X)] = E[X(X-x)(X-y)(X-z)] = \delta - (x+y+z)\gamma + (xy+xz+yz) = 0$, one obtains that z is solution of the quadratic equation $q(x)z^2 - C(x)z - D(x) = 0$, hence $z=x^*$ as defined in (5.8). One verifies that the map x^* is strictly increasing in x, and that $(x^*)^*=x$, which is the defining property of an involution. Since $D_4^{(3)}(a,b) \subset D_3^{(3)}(a,b)$ one knows by Theorem 5.2 that $(x,z) \in [a,c] \mathrm{x} [\bar{c},b]$. However if $x \in (b^*,c]$ then $z = x^* \in [b,\infty)$ and the support $\{x,\varphi(x,x^*),x^*\}$ is not feasible. Therefore one must have $x \in [a,b^*]$. Since * is a strictly increasing involution, the image of $[a,b^*]$ is $[a^*,b]$. It remains to show the validity of the formulas (5.10). One knows that $y = \varphi(x,x^*)$, $z = x^*$ are solutions of the quadratic equation $q(x)z^2 - C(x)z - D(x) = 0$. One calculates

$$\begin{aligned}
&(y-x)(z-x) = x^2 - (y+z)x + yz \\
&= x^2 - \frac{C(x)}{q(x)}x - \frac{D(x)}{q(x)} = -\frac{q(x)^2 + \Delta(1+x^2)}{q(x)}, \\
&1 + yz = -\frac{\Delta}{q(x)}.
\end{aligned}$$

From (4.7) one gets $p_x^{(3)} = \frac{1+yz}{(y-x)(z-x)} = \frac{\Delta}{q(x)^2 + \Delta(1+x^2)}$. The same calculations hold making cyclic permuations of x, y, z. This shows (5.10). ◊

Remarks 5.3. The following limiting cases are of interest :

$$\begin{aligned}
(5.11) \quad D_4^{(3)}(a,\infty) = \{\, & X \in D([a,\infty)) : X \text{ has support } \{x,\varphi(x,x^*),x^*\}, x \in [a,c], \\
& \text{and } E[X^3] = \gamma,\ E[X^4] = \delta \,\} \\
\cong S_4(a,\infty) = \{ & [a,c]; \varphi, ^* \}
\end{aligned}$$

$$\begin{aligned}
(5.12) \quad D_4^{(3)}(-\infty,\infty) = \{\, & X \in D((-\infty,\infty)) : X \text{ has support } \{x,\varphi(x,x^*),x^*\}, x \in (-\infty,c], \\
& \text{and } E[X^3] = \gamma,\ E[X^4] = \delta \,\} \\
\cong S_4(-\infty,\infty) = \{ & (-\infty,c]; \varphi, ^* \}
\end{aligned}$$

The algebraic-analytical structure of the following subset of $D_4^{(4)}(a,b)$ will be needed in Section III.2 :

(5.13) $$D_{4,2}^{(4)}(a,b)=\{X\in D(a,b) \text{ has a 4-atomic support } \{a,x,y,b\},\ a<x<y<b \text{ such that } E[X^3]=\gamma,\ E[X^4]=\delta\ \}$$

Note that the additional subscript 2 indicates that 2 atoms of the support, here the end points of the interval, are left fixed.

Theorem 5.4. (*Characterization of the set of standard four atomic random variables*) Suppose the moment inequalities (4.13) to (4.15) are fulfilled. Let $S_{4,2}^{(4)}(a,b)=\{[b^*,\varphi(a,a^*)];\psi\}$ be the algebraic set consisting of the real interval $[b^*,\varphi(a,a^*)]$, and the strictly increasing involution mapping ψ from $[b^*,\varphi(a,a^*)]$ to $[\varphi(b,b^*),a^*]$, which maps x to

(5.14) $$\psi(x;a,b)=\frac{\delta-(a+b+x)\gamma+(ab+ax+bx)}{\gamma-(a+b+x)-abx}.$$

Then the set $D_{4,2}^{(4)}(a,b)$ is isomorphic to the algebraic set $S_{4,2}^{(4)}(a,b)$. More precisely each $X\in D_{4,2}^{(4)}(a,b)$ is uniquely determined by its support $\{x_1,x_2,x_3,x_4\}=\{a,x,\psi(x;a,b),b\}$, $x\in[b^*,\varphi(a,a^*)]$, and the probabilities are given by the formulas

(5.15) $$p_{x_i}^{(4)}=E\left[\prod_{j\neq i}\left(\frac{X-x_j}{x_i-x_j}\right)\right],\quad i=1,...,4.$$

In particular one has $p_a^{(4)}=p_a^{(4)}(x,b)$, $p_x^{(4)}=p_x^{(4)}(a,b)$, $p_b^{(4)}=p_b^{(4)}(a,x)$, where one sets

(5.16) $$p_z^{(4)}(u,v)=\frac{\gamma-(u+v+\psi(x;a,b))-uv\psi(x;a,b)}{(z-u)(z-v)(z-\psi(x;a,b))}.$$

Moreover if $x=b^*$ then $\psi(b^*;a,b)=\varphi(b,b^*)$, $p_a^{(4)}=0$, and if $x=\varphi(a,a^*)$ then $\psi(\varphi(a,a^*);a,b)=a^*$, $p_b^{(4)}=0$.

Proof. Let $X\in D_{4,2}^{(4)}(a,b)$ has support $\{a,x,y,b\}, a\le x<y\le b$. From the moment condition $E[p_4(X)]=E[(X-a)(X-x)(X-y)(X-b)]=0$, valid by (1.3), one gets $y=\psi(x;a,b)$. Suppose that $b^*\le x\le\varphi(a,a^*)<\varphi(a,b)$. Then the denominator of $\psi(x;a,b)$ is strictly negative. Since

$$\psi(x;a,b)=\frac{(\gamma-(a+b))x-(\delta-(a+b)\gamma+ab)}{(1+ab)x-(\gamma-(a+b))},$$

it follows from

$$(\delta-(a+b)\gamma+ab)(1+ab)>(\gamma-(a+b))^2$$

that $\psi(x;a,b)$ is strictly increasing in x. The involution property is immediately verified. One checks that $\psi(b^*;a,b)=\varphi(b,b^*)$ and $\psi(\varphi(a,a^*);a,b)=a^*$. Thus ψ maps

$[b^*, \varphi(a,a^*)]$ to $[\varphi(b,b^*), a^*]$. To be feasible the probabilities must be non-negative. One shows that $p_a^{(4)} \ge 0$ only if $x \ge b^*$, $p_b^{(4)} \ge 0$ only if $x \le \varphi(a,a^*)$, $p_x^{(4)} \ge 0$ only if $y = \psi(x;a,b) \ge \varphi(a,b)$, and $p_y^{(4)} \ge 0$ only if $x \le \varphi(a,b)$. It follows that $\{a, x, \psi(x;a,b), b\}$ is uniquely determined by $x \in [b^*, \varphi(a,a^*)]$. ◊

By the same method it is certainly possible to obtain structural results for the sets $D_k^{(n)}(a,b)$ for higher orders of n and k. In concrete applications one has in general to rely on numerical methods to solve the orthogonal polynomial equations $p_n(x) = 0$. Our numerical experience with the Newton-Maehly algorithm (e.g. Stoer(1983), p.258-59), which is a suitable modification of the ordinary Newton algorithm, has been satisfying for solving AMP(n), n=6,...,15, when using a "quadruple" precision floating-point arithmetic.

6. Structure of finite atomic symmetric random variables by known kurtosis.

Consider a symmetric random variable X taking values in the interval $[A,B]$. Let $C = \frac{1}{2}(A+B)$ be its symmetry center, which equals the mean $\mu = C$. Then the variance necessarily satisfies the inequality $0 \le \sigma^2 \le E^2 = (\mu - A)(B - \mu) = \frac{1}{4}(B-A)^2$, which is the non-standardized version of the moment inequality (4.14). Making use of the standard transformation $Z = (X - \mu)/\sigma$, one sees that Z is standard symmetric with mean zero, variance one and range $[-a,a]$, $a = E/\sigma \ge 1$. Therefore it suffices to discuss the standardized case. Clearly the moments of odd order of standard symmetric random variables vanish, in particular $\gamma = 0$. Let $\gamma_2 = \delta - 3$ be the kurtosis, where $\delta = \mu_4$, and set $\Delta = \delta - 1$, which is the same parameter as in Sections 4 and 5 because $\gamma = 0$. By (4.16) one assumes that $\delta \ge 1$.

The whole set of standard symmetric random variables taking values in $[-a,a]$ is denoted by $D_S(a)$. For fixed $n,k \ge 2$ one considers the subsets of standard n-atomic symmetric random variables by given range $[-a,a]$ and known moments of even order up to the order $2k$, which are defined and denoted by

(6.1) $$D_{S,2k}^{(n)}(a;\mu_4,...,\mu_{2k}) = \{X \in D_S(a) \text{ has a finite } n\text{-atomic symmetric ordered support of the form } \{-x_m,...,-x_1,x_1,...,x_m\} \text{ if } n=2m \text{ and } \{-x_m,...,-x_1,0,x_1,...,x_m\} \text{ if } n=2m+1,\ x_i \in [-a,a], \text{ such that } E[X^{2j}] = \mu_{2j}, j=1,...,k\}$$

In case the moments are clear from the context, the simpler notation $D_{S,2k}^{(n)}(a)$ will be used. In Section III.4 the structure of the sets $D_{S,4}^{(2)}(a;\delta)$, $D_{S,4}^{(3)}(a;\delta)$, $D_{S,4}^{(4)}(a;\delta)$ as well as of the following subset of $D_{S,4}^{(5)}(a;\delta)$ will be needed :

(6.2) $$D_{S,4,2}^{(5)}(a;\delta) = \{X \in D_S(a) \text{ has a 5-atomic symmetric support } \{-a,-x,0,x,a\}, 0 < x < a, \text{ and } E[X^4] = \delta\}.$$

Theorem 6.1. (*Parametrization of standard symmetric random variables*) Suppose that $a \ge 1, \delta \ge 1$. The considered sets of finite atomic symmetric random variables are completely described as follows :

(S1) The set $D_{S,4}^{(2)}(a;\delta)$ is non-empty if and only if $\delta = 1$, and consists of the only diatomic random variable with support $\{-1,1\}$ and probabilities $\{\frac{1}{2},\frac{1}{2}\}$.

(S2) The set $D_{S,4}^{(3)}(a;\delta)$ is non-empty if and only if $\delta \in (1, a^2]$, and consists of the only triatomic random variable with support $\{-\sqrt{\delta}, 0, \sqrt{\delta}\}$ and probabilities $\left\{\frac{1}{2\delta}, \frac{\delta-1}{\delta}, \frac{1}{2\delta}\right\}$.

(S3) For each $\delta \in (1, a^2]$ the elements of the set $D_{S,4}^{(4)}(a;\delta)$ consist of the four atomic random variables with support $\{-x, -x^s, x^s, x\}$, $x \in (\sqrt{\delta}, a]$, and probabilities
$p_{-x}^{(4)} = p_x^{(4)} = \frac{1}{2}\cdot\frac{\Delta}{\Delta+(x^2-1)^2}$, $p_{-x^s}^{(4)} = p_{x^s}^{(4)} = \frac{1}{2}\cdot\frac{(x^2-1)^2}{\Delta+(x^2-1)^2}$,
where the formula $x^s = \sqrt{\frac{x^2-\delta}{x^2-1}}$ defines a strictly increasing involution mapping $(\cdot)^s$, which maps the interval $(\sqrt{\delta}, a]$ to the interval $(0, a^s]$. Alternatively x^s may be defined as the non-negative solution of the Δ-equation $[1-(x^s)^2]\cdot[x^2-1] = \Delta$.

(S4) For each $\delta \in (1, a^2]$ the elements of the set $D_{S,4,2}^{(5)}(a;\delta)$ consist of the five atomic random variables with support $\{-a, -\sqrt{\delta-x}, 0, \sqrt{\delta-x}, a\}$, $x \in (0, \delta-(a^s)^2)$, and probabilities
$p_{-a}^{(5)} = p_a^{(5)} = \frac{1}{2}\cdot\frac{x}{a^2\cdot(x+a^2-\delta)}$, $p_{-\sqrt{\delta-x}}^{(5)} = p_{\sqrt{\delta-x}}^{(5)} = \frac{1}{2}\cdot\frac{a^2-\delta}{(\delta-x)\cdot(x+a^2-\delta)}$,
$p_0^{(5)} = \frac{(a^2-1)\cdot(\delta-x-(a^s)^2)}{a^2\cdot(\delta-x)}$, with a^s as defined under (S3).

Proof. By Lemma 2.1 the atoms and probabilities of a finite atomic random variable solve an algebraic moment problem. In the special cases of standard n-atomic symmetric random variables, n=2,3,4,5, the atoms are solutions of the standard orthogonal polynomials obtained from the recurrence relation (2.5), which yields successively

(6.3) $$p_2(x) = x^2 - 1,$$

(6.4) $$p_3(x) = x\cdot(x^2-\delta), \quad \delta = \mu_4,$$

(6.5) $$p_4(x) = x^4 - t_2(1-x^2) - \delta, \quad t_2 = \frac{\delta-\alpha}{\delta-1}, \quad \alpha = \mu_6,$$

(6.6) $$p_4(x) = x\cdot(x^4 + t_3x^2 - \delta t_3 - \alpha), \quad t_3 = \frac{\beta-\delta\alpha}{\delta^2-\alpha}, \quad \beta = \mu_8, \quad \alpha = \mu_6.$$

The probabilities taken at the atoms are obtained from the formula (2.1). A more detailed derivation of the various cases follows.

Case (S1) :

Since the fourth order moment of the diatomic random variable with support $\{-1,1\}$ is $\delta-1$, the affirmation is immediate.
Case (S2) :

Clearly the only possible support is $\{-\sqrt{\delta},0,\sqrt{\delta}\}$. The restriction about δ follows from the range restriction.

Case (S3) :

Let $X \in D_{S,4}^{(4)}(a;\delta)$ has support $\{-x,-y,y,x\}, 0<y<x\le a$. From the condition $p_4(x)=0$ and (6.5), one gets the relation

(6.7) $$t_2=\frac{\delta-\alpha}{\delta-1}=\frac{x^4-\delta}{1-x^2},$$

which determines by given x the value of the moment $\alpha=\mu_6$. The atomic squares x^2, y^2, which are zeros of the quadratic equation $p_4(x)=0$ in x^2, satisfy by Vietà the relations

(6.8) $$x^2+y^2=-t_2=\frac{x^4-\delta}{x^2-1},\quad x^2y^2=-t_2-\delta=\frac{x^2\cdot(x^2-\delta)}{x^2-1},$$

hence $y^2=\dfrac{x^2-\delta}{x^2-1}$. Since $\delta\in(1,a^2]$ and the range of X is $[-a,a]$, one has $x\in(\sqrt{\delta},a]$.
One verifies that the square of the function $y=x^s=\sqrt{\dfrac{x^2-\delta}{x^2-1}}$ is the identity (involution property) and that $y=x^s$ is a strictly increasing function of x from $(\sqrt{\delta},a]$ to $(0,a^s]$. The probabilities are calculated without difficulty, for example

(6.9) $$p_x^{(4)}=\frac{E[(X+x)(X^2-(x^s)^2)]}{2x(x^2-(x^s)^2)}=\frac{1}{2}\cdot\frac{1-(x^s)^2}{x^2-(x^s)^2}=\frac{1}{2}\cdot\frac{\Delta}{\Delta+(x^2-1)^2}.$$

Case (S4) :

Let $X \in D_{S,4,2}^{(5)}(a;\delta)$ has support $\{-a,-y,0,y,a\}, 0<y<a$. From the condition $p_5(a)=0$ and (6.6), one gets the relation

(6.10) $$t_2=\frac{\beta-\delta\alpha}{\delta^2-\alpha}=\frac{a^4-\alpha}{\delta-a^2},$$

which determines the value of $\beta=\mu_8$ as function of $\delta, a^2, \alpha=\mu_6$. The atomic squares a^2, y^2, which are zeros of the quadratic equation $x^4+t_3x^2-\delta t_3-\alpha=0$ in x^2, satisfy

(6.11) $$a^2+y^2=-t_3=\frac{a^4-\alpha}{a^2-\delta},\quad a^2y^2=-\delta t_3-\alpha=\frac{a^2\cdot(\delta a^2-\alpha)}{a^2-\delta},$$

hence $y^2=\frac{\delta a^2-\alpha}{a^2-\delta}$, which determines $\alpha=\mu_6$ as function of δ, a, y. It remains to check for which values of y one obtains a feasible random variable. For this one calculates the probabilities using (2.1) :

$$p_{-a}^{(5)}=p_a^{(5)}=\frac{1}{2}\cdot\frac{\delta-y^2}{a^2\cdot(a^2-y)},\quad p_{-y}^{(5)}=p_y^{(5)}=\frac{1}{2}\cdot\frac{a^2-\delta}{y^2\cdot(a^2-y)},\ p_0^{(5)}=\frac{(a^2-1)\cdot(y^2-(a^s)^2)}{a^2\cdot y^2}.$$

These are positive provided $y\in(a^s,\sqrt{\delta})$. Setting $y=\sqrt{\delta-x}$ the affirmation follows. ◊

7. Notes.

The historical origin of the orthogonal polynomials goes back to Chebyshev, who was probably the first to recognize their orthogonal properties and their importance in Numerical and Data Analysis. A tribute to this contribution by R.Roy is found in Rassias et al.(1993).

The algebraic moment problem has been studied by Mammana(1954). Among the numerous monographs, which are entirely devoted to orthogonal polynomials and special functions, one may mention Askey(1975), Freud(1969), Rainville(1960) and Szegö(1967). Statistical applications of Hermite polynomials are found in the collected works of Cramèr(1994), papers no. 21, 25, 68. The Chebyshev polynomials of the first kind satisfy an optimal norm property (e.g. Karlin and Studden(1966), chap. IV, Theorem 4.2, or Schwarz(1986), Theorem 4.12) and solve a Lagrange interpolation problem (e.g. Demidovich and Maron(1987), p.553-554). Legendre polynomials find application in Gauss quadrature formulas (e.g. Schwarz(1986), Section 8.4).

The conditions under which their exist random variables on a finite interval with given moments to order four are known in the statistical literature (e.g. Jansen et al.(1986)). A recent general proof for the existence of moment spaces, or equivalently for the existence of random variables with known moments up to a given order, is in De Vylder(1996), II.Chapter 3.3. The lower bound on the kurtosis in dependence of the skewness has been given by Pearson(1916), Wilkins(1944) and Guiard(1980), while the upper bound is found in Simpson and Welch(1960), Jansen et al.(1986) and Teuscher and Guiard(1995). Information about the statistical meaning and interpretation of the skewness and kurtosis parameters can be found in Groeneveld(1991), Balanda and MacGillivray(1988/90), and their references.

The complete algebraic-analytical structure of the considered sets of finite atomic standardized random variables by given range and known moments to order four is implicit in Jansen et al.(1986), Section 2. However, by considering without loss of generality only standardized random variables, much calculation has been simplified and some results find improvement. Furthermore our proofs can be viewed as a direct application of the mathematical theory of orthogonal polynomials. This method allows us to find very simply the structure of the finite atomic symmetric random variables by known kurtosis.

CHAPTER II

BEST BOUNDS FOR EXPECTED VALUES BY KNOWN RANGE, MEAN AND VARIANCE

1. Introduction.

The present chapter deals with real random variables X taking values in a given interval I=[a,b], $-\infty \le a < b \le \infty$, and which have a known mean μ=E[X] and variance σ^2=Var[X]. The space of all such random variables with the characteristics I, μ, σ, is here denoted by D:=D(I,μ, σ). Given $X \in D$ and a transformed random variable f(X), where f(x) is a given real function, it has always been a task of practical interest to find the solutions to the extremal problems $\max_{X \in D}\{E[f(X)]\}$, $\min_{X \in D}\{E[f(X)]\}$, which consists to construct the best bounds for the expected values of a random function over the set of random variables with given range, mean and variance.

A general approach to these extremal problems is the well-known *majorant/minorant polynomial method*, which consists to bound f(x) by some quadratic polynomial q(x), and to construct a *finite atomic* random variable $X \in D$ such that all atoms of f(X) are simultaneously atoms of q(X). Indeed, suppose q(x) and a finite atomic $X \in D$ have been found such that Pr(q(X)=f(X))=1 and q(x)≥f(x) (resp. q(x)≤f(x)) for all $x \in I$. Then the expected value E[q(X)]=E[f(X)] depends only on μ, σ, and thus necessarily X maximizes E[f(X)] (respectively minimizes it), and the given extremal problems are solved.

There are important applications for which the described technique works, for example inequalities of Chebyshev type (Section 4) and expected values of stop-loss transform type (Section 5). This justifies the formulation in Sections 2 and 3 of a general algorithm for solving these extremal problems in case the function f(x) is a *piecewise linear* function. This apparently severe restriction is often a convenient working hypothesis in practice. Indeed, any function f(x) can be closely approximated by piecewise linear functions g(x) and h(x) such that g(x)≤f(x)≤h(x), which leads after optimization to practical upper and lower bounds $\min E[g(X)] \le E[f(X)] \le \max E[h(X)]$.

2. The quadratic polynomial method for piecewise linear functions.

The present and next Section deals with a comprehensive solution to the problem of the determination of best bounds for expected values E[f(X)], where f(x) is a *piecewise linear* function defined on I=[a,b] and $X \in D(a,b)$ is a standard random variable.

In this situation there exists a decomposition in subintervals

(2.1) $$I = \bigcup_{i=m}^{n} I_i \quad (-\infty \le m \le n \le \infty)$$

such that $I_i = [a_i, b_i]$, $a_m = a$, $a_{i+1} = b_i$, $i = m,\dots,n$, $b_n = b$, and

(2.2) $$f(x) = \ell_i(x), \quad x \in I_i, \text{ with } \ell_i(x) = \alpha_i + \beta_i x, \quad x \in R.$$

If there are only finitely many subintervals in $(-\infty,0]$, one can start with $m=0$. Otherwise one starts with $m=-\infty$. The abscissa of the point of intersection of two non-parallel lines $\ell_i(x) \neq \ell_j(x)$ is denoted by $d_{ij} = d_{ji} = \frac{\alpha_i - \alpha_j}{\beta_j - \beta_i}$.

By Lemma I.4.1, a diatomic random variable $X \in D_2^{(2)}(a,b)$ is determined by its support, a fact denoted by X={u,v}, where $(u,v) \in I_i \times I_j$ for some indices $i, j \in \{m,\dots,n\}$. Similarly, Lemma I.4.2 shows that a triatomic random variable $X \in D_2^{(3)}(a,b)$ is determined by its support, a fact denoted by X={u,v,w}, where $(u,v,w) \in I_i \times I_j \times I_k$ for some indices $i, j, k \in \{m,\dots,n\}$. Furthermore a diatomic random variable is viewed as a special triatomic random variable obtained by identifying $X = \{u,v\} \in D_2^{(2)}(a,b)$ with any $X = \{u,v,w\} \in D_2^{(3)}(a,b)$ such that $1+uv=0$ and w arbitrary.

The piecewise quadratic function $q(x)-f(x)$ is denoted by $Q(x)$. Note that $Q(x)$ coincides on I_i with the quadratic polynomial $Q_i(x) := q(x) - \ell_i(x)$. Use is made of the backward functional operator defined by $\nabla_{ij}\ell(x) := \ell_j(x) - \ell_i(x)$.

To apply the majorant/minorant quadratic polynomial method, it is necessary to determine the set of random variables X such that all atoms of the transformed random variable $f(X)$ are atoms of some quadratic random variable $q(X)$, where $q(x)$ is some quadratic polynomial, and such that $q(x) \geq f(x)$ on I for a maximum, respectively $q(x) \leq f(x)$ on I for a minimum. In a first step we restrict our attention to quadratic polynomials $q(x)$ with *non-zero* quadratic term such that $\Pr(q(X)=f(X))=1$. One observes that the piecewise quadratic function $Q(x)=q(x)-f(x)$ can have at most two zeros on each subinterval I_i (double zeros being counted twice). If an atom of X, say u, is an interior point of some I_i, then it must be a double zero of $Q_i(x)$. Indeed $q(x) \geq \ell_i(x)$ (resp. $q(x) \leq \ell_i(x)$) for $x \in I_i$ can only be fulfilled if the line $l_i(x)$ is tangent to $q(x)$ at u, that is $q'(u) = \ell_i'(u) = f'(u)$, hence u is a double zero. Therefore in a first step, one has to describe the following set of triatomic random variables

(2.3) $D_{f,q}^3 = \{\ X = \{u,v,w\} \in D_2^{(3)}(a,b)$: there exists a quadratic polynomial $q(x)$ with *non-zero* quadratic term such that $\Pr(q(X)=f(X))=1$ and $q'(x)=f'(x)$ if $x \in \{u,v,w\}$ is an interior point of some subinterval I_i $\}$.

In case $f(x)$ is piecewise linear, this set can be be described completely.

Theorem 2.1. (*Classification of quadratic polynomial majorants and minorants*) Let X={u,v,w} be a triatomic random variable such that $(u,v,w) \in I_i \times I_j \times I_k$. An element $X \in D_{f,q}^3$ belongs necessarily to one of the following *six different types*, where permutations of the atoms are allowed :

(D1) X={u,v} is diatomic with $u, v = \bar{u}$ double zeros of some $Q(x)$ such that
$$(u,v) = (d_{ij} \mp \sqrt{1+d_{ij}^2}, d_{ij} \pm \sqrt{1+d_{ij}^2}), \quad \beta_j \neq \beta_i.$$

(D2) X={u,v} is diatomic with v a rand point of I_j and $u=\bar{v}$ a double zero of some Q(x), such that either (i) $\beta_j \neq \beta_i$, $v \neq d_{ij}$ or (ii) $\beta_j = \beta_i$, $\alpha_j \neq \alpha_i$.

(T1) X={u,v,w} with u, v, w double zeros of some Q(x) such that $\beta_i, \beta_j, \beta_k$ are pairwise different, $d_{jk} - d_{ik} \neq 0$, $d_{ik} - d_{ij} \neq 0$, $d_{ij} - d_{jk} \neq 0$, and

$$u = d_{ij} + d_{ik} - d_{jk}$$
$$v = d_{jk} + d_{ij} - d_{ik}$$
$$w = d_{ik} + d_{jk} - d_{ij}$$

(T2) X={u,v,w} with w a rand point of I_k, u, v double zeros of some Q(x) such that $\beta_i, \beta_j, \beta_k$ are pairwise different, $w \neq d_{ik}, d_{jk}$, and

$$u = w - \frac{2}{(\beta_j - \beta_i)} \cdot \left\{ \nabla_{ik}\ell(w) - \operatorname{sgn}\left(\frac{w-v}{w-u}\right) \cdot \sqrt{\nabla_{ik}\ell(w) \cdot \nabla_{jk}\ell(w)} \right\},$$
$$v = w + \frac{2}{(\beta_j - \beta_i)} \cdot \left\{ \nabla_{jk}\ell(w) - \operatorname{sgn}\left(\frac{w-v}{w-u}\right) \cdot \sqrt{\nabla_{ik}\ell(w) \cdot \nabla_{jk}\ell(w)} \right\}.$$

(T3) X={u,v,w} with v, w rand points of I_j, I_k, u a double zero of some Q(x), such that either (i) $\beta_j \neq \beta_i$, $v \neq d_{ij}$, or (ii) $\beta_j = \beta_i$, $\alpha_j \neq \alpha_i$,
and either (iii) $\beta_k \neq \beta_i$, $w \neq d_{ik}$, or (iv) $\beta_k = \beta_i$, $\alpha_k \neq \alpha_i$, and

$$u = \frac{1}{2}(v+w), \quad \text{if} \quad \frac{\nabla_{ik}\ell(w)}{\nabla_{ij}\ell(v)} = 1,$$
$$u = v + \frac{w-v}{\operatorname{sgn}\left(\frac{w-u}{v-u}\right) \cdot \sqrt{\frac{\nabla_{ik}\ell(w)}{\nabla_{ij}\ell(v)}} - 1} \quad \text{if} \quad \frac{\nabla_{ik}\ell(w)}{\nabla_{ij}\ell(v)} \neq 1.$$

(T4) X={u,v,w} with u, v, w rand points of I_i, I_j, I_k, and either (i) $\beta_i, \beta_j, \beta_k$ not all equal, or (ii) $\alpha_i, \alpha_j, \alpha_k$ not all equal.

Proof. The definition (2.3) implies that an element $X \in D^3_{f,q}$ has either an atom u, which is double zero of Q(x) (types (D1), (D2), (T1), (T2), (T3)), or all three atoms of X are rand points of subintervals I_k (type (T4)). The stated specific forms of the different types are now derived.

Repeated use of the fact that a quadratic polynomial is uniquely determined by three conditions is made. If u is a double zero of $Q_i(x) = q(x) - \ell_i(x)$, one has for a zero v of $Q_j(x)$:

$$q(x) = c_{ij}(v) \cdot (x-u)^2 + \ell_i(x), \quad with$$

(2.4) $$c_{ij}(v) = \frac{\nabla_{ij}\ell(v)}{(v-u)^2} = \begin{cases} \frac{(\beta_j - \beta_i)(v-d_{ij})}{(v-u)^2}, & if \ \beta_j \neq \beta_i \\ \frac{\alpha_j - \alpha_i}{(v-u)^2}, & if \ \beta_j = \beta_i \end{cases}$$

Type D1 :

Since v is a double zero of $Q_j(x)$, the tangent line to $q(x)$ at v coincides with $\ell_j(x)$, which implies the condition $q'(v) = \ell_j'(v)$. Using (2.4) one gets

$$2c_{ij}(v) \cdot (v-u) = \beta_j - \beta_i.$$

If $\beta_j = \beta_i$ then $c_{ij}(v) = 0$, hence $\alpha_j = \alpha_i$, and $q(x) = \ell_i(x)$ has a vanishing quadratic term. Therefore only $\beta_j \neq \beta_i$ must be considered, which implies that $u + v = 2d_{ij}$. Since $v = \bar{u}$ one gets immediately the desired formulas for u, v.

Type D2 :

The formula (2.4) shows the existence of $q(x)$ and the conditions (i), (ii) assure that the quadratic term of $q(x)$ is non-zero.

Type T1 :

Since u, v, w are double zeros of $Q_i(x), Q_j(x), Q_k(x)$ respectively, cyclic permutations of i, j, k and u, v, w in (2.4) yield 3 different expressions for $q(x)$:

(i) $q(x) = c_{ij}(v) \cdot (x-u)^2 + \ell_i(x)$

(ii) $q(x) = c_{jk}(w) \cdot (x-v)^2 + \ell_j(x)$

(iii) $q(x) = c_{ki}(u) \cdot (x-w)^2 + \ell_k(x)$

The three necessary conditions $q'(v) = \ell_j'(v)$, $q'(w) = \ell_k'(w)$, $q'(u) = \ell_i'(u)$ yield

(i) $2c_{ij}(v) \cdot (v-u) = \beta_j - \beta_i$

(ii) $2c_{jk}(w) \cdot (w-v) = \beta_k - \beta_j$

(iii) $2c_{ki}(u) \cdot (u-w) = \beta_i - \beta_k$

One must have $\beta_i, \beta_j, \beta_k$ pairwise different. Otherwise $q(x)$ is a linear form (same argument as for type D1). One obtains the system of equations

(i) $u + v = 2d_{ij}$

(ii) $v + w = 2d_{jk}$

(iii) $w + u = 2d_{ik}$

with the indicated solution. Moreover one has $c_{ij}(v) \neq 0$, $c_{jk}(w) \neq 0$, $c_{ki}(u) \neq 0$, hence $v - d_{ij} = d_{jk} - d_{ik}$, $w - d_{jk} = d_{ik} - d_{ij}$, $u - d_{ik} = d_{ij} - d_{jk}$ are all different from zero.

Type T2 :

In case u, v are double zeros of $Q_i(x), Q_j(x)$, one considers the two different expressions :

(i) $$q(x) = c_{ik}(w)\cdot(x-u)^2 + \ell_i(x)$$

(ii) $$q(x) = c_{jk}(w)\cdot(x-v)^2 + \ell_j(x)$$

The additional conditions $q'(v) = \ell_j'(v)$, $q'(u) = \ell_i'(u)$ imply the equations

(i) $$2c_{ik}(w)\cdot(v-u) = \beta_j - \beta_i$$

(ii) $$2c_{jk}(w)\cdot(u-v) = \beta_i - \beta_j$$

If $\beta_j = \beta_i$ one has $c_{ik}(w) = c_{jk}(w) = 0$, hence $q(x)$ is a linear form. Thus one has $\beta_j \neq \beta_i$. Since $c_{ik}(w) \neq 0$, $c_{jk}(w) \neq 0$ one has also $\beta_k \neq \beta_i$, $w \neq d_{ik}$, $\beta_k \neq \beta_j$, $w \neq d_{jk}$. Rearranging (i), (ii) one has equivalently

(i) $$\frac{1}{2}\cdot\left(\frac{\beta_j - \beta_i}{v-u}\right) = \frac{\nabla_{ik}\ell(w)}{(w-u)^2}$$

(ii) $$\frac{1}{2}\cdot\left(\frac{\beta_j - \beta_i}{v-u}\right) = \frac{\nabla_{jk}\ell(w)}{(w-v)^2}$$

Through comparison one gets the relation

$$\frac{w-v}{w-u} = \operatorname{sgn}\left(\frac{w-v}{w-u}\right)\cdot\sqrt{\frac{\nabla_{jk}\ell(w)}{\nabla_{ik}\ell(w)}}\,.$$

Now rewrite (i) in the form

$$(u-w)^2 = -\left(\frac{2}{\beta_j - \beta_i}\right)\cdot\nabla_{ik}\ell(w)\cdot\{(u-w)+(w-v)\}.$$

Divide by $(u-w)$ and use the obtained relation to get the desired formula for u. The expression for v is obtained similarly.

Type T3 :

Using (2.4) the condition $q(w) = \ell_k(w)$ can be written as

$$\nabla_{ij}\ell(v)\cdot(w-u)^2 = \nabla_{ik}\ell(w)\cdot(v-u)^2\,.$$

In case the constraints (i) to (iv) are not fulfilled, $q(x)$ is linear. Otherwise one gets

$$\frac{w-u}{v-u} = \mathrm{sgn}(\frac{w-u}{v-u}) \cdot \sqrt{\frac{\nabla_{ik}\ell(w)}{\nabla_{ij}\ell(v)}},$$

which implies the formula for the atom u.

Type T4 :

If the constraints are not fulfilled, then q(x) is linear. Otherwise $\ell_i(u), \ell_j(v), \ell_k(w)$ do not lie on the same line and there exists always a q(x) through these points. ◊

In the situation that f(x) is composed of only *finitely* many piecewise linear segments, the formulas of Theorem 2.1 show that the set $D^3_{f,q}$, among which global extrema are expected to be found, is *finite*. An *algorithm* to determine the global extrema involves the following steps. For each $X \in D^3_{f,q}$ with corresponding q(x) such that $\Pr(q(X) = f(X)) = 1$, test if q(x) is *QP-admissible* (read quadratic polynomial admissible), which means that q(x) is either a *QP-majorant* (read quadratic polynomial majorant) such that $q(x) \geq f(x)$ on I, or it is a *QP-minorant* (read quadratic polynomial minorant) such that $q(x) \leq f(x)$ on I. If q(x) is a QP-majorant (resp. a QP-minorant) then the global maximum (resp. minimum) is attained at X, and X induces a so-called *QP-global maximum* (resp. *QP-global minimum*). If for all $X \in D^3_{f,q}$ the described test fails, and there exists global triatomic extremal random variables, then there must exist a linear function $\ell(x)$ and triatomic random variables X such that Pr($\ell(X)$ =f(X))=1 and $\ell(x) \geq f(x)$ on I for a maximum (resp. $\ell(x) \leq f(x)$ on I for a minimum). This follows because the set $D^3_{f,\ell}$ of such random variables has been excluded from $D^3_{f,q}$. Observe that these linear types of global extrema are usually not difficult to find (e.g. Proposition 3.1). To design a (possibly) *efficient algorithm*, it remains to formulate simple conditions, which guarantee that a given q(x) is QP-admissible. This is done in the next Section.

3. Global triatomic extremal random variables for expected piecewise linear transforms.

The same notations as in Section 2 are used. The conditions under which a given quadratic polynomial is QP-admissible are determined. The general idea is as follows. If X={u,v,w} with $(u,v,w) \in I_i \times I_j \times I_k$, one determines first the *condition*, say (C1), under which $Q_i(x), Q_j(x), Q_k(x) \geq 0$ (resp. ≤ 0). Then, given an index $s \neq i,j,k$, one imposes the condition that q(x) does not intersect with the open line segment defined by $\ell_s(x) = \beta_s x + \alpha_s, \quad x \in \mathring{I}_s$. Geometrically this last condition can be fulfilled in two logically distinct ways :

(C2) $Q_s(x) \geq 0$ (resp. ≤ 0), that is q(x) has at most one point of intersection with $\ell_s(x)$. This holds exactly when the discriminant of $Q_s(x)$ is non-positive.

(C3) The quadratic polynomial q(x) has two distinct points of intersection with $\ell_s(x)$, whose first coordinates lie necessarily outside the open interval $\mathring{I}_s$, that is $\{\xi,\eta : Q_s(\xi) = Q_s(\eta), \xi \neq \eta\} \not\subset \mathring{I}_s$.

Two cases must be distinguished.

Case 1 : one of $Q_s(x)$, s=i,j,k, has a double zero

Permuting the indices if necessary, one can assume that u is a double zero of $Q_i(x)$. One has $Q_i(x) = q(x) - \ell_i(x) = c_{ij}(v) \cdot (x-u)^2$ and for s≠i,j one has

$$Q_s(x) = q(x) - \ell_s(x) = Q_i(x) - \nabla_{is}\ell(x) \\ = c_{ij}(v) \cdot (x-u)^2 + (\beta_i - \beta_s) \cdot (x-u) - \nabla_{is}\ell(u) \tag{3.1}$$

Its discriminant equals

$$\Delta_{ijs}(u,v) = (\beta_s - \beta_i)^2 + 4 \cdot \frac{\nabla_{ij}\ell(v) \cdot \nabla_{is}\ell(u)}{(v-u)^2}. \tag{3.2}$$

Case 2 : u, v, w are simple zeros of $Q_s(x)$, s=i,j,k

By assumption $Q_i(x)$ has besides u a second zero, say $z_i = z_{ijk}(u,v,w)$. One can set

$$Q_i(x) = q(x) - \ell_i(x) = c_{ijk}(u,v,w) \cdot (x-u) \cdot (x-z_i),$$

where the unknown constants $c := c_{ijk}(u,v,w)$, $z := z_i$ are determined by the conditions $q(v) = \ell_j(v)$, $q(w) = \ell_k(w)$, which yield the equations

$$c(v-u)(x-z) = \nabla_{ij}\ell(v) \tag{3.3}$$

$$c(w-u)(w-z) = \nabla_{ik}\ell(w) \tag{3.4}$$

Rewrite (3.4) as

$$c(w-z) = \frac{\nabla_{ik}\ell(w)}{w-u}. \tag{3.5}$$

From (3.3) one gets

$$c(v-u)(v-w) + c(v-u)(w-z) = \nabla_{ij}\ell(v).$$

Inserting (3.3) it follows that

$$c = \left(\frac{1}{w-v}\right) \cdot \left(\frac{\nabla_{ik}\ell(w)}{w-u} - \frac{\nabla_{ij}\ell(v)}{v-u}\right),$$

which can be transformed to the equivalent form

$$c = c_{ijk}(u,v,w) = \left(\frac{1}{w-u}\right) \cdot \left(\frac{\nabla_{jk}\ell(w)}{w-v} - \frac{\nabla_{ij}\ell(u)}{v-u}\right). \tag{3.6}$$

Insert (3.6) into (3.5) to obtain

(3.7) $$z_i = z_{ijk}(u,v,w) = w - \frac{\nabla_{ik}\ell(w)}{\frac{\nabla_{jk}\ell(w)}{w-v} - \frac{\nabla_{ij}\ell(u)}{v-u}}.$$

For $s \neq i,j,k$ one considers now the quadratic polynomial

$$Q_s(x) = q(x) - \ell_s(x) = Q_i(x) + \nabla_{is}\ell(x),$$

that is written out

(3.8) $$Q_s(x) = c_{ijk}\cdot(x-u)^2 + \left(\beta_i - \beta_s + c_{ijk}\cdot(u - z_{ijk})\right)\cdot(x-u) - \nabla_{is}\ell(u).$$

Its discriminant equals

(3.9) $$\Delta_{ijks}(u,v,w) = \left(\beta_i - \beta_s + c_{ijk}\cdot(u - z_{ijk})\right)^2 + 4c_{ijk}\cdot\nabla_{is}\ell(u),$$

where one uses the expression

(3.10) $$c_{ijk}\cdot(u - z_{ijk}) = c(w-z) - c(w-u) = \frac{\nabla_{ik}\ell(w)}{w-u} + \frac{\nabla_{ij}\ell(v)}{v-u} - \frac{\nabla_{jk}\ell(v)}{w-u}.$$

Making use of these preliminaries, the set of *QP-global extrema* for the expected piecewise linear transform E[f(X)], described as the subset of $D^3_{f,q}$ of those random variables leading to a QP-admissible quadratic polynomial, is determined as follows.

Theorem 3.1 (*Characterization of QP-global extremal random variables*) The quadratic polynomial q(x) associated to a triatomic random variable $X=\{u,v,w\}\in D^3_{f,q}$, $(u,v,w)\in I_i \times I_j \times I_k$, is a QP-majorant (resp. a QP-minorant) if and only if the following conditions hold :

I. Diatomic types D1, D2

(C1) $Q_i(x), Q_j(x) \geq 0$ (resp. ≤ 0), type D1 : $\beta_j > \beta_i$ (resp. $\beta_j < \beta_i$)

(C1) $Q_i(x)$ (resp. ≤ 0), type D2 :
(a1) $\beta_j > \beta_i$ (resp. $\beta_j < \beta_i$), if $\beta_j \neq \beta_i$
(b1) $\alpha_j > \alpha_i$ (resp. $\alpha_j < \alpha_i$), if $\beta_j = \beta_i$

(C1) $Q_j(x) \geq 0$ (resp. ≤ 0), type D2 :
(a2) $\beta_j > \beta_i$ (resp. $\beta_j < \beta_i$), and $\eta_j := d_{ij} + \frac{(d_{ij}-u)^2}{v-d_{ij}} \notin \overset{\circ}{I}_j$, if $\beta_j \neq \beta_i$
(b2) $\alpha_j > \alpha_i$ (resp. $\alpha_j < \alpha_i$), and $\eta_j := 2u - v \notin \overset{\circ}{I}_j$ if $\beta_j = \beta_i$

For all $s \neq i,j$ one has either

(C2) $\Delta := \Delta_{ijs}(u,v) \le 0$, or

(C3) $\Delta > 0$ and $\xi_s, \eta_s := \dfrac{\beta_s - \beta_i \pm \sqrt{\Delta}}{2c_{ij}(v)} \notin \overset{o}{I}_s$

II. Triatomic types T1, T2, T3, T4

(C1) $Q_i(x), Q_j(x), Q_k(x) \ge 0$ (resp. ≤ 0) :

Type T1 : $\operatorname{sgn}\left(\dfrac{\beta_j - \beta_i}{d_{jk} - d_{ik}}\right) = \operatorname{sgn}\left(\dfrac{\beta_k - \beta_i}{d_{ik} - d_{ij}}\right) = \operatorname{sgn}\left(\dfrac{\beta_i - \beta_k}{d_{ij} - d_{jk}}\right) = 1$ (resp. =-1)

Type T2 : $\operatorname{sgn}\left(\dfrac{\beta_k - \beta_i}{w - d_{ik}}\right) = \operatorname{sgn}\left(\dfrac{\beta_k - \beta_j}{w - d_{jk}}\right) = 1$ (resp. =-1), and

$\eta_k := d_{jk} + \dfrac{(d_{jk} - v)^2}{w - d_{jk}} \notin \overset{o}{I}_k$

Type T3 : (a1) $\operatorname{sgn}\left(\dfrac{\beta_j - \beta_i}{v - d_{ij}}\right) = 1$ (resp. =-1), if $\beta_j \neq \beta_i$,

(b1) $\alpha_j > \alpha_i$ (resp. $\alpha_j < \alpha_i$), if $\beta_j = \beta_i$,

(a2) $\operatorname{sgn}\left(\dfrac{\beta_k - \beta_i}{w - d_{ik}}\right) = 1$ (resp. =-1), if $\beta_k \neq \beta_i$,

(b2) $\alpha_k > \alpha_i$ (resp. $\alpha_k < \alpha_i$), if $\beta_k = \beta_i$,

and furthermore

(a3) $\eta_j := d_{ij} + \dfrac{(d_{ij} - u)^2}{v - d_{ij}} \notin \overset{o}{I}_j$, if $\beta_j \neq \beta_i$,

(b3) $\eta_j := 2u - v \notin \overset{o}{I}_j$, if $\beta_j = \beta_i$,

(a4) $\eta_k := d_{ik} + \dfrac{(d_{ik} - u)^2}{w - d_{ik}} \notin \overset{o}{I}_k$, if $\beta_k \neq \beta_i$,

(b4) $\eta_k := 2u - w \notin \overset{o}{I}_k$, if $\beta_k = \beta_i$.

Type T4 : $\operatorname{sgn}\{c_{ijk}(u,v,w)\} = \operatorname{sgn}\{c_{jki}(v,w,u)\} = \operatorname{sgn}\{c_{kij}(w,u,v)\} = 1$ (resp. =-1),
and furthermore
$\eta_i := z_{ijk}(u,v,w) \notin \overset{o}{I}_i$, $\eta_j := z_{jki}(v,w,u) \notin \overset{o}{I}_j$, $\eta_k := z_{kij}(w,u,v) \notin \overset{o}{I}_k$

(C2), (C3) for Types T1, T2, T3 :

For all $s\neq i,j,k$ one has either $\Delta := \Delta_{ijs}(u,v)\le 0$, or

$$\Delta > 0 \text{ and } \xi_s, \eta_s := \frac{\beta_s - \beta_i \pm \sqrt{\Delta}}{2c_{ij}(v)} \notin \mathring{I}_s$$

(C2), (C3) for Type T4 :

For all $s\neq i,j,k$ one has either $\Delta := \Delta_{ijks}(u,v,w)\le 0$, or

$$\Delta > 0 \text{ and } \xi_s, \eta_s := \frac{\beta_s - \beta_i + c_{ijk}\cdot(z_{ijk}-u) \pm \sqrt{\Delta}}{2c_{ijk}} \notin \mathring{I}_s$$

Proof. One proceeds case by case.

Case I : diatomic types

(C1) type D1 :

Use (2.4) and its permuted version obtained by replacing u by v to get

$$Q_i(x) = (\beta_j - \beta_i)\cdot\frac{(v-d_{ij})}{(v-u)^2}\cdot(x-u)^2 = \frac{(\beta_j-\beta_i)}{4\sqrt{1+d_{ij}^2}}\cdot(x-u)^2,$$
$$Q_j(x) = (\beta_j - \beta_i)\cdot\frac{(d_{ij}-u)}{(u-v)^2}\cdot(x-v)^2 = \frac{(\beta_j-\beta_i)}{4\sqrt{1+d_{ij}^2}}\cdot(x-v)^2,$$

which implies the displayed condition.

(C1) $Q_i(x)\ge 0$ (resp. ≤ 0), type D2 :

If $\beta_j \neq \beta_i$ one argues as for type D1, hence (a1). Otherwise one has

$$Q_i(x) = \frac{(\alpha_j - \alpha_i)}{(v-u)^2}\cdot(x-u)^2,$$

which shows (b1).

(C1) $Q_j(x)$ (resp. ≤ 0), type D2 :

Besides ξ=v the quadratic polynomial $Q_j(x)$ has a second zero η, which is solution of the equation $q(\eta)=\ell_j(\eta)$, and which must lie outside the open interval $\mathring{I}_j$. Using (2.4) one has to solve the equation

$$\nabla_{ij}\ell(v)\cdot(\eta-u)^2 = \nabla_{ij}\ell(\eta)\cdot(v-u)^2.$$

One finds

$$\eta = \begin{cases} d_{ij} + \dfrac{(d_{ij}-u)^2}{v-d_{ij}}, if\ \beta_j \neq \beta_i \\ 2u-v, if\ \beta_j = \beta_i \end{cases}$$

Furthermore one has $Q_j(x) = c_{ij}(v)\cdot(x-u)^2 + \nabla_{ji}l(x)$ and the sign of $Q_j(x)$ is determined by the sign of $c_{ij}(v)$, leading to the same conditions as for $Q_i(x)$.

The conditions (C2) and (C3) follow immediately using the formulas (3.1) and (3.2) described in the text under Case 1.

Case II : triatomic types

(C1) type T1 :

From the proof of Theorem 2.1 one borrows the formulas

$$Q_i(x) = c_{ij}(v)\cdot(x-u)^2,\ Q_j(x) = c_{jk}(w)\cdot(x-v)^2,\ Q_k(x) = c_{ki}(u)\cdot(x-w)^2,$$

which imply the desired condition.
(C1) type T2 :

The following formulas are found in the proof of Theorem 2.1 :

$$\begin{aligned} &Q_i(x) = c_{ik}(w)\cdot(x-u)^2, \\ &Q_j(x) = c_{jk}(w)\cdot(x-v)^2, \\ &Q_k(x) = Q_j(x) - \ell_k(x) = c_{jk}(w)\cdot(x-v)^2 - \nabla_{jk}\ell(x). \end{aligned}$$

The sign of these quadratic polynomials is determined by the sign of its quadratic terms, which implies the first statement. On the other side $Q_k(x)$ has besides ξ=w a second zero η, which must lie outside the open interval $\overset{\circ}{I}_k$. The equation $Q_k(\eta) = 0$ implies the relation

$$\nabla_{jk}\ell(w)\cdot(\eta-v)^2 = \nabla_{jk}\ell(\eta)\cdot(w-v)^2,$$

which has the unique solution

$$\eta = d_{jk} + \frac{(d_{jk}-v)^2}{w-d_{jk}}.$$

This implies the second statement.

(C1) type T3 :

One has the formulas

$$Q_i(x) = c_{ij}(v)\cdot(x-u)^2 = c_{ik}(w)\cdot(x-u)^2,$$
$$Q_j(x) = Q_i(x) - \ell_j(x) = c_{ij}(v)\cdot(x-u)^2 - \nabla_{ij}\ell(x),$$
$$Q_k(x) = Q_j(x) - \ell_k(x) = c_{ik}(w)\cdot(x-u)^2 - \nabla_{ik}\ell(x).$$

Looking at the sign of the quadratic terms implies the first statement. Besides $\xi_j = v$ the second zero η_j of $Q_j(x)$ must lie outside $\overset{o}{I}_j$. Simlarly $Q_k(x)$ has two zeros $\xi_k = v, \eta_k$, of which the second one must lie outside $\overset{o}{I}_k$. The above formulas imply the following equivalent statements

$$Q_j(\eta_j) = 0 \Leftrightarrow \nabla_{ij}\ell(v)\cdot(\eta_j - u)^2 = \nabla_{ij}\ell(\eta_j)\cdot(v-u)^2 \Leftrightarrow$$
$$\eta_j = d_{ij} + \frac{(d_{ij}-u)^2}{v-d_{ij}}, \quad if\ \beta_j \neq \beta_i, \quad \eta_j = 2u - v, \quad if\ \beta_j = \beta_i$$
$$Q_k(\eta_k) = 0 \Leftrightarrow \nabla_{ik}\ell(w)\cdot(\eta_k - u)^2 = \nabla_{ik}\ell(\eta_k)\cdot(w-u)^2 \Leftrightarrow$$
$$\eta_k = d_{ik} + \frac{(d_{ik}-u)^2}{w-d_{ik}},\ if\ \beta_k \neq \beta_i,\ \eta_k = 2u - w,\ if\ \beta_k = \beta_i$$

from which the required conditions are shown.

(C1) type T4 :

The formulas in the text under Case 2 show through permutation of indices that

$$Q_i(x) = c_{ijk}(u,v,w)\cdot(x-\xi_i)\cdot(x-\eta_i), \quad \xi_i = u, \quad \eta_i = z_{ijk}(u,v,w),$$
$$Q_j(x) = c_{jki}(v,w,u)\cdot(x-\xi_j)\cdot(x-\eta_j), \quad \xi_j = u, \quad \eta_j = z_{jki}(v,w,u),$$
$$Q_k(x) = c_{kij}(w,u,v)\cdot(x-\xi_k)\cdot(x-\eta_k), \quad \xi_k = u, \quad \eta_k = z_{kij}(w,u,v).$$

The signs of the quadratic terms imply the first statement. The second affirmation is the fact that the corresponding zeros must lie outside the displayed open intervals.

Finally, the conditions (C2) and (C3) are clear from the distinction in the text between Cases 1 and 2. ◊

For specific choices of transforms and/or triatomic random variables, it is sometimes possible to derive general rules, which are useful in the optimization process. To illustrate let us derive some minimizing decision criteria. These have been applied to handle the minimum problem for the "two-layers stop-loss transform" in Section 5.

Proposition 3.1. Assume the function $f(x)$ is *piecewise linear convex* on I. Suppose there exists a triatomic random variable $X_* = \{x,y,z\} \in D^3_{f,f_i}$ such that $\Pr(f_i(X_*) = f(X_*)) = 1$ and that $0 \in I_i$. Then X_* is a minimizing solution of the extremal problem :

$$\min_{X\in D}\{E[f(X)]\} = E[f(X_*)] = f_i(0).$$

Proof. Since f(x) is convex on I, one has from Jensen's inequality and using the fact $\mu = 0 \in I_i$ that $E[f(X)] \ge f(\mu) = f_i(0)$ for all $X \in D$. By assumption all the mass points of X_* belong to I_i and since $f(x) = f_i(x)$ on I_i, one gets $E[f(X_*)] = f_i(0)$. Therefore the lower bound is attained. ◊

Proposition 3.2. Assume the payoff function f(x) is *piecewise linear convex* on I. Suppose $X \in D^3_{f,q}$ is not a type T4. Then X cannot minimize E[f(X)].

Proof. Without loss of generality let us assume that X={u,v} or X={u,v,w} with $u \in I_i$ a double zero of $Q_i(x) = q(x) - \ell_i(x),\ v \in I_j$. A straightforward calculation shows that $q(x) = c_{ij}(v) \cdot (x-u)^2 + \ell_i(x)$, where

$$c_{ij}(v) = \frac{\nabla_{ij}\ell(v)}{(v-u)^2} = \frac{f'(u) - h(v,u)}{u-v} = \frac{h(u,v) - f'(u)}{v-u}, \text{ with } h(u,v) = h(v,u) = \frac{f(v) - f(u)}{v-u}.$$

Let us distinguish between two subcases.

<u>Case 1</u> : v<u

Since f(x) is convex on I, one has for all x such that v<u<x the inequality $h(v,u) \le \frac{f(u) - f(x)}{u-x}$. Taking limits as $x \to u$ one has also $h(v,u) \le f'(u)$, hence $c_{ij}(v) \ge 0$.

<u>Case 2</u> : v>u

Similarly for all x such that x<u<v one has the inequality $h(u,v) \le \frac{f(u) - f(x)}{u-x}$, and in the limit as $x \to u$ one has also $h(u,v) \ge f'(u)$, hence $c_{ij}(v) \ge 0$. In both cases one has $q(x) \ge \ell_i(x)$. This implies that $q(x) \le f(x)$ cannot hold, hence X cannot minimize E[f(X)]. ◊

Combining both results, it is possible to restrict considerably the set of triatomic random variables, which can minimize the expected transform.

Corollary 3.1. Suppose the function f(x) is *piecewise linear convex*. Then the minimal expected value $\min_{X \in D}\{E[f(X)]\} = E[f(X_*)]$ is attained either for $X_* \in D^3_{f,q}$ of type T4 or for $X_* \in D^3_{f,f_i},\ \ 0 \in I_i,\ \ m \le i \le n$.

4. Inequalities of Chebyshev type.

Let (Ω, A, P) be a probability space such that Ω is the sample space, A the σ-field of events of Ω and P the probability measure. For a given event $E \in A$, an *inequality of Chebyshev type* gives precise conditions under which the probability P(E) is maximal respectively minimal, where P satisfies some properties.

Denoting by $I_E(X)$ the indicator function of an event E, and setting $f(X)=I_E(X)$, one observes that the probability of an event E identifies with the expected value

$P(E) = E[f(X)]$. In case E is a finite union of subintervals of the real numbers, the indicator function $f(x)=I_E(x)$ is piecewise linear, and thus inequalities of the Chebyshev type can in principle be determined by applying the algorithm to construct quadratic polynomial majorants and minorants, which has been systematically studied in Sections 2 and 3. To illustrate the method, another simple proof of the inequality of Selberg(1942) is presented.

Theorem 4.1. Let $X \in D = D((-\infty,\infty);\mu,\sigma)$ be a real random variable with known mean μ and variance σ^2, and consider the event

(4.1) $$E = \{X \le \mu - \alpha\} \cup \{X \ge \mu + \beta\}, \; -\alpha < \beta.$$

Then the maximum of the probability $P(E)$ over D is given and attained as in Table 4.1.

Remark 4.1. The inequality of Selberg generalizes the classical inequality of Chebyshev

(4.2) $$P(|X-\mu| \ge \alpha) \le \min\left\{\left(\tfrac{\sigma}{\alpha}\right)^2, 1\right\}$$

obtained by setting $\beta = \alpha$ in Table 4.1. Moreover, letting α, β tend to infinity, one gets the well-known *one-sided Chebyshev inequalities* (see also Section III.4)

(4.3) $$P(X \le \mu - \alpha) \le \frac{\sigma^2}{\sigma^2 + \alpha^2} \quad (\beta \to \infty)$$

(4.4) $$P(X \le \mu + \beta) \ge 1 - \frac{\sigma^2}{\sigma^2 + \beta^2} \quad (\alpha \to \infty)$$

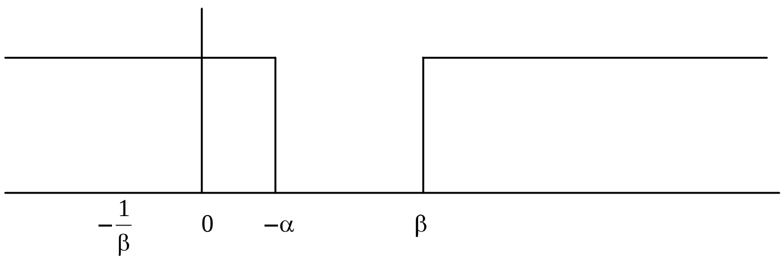
Table 4.1 : inequality of Selberg, $\gamma = \min(\alpha, \beta)$, $\alpha + \beta > 0$

condition	maximum	support of finite atomic extremal random variable
(1) $\alpha \le 0$	1	$\left\{\mu - \frac{\sigma^2}{x-\mu}, x\right\}$, $x \in [\beta, \infty)$
(2) $\beta \le 0$	1	$\left\{x, \mu + \frac{\sigma^2}{\mu - x}\right\}$, $x \in (-\infty, -\alpha]$
(3) $\alpha, \beta > 0$		
(3a) $\alpha\beta \le \sigma^2$	1	$\left\{\mu - \frac{\sigma^2}{x-\mu}, x\right\}$, $x \in \left[\beta, \frac{1}{\alpha}\right]$
(3b) $\alpha\beta - \gamma^2 \le 2\sigma^2 \le 2\alpha\beta$	$\frac{(\beta-\alpha)^2 + 4\sigma^2}{(\alpha+\beta)^2}$	$\{\mu - \alpha, \mu + \frac{1}{2}(\beta - \alpha), \mu + \beta\}$
(3c) $2\sigma^2 \le \alpha\beta - \gamma^2$	$\frac{\sigma^2}{\sigma^2 + \gamma^2}$	$\left\{\mu - \alpha, \mu + \frac{\sigma^2}{\alpha}\right\}$, *if* $\beta \ge \alpha$ $\left\{\mu - \frac{\sigma^2}{\beta}, \mu + \beta\right\}$, *if* $\beta \le \alpha$

Proof of Theorem 4.1. Making use of the standard location-scale transformation, it suffices to consider the special situation $\mu = 0, \sigma = 1$. We proceed case by case. Notations and conventions are those of Section 2.

Case (1) : $\alpha \le 0$

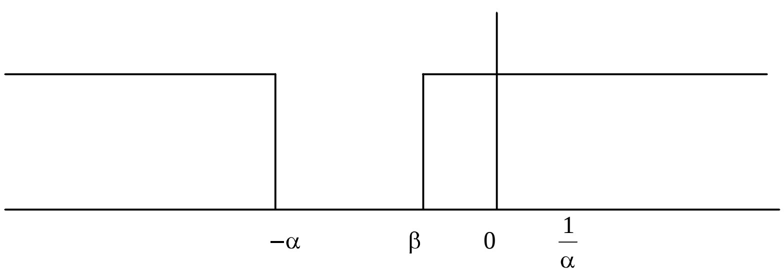
Choose a diatomic random variable X with support $\{\bar{x}, x\}$, $x \ge \beta$, to see that $P(f(X) = \ell(X)) = 1$, where $\ell(X) = 1$ as in the following figure :



Since $f(X) \le \ell(X)$ one has $\max\{P(E)\} = 1$.

Case (2) : $\beta \le 0$

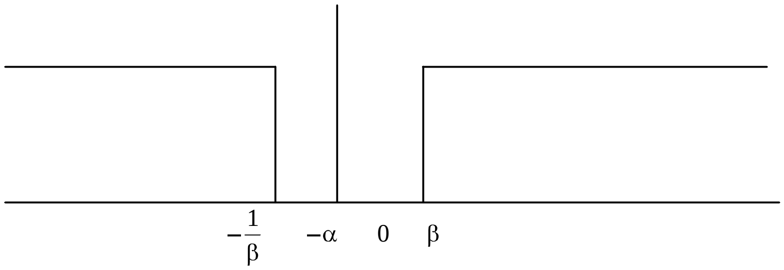
Choose a diatomic random variable X with support $\{x, \bar{x}\}$, $x \le -\alpha$, to see that $P(f(X) = \ell(X)) = 1$, where $\ell(X) = 1$ as in the following figure :



Since $f(X) \le \ell(X)$ one has $\max\{P(E)\} = 1$.

Case (3a) : $\alpha\beta \le 1$

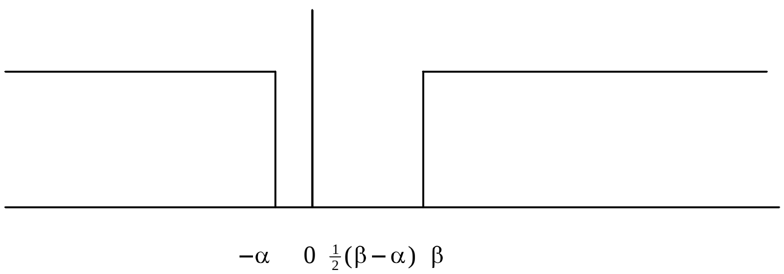
Choose a diatomic random variable X with support $\{\bar{x}, x\}$, $\beta \le x \le \frac{1}{\alpha}$, to see that $P(f(X) = \ell(X)) = 1$, where $\ell(X) = 1$ as in the following figure :



Since $f(X) \le \ell(X)$ one has $\max\{P(E)\} = 1$.

Case (3b) : $\alpha(\beta-\alpha) \le 2 \le \alpha\beta$, $\beta \ge \alpha$ (the case $\beta \le \alpha$ is shown by symmetry)

One constructs a quadratic polynomial q(x) such that f(x)≤q(x), and a triatomic random variable with the property $P(f(X) = q(X)) = 1$ as in the following figure :



Consider the decomposition of the real axis in subintervals $I_i x I_j x I_k$ such that $I_i = [-\alpha, \beta]$, $I_j = (-\infty, -\alpha]$, $I_k = [\beta, \infty)$. Then the piecewise linear function f(x) is given by

$$f(x) = \begin{cases} \ell_j(x) = 1, & x \in I_j \\ \ell_i(x) = 0, & x \in I_i \\ \ell_k(x) = 1, & x \in I_k \end{cases}$$

In the notations of Section 2, one has $\alpha_i = \beta_i = 0, \alpha_j = 1, \beta_j = 0, \alpha_k = 1, \beta_k = 0$. One considers a triatomic X with support $\{u, v, w\}, v = -\alpha, w = \beta$ rand points of I_j, I_k. Then X is of type (T3) in Theorem 2.1. Since

$$\frac{\nabla_{ik}\ell(w)}{\nabla_{ij}\ell(v)} = \frac{\alpha_k - \alpha_i}{\alpha_j - \alpha_i} = 1,$$

one has necessarily $u = \frac{1}{2}(v+w) = \frac{1}{2}(\beta-\alpha)$. The corresponding quadratic polynomial is

$$q(x) = \left\{\frac{2x - (\beta-\alpha)}{\alpha+\beta}\right\}^2.$$

When is X a feasible triatomic random variable ? By Lemma I.4.2, this is the case exactly when $u \le \overline{w} < 0$, $\overline{w} \le v \le \overline{u}$. These inequalities imply the condition $\alpha(\beta-\alpha) \le 2 \le \alpha\beta$. Using Theorem 3.1, it remains to verify that $q(x)$ is QP-admissible, that is $f(x) \le q(x)$. For X of type (T3) this follows from the following facts :

condition (C1) in Theorem 3.1 :

(b1) is fulfilled : $\beta_j = \beta_i = 0, \quad \alpha_j = 1 > \alpha_i = 0$
(b2) is fulfilled : $\beta_k = \beta_i = 0, \quad \alpha_k = 1 > \alpha_i = 0$
(b3) is fulfilled : $\beta_j = \beta_i, \quad \eta_j = 2u - v = \beta \notin \overset{o}{I}_j = (-\infty, -\alpha)$
(b4) is fulfilled : $\beta_k = \beta_i, \quad \eta_k = 2u - w = -\alpha \notin \overset{o}{I}_k = (\beta, \infty)$

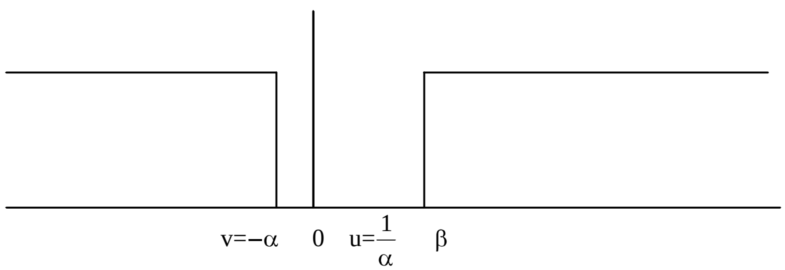
Finally the value of the maximum equals

$$E[q(X)] = \frac{E\left[4X^2 - 4(\beta-\alpha)X + (\beta-\alpha)^2\right]}{(\alpha+\beta)^2} = \frac{(\beta-\alpha)^2 + 4}{(\alpha+\beta)^2},$$

which is the required value in the standard case $\mu = 0, \sigma = 1$.

<u>Case (3c)</u> : $2 \le \alpha(\beta-\alpha), \quad \beta \ge \alpha$ (the case $\beta \le \alpha$ is shown by symmetry)

An appropriate quadratic polynomial is displayed in the following figure :



The same decomposition in subintervals as in case (3b) applies. With $v=-\alpha$ a rand point, one considers a diatomic random variable X with support $\{u, v\}$, $u = \frac{1}{\alpha}$. In Theorem 2.1, it is of type (D2). The corresponding quadratic polynomial equals

$$q(x) = \left\{\frac{\alpha x - 1}{1+\alpha^2}\right\}^2.$$

Under which condition is $q(x)$ QP-admissible ? One must check conditions (C1), (C2) in Theorem 3.1 :

condition (C1) :

(b1) is fulfilled : $\beta_j = \beta_i = 0, \quad \alpha_j = 1 > \alpha_i = 0$
(b2) is fulfilled : $\alpha_j > \alpha_i, \quad \eta_j = 2u - v = \frac{2+\alpha^2}{\alpha} \notin \overset{o}{I}_j = (-\infty, -\alpha)$

condition (C2) :

We determine the condition under which $q(x)$ does not intersect with the line segment $l_k(x)=1,\quad x\in \overset{o}{I}_k=(\beta,\infty)$. The zeros of the quadratic polynomial $Q_k(x)=q(x)-l_k(x)$ $=\left\{\frac{\alpha x-1}{1+\alpha^2}\right\}^2-1$ are $\xi_k=-\alpha,\quad \eta_k=\frac{2+\alpha^2}{\alpha}$. They are not in $\overset{o}{I}_k=(\beta,\infty)$ exactly when $2\le\alpha(\beta-\alpha)$, the defining condition in case (3c). Furthermore X is a feasible diatomic random variable provided $u=\frac{1}{\alpha}\in[-\alpha,\beta]$. Since $\alpha\beta\ge 2+\alpha^2\ge 1$ this condition is fulfilled. Finally the value of the maximum is

$$E[q(X)]=\frac{E[\alpha^2X^2-2\alpha X+1]}{(1+\alpha^2)^2}=\frac{1}{1+\alpha^2},$$

the required value in the standard case. ◊

5. Best bounds for expected values of stop-loss transform type.

Let X be a random variable defined on $I=[a,b]$ with survival function $\overline{F}(x)$. The *stop-loss transform* of X is defined and denoted by $\pi(d)=E[(X-d)_+]=\int_d^b\overline{F}(t)dt$ provided this quantity exists for all $d\in I$. The corresponding function $f(x)=(x-d)_+$, which equals $(x-d)$ if $x>d$ and zero otherwise, is clearly piecewise linear. Since $\pi'(x)=-\overline{F}(x)$ there is a one-to-one correspondence between a distribution and its stop-loss transform, which both characterize a random variable and are thus equally well important mathematical notions.

In applications, the stop-loss transform serves as prototype of the financial instruments, called derivatives, extensively encountered in Reinsurance and Option Markets under the names *stop-loss contract* and *call option*. The following closely related financial instruments are important modifications of the stop-loss contract, whose corresponding functions remain in the piecewise linear class. The *limited stop-loss contract* is defined by the financial payoff $f(x)=(x-d)_+-(x-L)_+$, $L>d$, whose limited maximal payment is the amount $L-d$. Its expected value $E[f(X)]=\pi(d)-\pi(L)$ consists of a difference in stop-loss transforms. The *franchise deductible contract* is defined by $f(x)=d\cdot I_{\{X>d\}}(x)+(x-d)_+$ with expected value $E[f(X)]=d\cdot\overline{F}(d)+\pi(d)$, while the *disappearing deductible contract* is generated by the payoff function $f(x)=\frac{1}{(d_2-d_1)}\cdot\{d_2\cdot(x-d_1)_+-d_1\cdot(x-d_2)_+\}$, $d_1<d_2$. Finally a *two-layers stop-loss contract* is defined by the piecewise linear convex payoff function $f(x)=r(x-L)_++(1-r)(x-M)_+$, $0<r<1$, $a<L<M<b$. It is a special case of the *n-layers stop-loss contract* defined by $f(x)=\sum r_i(x-d_i)_+$, $\sum r_i\le 1$, $r_i\ge 0$, $a=d_0<d_1<\ldots<d_n<b=d_{n+1}$, whose payment increases proportionally in each layer $[d_{i-1},d_i]$, $i=1,\ldots,n+1$. In general one assumes that the financial payoff $f(x)$ of a reinsurance contract satisfies the constraint $0\le f(x)\le x$. This restriction is of relevance in Section 5.3.

Based on the majorant/minorant quadratic polynomial method for piecewise linear functions, it is possible to derive and present in an unified way best bounds for these modified stop-loss transforms. After a standard location-scale transformation has been made, one can assume optimization takes place over the set D(a,b) of all standard random variables defined on the interval $[a,b]$.

5.1. The stop-loss transform.

The optimization problems for the "pure" stop-loss transform defined by $f(x)=(x-d)_+$ are solved in Tables 5.1 and 5.2. Application of the majorant/minorant quadratic polynomial method is straightforward. For the maximum consult for example Jansen et al.(1986), and for the minimum use Corollary 3.1. Details are left to the reader.

Table 5.1 : maximum stop-loss transform for standard random variables on $[a,b]$

conditions	maximum	extremal support
$a \le d \le \frac{1}{2}(a+\bar{a})$	$(-a)\cdot\frac{1+ad}{1+a^2}$	$\{a,\bar{a}\}$
$\frac{1}{2}(a+\bar{a}) \le d \le \frac{1}{2}(b+\bar{b})$	$\frac{1}{2}(\sqrt{1+d^2}-d)$	$\{d-\sqrt{1+d^2}, d+\sqrt{1+d^2}\}$
$\frac{1}{2}(b+\bar{b}) \le d \le b$	$\frac{b-d}{1+b^2}$	$\{\bar{b},b\}$

Table 5.2 : minimum stop-loss transform for standard random variables on $[a,b]$

conditions	minimum	extremal support
$d > \bar{a}$	0	$\{\bar{d},d\}$
$d < \bar{b}$	-d	$\{d,\bar{d}\}$
$\bar{b} \le d \le \bar{a}$	$\frac{1+ad}{b-a}$	$\{a,d,b\}$

Remarks 5.1.

(i) The global extrema in the non-standardized scale with arbitrary μ, σ are obtained easily from the stop-loss transform relationship $\pi_X(D)=\sigma\cdot\pi_Z(d)$, $Z=\frac{X-\mu}{\sigma}$, $d=\frac{D-\mu}{\sigma}$.

(ii) Applying a different method these best bounds have been obtained firstly by De Vylder and Goovaerts(1982) (see also Goovaerts et al.(1984), p. 316). In the present form, Table 5.1 appears in Jansen et al.(1986), theorem 2 (with a misprint in case 3). Table 5.2 is the generalized version of theorem X.2.4 in Kaas et al.(1994).

(iii) In the limiting case as $a\to-\infty, b\to\infty$, the global extrema are attained by diatomic random variables, the maximum at $X=\{d-\sqrt{1+d^2}, d+\sqrt{1+d^2}\}$ (so-called inequality of Bowers(1969)) and the minimum at $X=\{d,\bar{d}\}$ if d<0 and at $X=\{\bar{d},d\}$ if d>0.

5.2. The limited stop-loss transform.

Best bounds for the limited stop-loss transform $E[f(X)]$ with $f(x)=(x-d)_+-(x-L)_+$, L>d, have been given in Goovaerts et al.(1984). For standard random variables, the optimal solutions are displayed in Tables 5.3 and 5.4. The simpler limiting case $a \to -\infty, b \to \infty$ is summarized in Tables 5.3' and 5.4'. Since the results are known, the details of the majorant/minorant quadratic polynomial method are left to the reader. In a different more complicated and less structured form one finds Tables 5.3 and 5.4 in Goovaerts et al.(1984), p. 357-58. Note that for Table 5.3 the subcase defined by $L > \bar{a}, \frac{1}{2}(a+L) \le d \le \frac{1}{2}(L+\bar{L})$, which is actually part of (3b), is misprinted there.

Table 5.3 : maximum limited stop-loss transform for standard random variables on [a,b]

conditions	maximum	extremal support
(1) $\bar{b} \le L \le \bar{a}$	$\frac{(b-a)-(\bar{a}-L)}{\bar{a}(L-a)(b-a)} \cdot (L-d)$	$\{a, L, b\}$
(2) $L < \bar{b}$	L-d	$\{L, \bar{L}\}$
(3) $L > \bar{a}$:		
(3a) $d \le \frac{1}{2}(a+\bar{a})$	$\frac{1+ad}{\bar{a}-a}$	$\{a, \bar{a}\}$
(3b) $\frac{1}{2}(a+\bar{a}) \le d \le \frac{1}{2}(L+\bar{L})$	$\frac{1}{2}(\sqrt{1+d^2}-d)$	$\{d-\sqrt{1+d^2}, d+\sqrt{1+d^2}\}$
(3c) $d \ge \frac{1}{2}(L+\bar{L})$	$\frac{L-d}{1+L^2}$	$\{\bar{L}, L\}$

Table 5.4 : minimum limited stop-loss transform for standard random variables on [a,b]

conditions	minimum	extremal support
(1) $\bar{b} \le d \le \bar{a}$	$\frac{1+ad}{(b-a)(b-d)} \cdot (L-d)$	$\{a, d, b\}$
(2) $d > \bar{a}$	0	$\{\bar{d}, d\}$
(3) $d < \bar{b}$:		
(3a) $L \le \frac{1}{2}(d+\bar{d})$	$\frac{d^2}{1+d^2} \cdot (L-d)$	$\{d, \bar{d}\}$
(3b) $\frac{1}{2}(d+\bar{d}) \le L \le \frac{1}{2}(b+\bar{b})$	$\frac{1}{2}(L-2d-\sqrt{1+L^2})$	$\{L-\sqrt{1+L^2}, L+\sqrt{1+L^2}\}$
(3c) $L \ge \frac{1}{2}(b+\bar{b})$	$L-d-\left(\frac{1+bL}{1+b^2}\right)b$	$\{\bar{b}, b\}$

Table 5.3' : maximum limited stop-loss transform for standard random variables on $(-\infty,\infty)$

conditions	maximum	extremal support
(1) L=0	–d	{0}
(2) L<0	L–d	$\{L,\overline{L}\}$
(3) L>0 :		
(3a) $d \le \frac{1}{2}(L+\overline{L})$	$\frac{1}{2}(\sqrt{1+d^2}-d)$	$\{d-\sqrt{1+d^2}, d+\sqrt{1+d^2}\}$
(3b) $d \ge \frac{1}{2}(L+\overline{L})$	$\frac{L-d}{1+L^2}$	$\{\overline{L},L\}$

Table 5.4' : minimum limited stop-loss transform for standard random variables on $(-\infty,\infty)$

conditions	minimum	extremal support
(1) d=0	0	{0}
(2) d>0	0	$\{\overline{d},d\}$
(3) d<0 :		
(3a) $L \le \frac{1}{2}(d+\overline{d})$	$\frac{d^2}{1+d^2}\cdot(L-d)$	$\{d,\overline{d}\}$
(3b) $L \ge \frac{1}{2}(d+\overline{d})$	$\frac{1}{2}(L-2d-\sqrt{1+L^2})$	$\{L-\sqrt{1+L^2}, L+\sqrt{1+L^2}\}$

5.3. The franchise deductible transform.

For $X \in D([A,B];\mu,\sigma)$ the franchise deductible transform $E[f_X(X)]$ is defined by the piecewise linear function $f_X(x) = D\cdot I_{\{X>D\}}(x) + (x-D)_+$ with D≥0. An optimization problem over the set of all standard random variables $Z=\frac{X-\mu}{\sigma}\in D(a,b)$, $a=\frac{A-\mu}{\sigma}$, $b=\frac{B-\mu}{\sigma}$, is obtained provided the (standard) franchise deductible transform $E[f_Z(Z)]$ is defined by $f_Z(z)=\sigma(z-\rho)\cdot I_{\{Z>d\}}(z)$ with $\rho=-\frac{\mu}{\sigma}$, $d=\frac{D-\mu}{\sigma}$. Note the *scale invariant* property $E[f_X(X)]=E[f_Z(Z)]$. In the special case A=0, respectively $a=\rho$ in the standardized scale, the maximum franchise deductible transform has been determined by Heijnen and Goovaerts(1989). The general case is much more complex.

A detailed analysis shows that the relevant di- and triatomic random variables are those displayed in Table 5.5. Subsequent use is made of the simplifying notations :

(5.1)
$$\omega(x)=\tfrac{1}{2}(x+\overline{x}),\quad \alpha=\omega(a),\quad \beta=\omega(b),$$
$$x_\xi=\xi-\sqrt{1+\xi^2},\quad \overline{x}_\xi=\xi+\sqrt{1+\xi^2}.$$

Table 5.5 : triatomic random variables and their feasible domains

feasible domain	feasible support
(1a) $d \geq \bar{a}, \omega(d) \leq \rho \leq \beta$	$\{x_\rho, \bar{x}_\rho\}$
(1b) $\bar{b} \leq d \leq \bar{a}, \alpha \leq \rho \leq \beta$	$\{x_\rho, \bar{x}_\rho\}$
(1c) $d \leq b^*, \alpha \leq \rho \leq \omega(d)$	$\{x_\rho, \bar{x}_\rho\}$
(2) $d \geq \bar{a}$	$\{\bar{d}, d\}$
(3) $d \leq \bar{b}$	$\{d, \bar{d}\}$
(4) $d \leq \bar{a}$	$\{a, \bar{a}\}$
(5) $d \geq \bar{b}$	$\{\bar{b}, b\}$
(6) $\bar{b} \leq d \leq \bar{a}$	$\{a, d, b\}$

A detailed case by case construction of QP-majorants $q(x) \geq f_Z(x)$ on [a,b] is presented. It is recommended that the reader draws for himself a geometrical figure of each situation, which is of great help in this analytical method.

Case 1 : $\{x_\rho, \bar{x}_\rho\} = \left\{\rho - \sqrt{1+\rho^2}, \rho + \sqrt{1+\rho^2}\right\}$

A QP-majorant q(x) through the point $(u, v = \bar{u})$ must have the properties q(u)=0, q'(u)=0, $q(v) = \sigma(v-\rho)$, $q'(v) = \sigma$ (u,v double zeros of $q(x) - f_Z(x)$). The unique solution is

(5.2)
$$q(x) = \frac{\sigma(x-u)^2}{2(v-u)}, \quad u + v = 2\rho.$$

Since $v = \bar{u}$ one sees that $u = x_\rho, v = \bar{x}_\rho$. Clearly $q(x) \geq f_Z(x)$ on [a,b]. The only restriction is $\{x_\rho, \bar{x}_\rho\} \in [a,d]x[d,b]$, which leads to the feasible domain given in Table 5.5.

Case 2 : $\{\bar{d}, d\}$, $d \geq \bar{a}$

A QP-majorant q(x), which satisfies the conditions $q(\bar{d}) = 0, q'(\bar{d}) = 0, q(d) = \sigma(d-\rho)$, is

(5.3)
$$q(x) = \frac{\sigma(d-\rho)(x-\bar{d})^2}{(d-\bar{d})^2}.$$

To be a QP-majorant q(x) must lie above the line $l(x) = \sigma(x-\rho)$ on [d,b], hence q'(d)≥σ, which implies the restriction $\rho \leq \omega(d)$.

Case 3 : $\{d, \bar{d}\}$, $d \leq \bar{b}$

The degenerate quadratic polynomial q(x)=$l(x) = \sigma(x-\rho)$ goes through (d,d*). Under the restriction $\rho \leq a$ one has further q(x)≥0 on [a,d], hence $q(x) \geq R_Z(x)$ on [a,b].

Case 4 : $\{a,\bar{a}\}$, $d \le \bar{a}$

Set $u=a$, $v=\bar{a}$. A QP-majorant q(x) through (u,v) satisfies q(u)=0, $q(v)=\sigma(v-\rho)$, $q'(v)=\sigma$, and the second zero z of q(x) lies in the interval $(-\infty,a]$. The unique solution is

(5.4) $$q(x)=c(x-v)^2+\sigma(x-v)+\sigma(v-\rho), \quad c=\frac{\sigma(\rho-u)}{(v-u)^2}.$$

The additional condition q(z)=0 yields the relation

(5.5) $$\rho=\rho(z)=\frac{uz-v^2}{u+z-2v}.$$

This increasing function lies for $z\in(-\infty,a]$ between the two bounds $a\le\rho\le\alpha$.

Case 5 : $\{\bar{b},b\}$, $d\ge\bar{b}$

Setting $u=\bar{b}$, v=b, a QP-majorant q(x) through (u,v) satisfies the conditions q(u)=0, q'(u)=0, $q(v)=\sigma(v-\rho)$, and the second point of intersection z of q(x) with the line $l(x)=\sigma(x-\rho)$ lies in the interval $[b,\infty)$. The unique solution is

(5.6) $$q(x)=c(x-u)^2, \quad c=\frac{\sigma(v-\rho)}{(v-u)^2}.$$

Solving $q(z)=\sigma(z-\rho)$ implies the monotone relation

(5.7) $$\rho=\rho(z)=\frac{vz-u^2}{v+z-2u},$$

from which one obtains the restriciton $\beta\le\rho$.

Case 6 : $\{a,d,b\}$, $\bar{b}\le d\le\bar{a}$

A QP-majorant q(x) through (a,d,b) satisfies the conditions q(a)=0, $q(d)=\sigma(d-\rho)$, $q(b)=\sigma(b-\rho)$, and the second zero z of q(x) lies in the interval $[b,\infty)$. One finds

(5.8) $$q(x)=c(x-a)(x-z), \quad c=\frac{\sigma(b-\rho)}{(b-a)(b-z)}=\frac{\sigma(d-\rho)}{(d-a)(d-z)},$$
$$z=\rho+\frac{(d-\rho)(b-\rho)}{(a-\rho)}.$$

Under the constraint d≥a one has z≥b if and only if $\rho\le a$.

The above construction of QP-majorants is summarized in Table 5.6. The only missing case occurs for $d\le\bar{b}, \rho\ge\omega(d)$. But in this situation $\bar{d}>d$, hence $\rho\ge\omega(d)>d$. But in the non-standardized scale $D=\sigma(d-\rho)<0$, and $f_X(X)$ does not define a feasible reinsurance payment because the usual constraint $0\le f_X(X)\le X$ is not satisfied. Table 5.7 summarizes the special case $\rho=a$ discussed by Heijnen and Goovaerts(1989).

Table 5.6 : maximum franchise deductible transform for standard random variables on $[a,b]$

conditions	maximum	extremal support
(1) $d \ge \bar{a}$:		
(1a) $\rho \le \omega(d)$	$\sigma \cdot \left(\frac{d-\rho}{1+d^2} \right)$	$\{\bar{d}, d\}$
(1b) $\omega(d) \le \rho \le \beta$	$-\frac{1}{2}\sigma x_\rho$	$\{x_\rho, \bar{x}_\rho\}$
(1c) $\rho \ge \beta$	$\sigma \cdot \left(\frac{b-\rho}{1+b^2} \right)$	$\{\bar{b}, b\}$
(2) $\bar{b} \le d \le \bar{a}$:		
(2a) $\rho \le a$	$\left(\frac{\sigma}{b-d} \right) \cdot \left(\frac{(1+ad)(b-\rho)}{b-a} - \frac{(1+ab)(d-\rho)}{d-a} \right)$	$\{a, d, b\}$
(2b) $a \le \rho \le \alpha$	$(-a) \cdot \sigma \cdot \left(\frac{1+\rho a}{1+a^2} \right)$	$\{a, \bar{a}\}$
(2c) $\alpha \le \rho \le \beta$	$-\frac{1}{2}\sigma x_\rho$	$\{x_\rho, \bar{x}_\rho\}$
(2d) $\rho \ge \beta$	$\sigma \cdot \left(\frac{b-\rho}{1+b^2} \right)$	$\{\bar{b}, b\}$
(3) $d \le \bar{b}$:		
(3a) $\rho \le a$	$\sigma(\mu - \rho)$	$\{d, \bar{d}\}$
(3b) $a \le \rho \le \alpha$	$(-a) \cdot \sigma \cdot \left(\frac{1+\rho a}{1+a^2} \right)$	$\{a, \bar{a}\}$
(3c) $\alpha \le \rho \le \omega(d)$	$-\frac{1}{2}\sigma x_\rho$	$\{x_\rho, \bar{x}_\rho\}$
(3d) $\rho \ge \omega(d)$	random function is not feasible	

Table 5.7 : special case $\rho = a$

conditions	maximum	extremal support
$d \ge \bar{a}$	$\frac{\mu + \sigma d}{1+d^2}$	$\{\bar{d}, d\}$
$d \le \bar{a}$	μ	$\{a, \bar{a}\}$

5.4. The disappearing deductible transform.

For $X \in D([A,B];\mu,\sigma)$ the disappearing deductible transform $E[f_X(X)]$ is defined by the piecewise linear function

(5.9) $$f_X(x) = r(x-d_1)_+ + (1-r)(x-d_2)_+, \quad r = \frac{d_2}{d_2 - d_1} \ge 1, \quad 0 \le d_1 < d_2.$$

In the standardized scale, this takes the form

$$(5.10)\qquad f_Z(z) = \sigma r(z-L)_+ + \sigma(1-r)(z-M)_+ = \begin{cases} 0, & z \le L, \\ \sigma r(z-L), & L \le z \le M, \\ \sigma(z-\rho), & z \ge M, \end{cases}$$

where one sets

$$(5.11)\qquad L = \frac{d_1 - \mu}{\sigma},\quad M = \frac{d_2 - \mu}{\sigma},\quad \rho = rL + (1-r)M = -\frac{\mu}{\sigma} < 0.$$

Up to the factor σ, this transform function looks formally like a "two-layers stop-loss transform" with however $r \ge 1$ fixed (see Section 5.5). This fact implies the statements :

(i) In the interval [M,b] the line segment $\sigma r(x-L)$ lies *above* the segment $\sigma(x-\rho)$.
(ii) In the interval [a,M] the line segment $\sigma r(x-L)$ lies *under* the segment $\sigma(x-\rho)$.

Using these geometric properties, a look at the QP-majorants of the "franchise deductible" and "stop-loss" transforms show that the maximum expected value is attained as follows :

(i) If $M \ge \bar{a}$ the best upper bounds are taken from the "stop-loss" Table 5.1 by changing b into M, d into L, and multiplying the stop-loss maxima with σr.
(ii) If $M \le \bar{a}$ take the values of the "franchise deductible" in Table 5.6 changing d to M.

The result, summarized in Table 5.1, generalizes that of Heijnen and Goovaerts(1989) obtained for the special case $\rho = a$.

Table 5.8 : maximum disappearing deductible transform for standard random variables

conditions	maximum	extremal support
(1) $M \ge \bar{a}$: (1a) $L \le \alpha$	$(-a)\cdot\left(\frac{1+aL}{1+a^2}\right)\cdot\sigma r$	$\{a,\bar{a}\}$
(1b) $\alpha \le L \le \omega(M)$	$-\frac{1}{2}x_L\cdot\sigma r$	$\{x_L,\bar{x}_L\}$
(1c) $\omega(M) \le L < M$	$\left(\frac{M-L}{1+M^2}\right)\cdot\sigma r$	$\{\bar{M},M\}$
(2) $\bar{b} \le M \le \bar{a}$:		
(2a) $\rho \le a$	$\left(\frac{\sigma}{b-M}\right)\cdot\left(\frac{(1+aM)(b-\rho)}{b-a}-\frac{(1+ab)(M-\rho)}{M-a}\right)$	$\{a,M,b\}$
(2b) $a \le \rho \le \alpha$	$(-a)\cdot\sigma\cdot\left(\frac{1+\rho a}{1+a^2}\right)$	$\{a,\bar{a}\}$
(2c) $\alpha \le \rho \le \beta$	$-\frac{1}{2}\sigma x_\rho$	$\{x_\rho,\bar{x}_\rho\}$
(2d) $\rho \ge \beta$	$\sigma\cdot\left(\frac{b-\rho}{1+b^2}\right)$	$\{\bar{b},b\}$
(3) $M \le \bar{b}$:		
(3a) $\rho \le a$	$\sigma(\mu-\rho)$	$\{M,\bar{M}\}$

(3b) $a \le \rho \le \alpha$	$(-a)\cdot\sigma\cdot\left(\frac{1+\rho a}{1+a^2}\right)$	$\{a, \bar{a}\}$
(3c) $\alpha \le \rho \le \omega(M)$	$-\frac{1}{2}\sigma x_\rho$	$\{x_\rho, \bar{x}_\rho\}$
(3d) $\rho \ge \omega(M)$	random function is not feasible	

5.5. The two-layers stop-loss transform.

A *two-layers stop-loss transform* is defined by the piecewise linear convex function

(5.12) $$f(x) = r(x-L)_+ + (1-r)(x-M)_+, \quad 0 < r < 1, \quad a < L < M < b.$$

It suffices to solve the standard optimization problem over D(a,b). The interval I=[a,b] is partitioned into the pieces $I_0 = [a,L], I_1 = [L,M], I_2 = [M,b]$. The piecewise linear segments are described by $f_i(x) = \beta_i x + \alpha_i, \beta_0 = 0, \beta_1 = r, \beta_2 = 1, \alpha_0 = 0, \alpha_1 = -rL, \alpha_2 = -d$, where d=rL+(1-r)M is interpreted as the "maximum deductible" of the corresponding two-layers stop-loss contract.

In a first part, the minimum two-layers stop-loss transform is determined. According to Corollary 3.1, it suffices to construct L-minorants for some $X \in D^3_{f,f_i}$, i=0,1,2, and QP-minorants for triatomic random variables of the type (T4).

L-minorants :

The diatomic random variable X=$\{\bar{L}, L\}$ belongs to D^3_{f,f_0} provided $L > \bar{a}$. By Proposition 3.1, in this domain of definition, the minimum is necessarily $E[f(X)] = f_0(0) = 0$. Similarly X=$\{L, \bar{L}\}$, $\{\bar{M}, M\}$ belong to D^3_{f,f_1} if $M > \bar{L} > 0$ and the minimum equals $E[f(X)] = f_1(0) = -rL$. Finally in D^3_{f,f_3} one considers X=$\{M, \bar{M}\}$, which is feasible provided $M < \bar{b}$ and leads to the minimum value $E[f(X)] = f_2(0) = -d$. These results are reported in Table 5.11.

QP-minorants :

It remains to determine the minimum in the following regions :

(1) $0 < L \le \bar{a}$
(2) $0 \le M \le \bar{L}$
(3) $\bar{b} \le M < 0$

One has to construct QP-minorants for triatomic random variables of the type (T4), which are listed in Table 5.9. Their feasible domains suggest to subdivide regions (1), (3) into 2 subregions and region (2) into 4 subregions. This subdivision with the corresponding feasible triatomic random variables is found in Table 5.10.

Table 5.9 : triatomic random variables of type (T4)

feasible support	feasible domain
$X_1 = \{a, L, b\}$	$\overline{b} \le L \le \overline{a}$
$X_2 = \{a, M, b\}$	$\overline{b} \le M \le \overline{a}$
$X_3 = \{L, M, b\}$	$L \le \overline{b} \le M \le \overline{L}$
$X_4 = \{a, L, M\}$	$L \le \overline{a} \le M \le \overline{L}$

Table 5.10 : triatomic random variables in the subregions

subregion	type (T4)
(1.1) $0 < L \le \overline{a}, \quad M \le \overline{a}$	$X_1, \quad X_2$
(1.2) $0 < L \le \overline{a}, \quad M \ge \overline{a}$	$X_1, \quad X_4$
(2.1) $L < \overline{b}, \quad M > \overline{a}, \quad M \le \overline{L}$	$X_3, \quad X_4$
(2.2) $\overline{b} \le L \le 0, \quad M > \overline{a}$	$X_1, \quad X_4$
(2.3) $L < \overline{b}, \quad 0 \le M \le \overline{a}$	$X_2, \quad X_3$
(2.4) $\overline{b} \le L \le 0, \quad 0 \le M \le \overline{a}$	$X_1, \quad X_2$
(3.1) $\overline{b} \le L \le 0, \quad \overline{b} \le M < 0$	$X_1, \quad X_2$
(3.2) $L \le \overline{b}, \quad \overline{b} \le M < 0$	$X_2, \quad X_3$

Applying the QP-method it is required to construct quadratic polynomials $q_i(x) \le f(x), \quad i = 1,2,3,4,$ such that the zeros of $Q^{(i)} = q_i(x) - f(x)$ are the atoms of X_i. Drawing for help pictures of the situation for i=1,2,3,4, which is left to the reader, one gets through elementary calculations the following formulas :

$$q_1(x) = \frac{(x-a)(x-L)(b-d)}{(b-a)(b-L)}$$

$$q_2(x) = c(x-a)(x-z), \quad c = \frac{(b-d)}{(b-a)(b-z)} = \frac{(M-d)}{(M-a)(M-z)}, \quad z = \frac{d(b-a) - M(b-d)}{d-a}$$

$$q_3(x) = e(x-L)(x-y), \quad e = \frac{(b-d)}{(b-L)(b-y)} = \frac{(M-d)}{(M-L)(M-y)}, \quad z = \frac{d(b-L) - M(b-d)}{d-L}$$

$$q_4(x) = \frac{(x-a)(x-L)(M-d)}{(M-a)(M-L)}$$

By application of Theorem 3.1 it is possible to determine when the $q_i(x)$'s are QP-admissible. However, in this relatively simple situation, the graphs of the $q_i(x)$'s show that the following equivalent criteria hold :

$$\begin{aligned}
q_1(x) \le f(x) \quad &\Leftrightarrow \quad q_1(M) \le f(M) = M - d \quad \Leftrightarrow \quad d \le \xi, \\
q_2(x) \le f(x) \quad &\Leftrightarrow \quad q_2(L) \le f(L) = 0 \quad \Leftrightarrow \quad d \ge \xi, \\
q_3(x) \le f(x) \quad &\Leftrightarrow \quad q_3(a) \le f(a) = 0 \quad \Leftrightarrow \quad d \le \xi, \\
q_4(x) \le f(x) \quad &\Leftrightarrow \quad q_4(b) \le f(b) = b - d \quad \Leftrightarrow \quad d \ge \xi,
\end{aligned}$$

where $\xi = \dfrac{bM - aL}{b + M - (a + L)}$ has been setted. Use these criteria for each subregion in Table 5.10 to get the remaining minimum values as displayed in Table 5.11.

In a second part, let us show that the maximum two-layers stop-loss transform can only be obtained numerically through algorithmic evaluation. Applying Theorem 2.1 one determines first the possible types of triatomic random variables for which the maximum may be attained.

Proposition 5.1. Triatomic random variables $X \in D^3_{f,q}$, for which there may exist a QP-majorant, are necessarily of the following types :

(D1)
$$\{x_L, \bar{x}_L\},\ (x_L, \bar{x}_L) \in [a, L]x[L, M]$$
$$\{x_d, \bar{x}_d\},\ (x_d, \bar{x}_d) \in [a, L]x[M, b]$$
$$\{x_M, \bar{x}_M\},\ (x_M, \bar{x}_M) \in [L, M]x[M, b]$$

(D2) $\{a, \bar{a}\}, \{\bar{b}, b\}$

(T1) $\{u, v, w\}$ such that $u := L - r(M - L) \ge a$, $v := L + r(M - L)$, $w := M + (1 - r)(M - L) \le b$, and $u \le \overline{w} < 0$, $\overline{w} \le v \le \overline{u}$

(T2) $\{a, v_a, w_a\}$ such that
$$v_a := a + \left(\frac{2}{1-r}\right) \cdot \left(-r(L-a) + \sqrt{r(L-a)(d-a)}\right) \in [L, M]$$
$$w_a := a - \left(\frac{2}{1-r}\right) \cdot \left(-(d-a) + \sqrt{r(L-a)(d-a)}\right) \in [M, b]$$
and $a \le \overline{w}_a < 0$, $\overline{w}_a \le v_a \le \bar{a}$

$\{u_b, v_b, b\}$ such that
$$u_b := b - \frac{2}{r} \cdot \left(b - d - \sqrt{(1-r)(b-M)(b-d)}\right) \in [a, L]$$
$$v_b := b + \frac{2}{r} \cdot \left((1-r)(b-M) - \sqrt{(1-r)(b-M)(b-d)}\right) \in [L, M]$$
and $u_b \le \bar{b} < 0$, $\bar{b} \le v_b \le \overline{u}_b$

(T3) $\{a, v_{a,b}, b\}$ such that $v_{a,b} := \dfrac{a\sqrt{(1-r)(b-M)} + b\sqrt{r(L-a)}}{\sqrt{(1-r)(b-M)} + \sqrt{r(L-a)}} \in [L, M]$,
and $\bar{b} \le v_{a,b} \le \bar{a}$

Table 5.11 : minimum two-layers stop-loss transform

conditions	minimum	extremal support
$L > \overline{a}$	0	$\{\overline{L}, L\}$
$M > \overline{L} > 0$	$-rL$	$\{L, \overline{L}\},\{\overline{M}, M\}$
$M < \overline{b}$	$-d$	$\{M, \overline{M}\}$
$d \le \frac{bM - aL}{b + M - (a + L)}$:		
$0 < L \le \overline{a}$ $\overline{b} \le L \le 0, \quad \overline{b} \le M$	$\frac{1 + aL}{(b - L)(b - a)} \cdot (b - d)$	$\{a, L, b\}$
$L \le \overline{b} \le M \le \overline{L}$	$\frac{1 + Ly}{(M - L)(M - y)} \cdot (M - d)$ $y = \frac{d(b - L) - M(b - d)}{d - L}$	$\{L, M, b\}$
$d \ge \frac{bM - aL}{b + M - (a + L)}$:		
$0 < L \le \overline{a}, \quad M \le \overline{a}$ $L \le 0, \quad 0 \le M \le \overline{a}$ $\overline{b} \le M \le 0$	$\frac{1 + az}{(b - a)(b - z)} \cdot (b - d)$ $z = \frac{d(b - a) - M(b - d)}{d - a}$	$\{a, M, b\}$
$0 < L \le \overline{a}, \quad M \ge \overline{a}$ $\overline{a} \le M \le \overline{L}$	$\frac{1 + aL}{(M - a)(M - L)} \cdot (M - d)$	$\{a, L, M\}$

Proof of Proposition 5.1. Leading to a minimum, the type (T4) can be eliminated. Type (D1) is immediately settled observing that $d_{01} = L, \ d_{02} = d, \ d_{12} = M$. For the type (D2) the cases where L, M are rand points are included as limiting cases of the type (D1) and thus omitted. Type (T1) is clear. For (T2) three cases must be distinguished. If $w \in I_0$ then necessarily w=a because $w \ne L, d$. The case $w \in I_1$ is impossible because one should have $w \ne L, M$. If $w \in I_2$ then w=b because $w \ne d, M$. Similarly type (T3) may be possible in three ways. If $(v, w) \in I_0 x I_1$ then v=a, w=L because $v \ne d, w \ne M$. If $(v, w) \in I_0 x I_2$ then v=a, w=b because $v \ne L, w \ne M$. If $(v, w) \in I_1 x I_2$ then v=M, w=b because $v \ne L, w \ne d$. However, the two types $\{u, M, b\},\{a, L, u\}$ can be eliminated. Indeed drawing a graph in these situations shows that no QP-majorant can be constructed. The additional constraints on the atoms follow from Lemma I.4.2. ◊

The precise conditions under which the triatomic random variables of Proposition 5.1 allow the construction of a QP-majorant follow from Theorem 3.1 and are displayed in Table 5.12. It suffices to choose appropriately $(u, v, w) \in I_i x I_j x I_k$ such that Theorem 3.1 applies. Details are left to the reader. Drawing pictures shows that the double-sided interval constraints are in fact one-sided constraints.

Table 5.12 : maximizing QP-admissible triatomic random variables

support	value of Δ	values of ξ, η	conditions
$\{x_L, \overline{x}_L\}$	$\Delta_{012}(x_L, \overline{x}_L)$	none $\eta_2 = \dfrac{\beta_2 - \beta_0 - \sqrt{\Delta}}{2c_{01}(\overline{x}_L)}$	$\Delta \le 0$ $\Delta > 0$ and $\eta_2 \ge b$
$\{x_d, \overline{x}_d\}$	$\Delta_{021}(x_d, \overline{x}_d)$	none	$\Delta \le 0$
$\{x_M, \overline{x}_M\}$	$\Delta_{120}(x_M, \overline{x}_M)$	none $\xi_0 = \dfrac{\beta_0 - \beta_1 + \sqrt{\Delta}}{2c_{12}(\overline{x}_M)}$	$\Delta \le 0$ $\Delta > 0$ and $\xi_0 \le a$
$\{a, \overline{a}\}$	$\Delta_{201}(\overline{a}, a)$ $\Delta_{102}(\overline{a}, a)$	$\eta_0 = d - \dfrac{(d-\overline{a})^2}{d-a}$ none $\eta_2 = \dfrac{\beta_2 - \beta_1 - \sqrt{\Delta}}{2c_{10}(a)}$	$M \le \overline{a}$, $\Delta \le 0$, $\eta_0 \le a$ $L \le \overline{a} < M$, $\Delta \le 0$ $L \le \overline{a} < M$, $\Delta > 0$, $\eta_2 \ge b$
$\{\overline{b}, b\}$	$\Delta_{021}(\overline{b}, b)$ $\Delta_{120}(\overline{b}, b)$	$\eta_2 = d + \dfrac{(d-\overline{b})^2}{b-d}$ none $\xi_0 = \dfrac{\beta_0 - \beta_1 + \sqrt{\Delta}}{2c_{12}(b)}$	$\overline{b} \le L$, $\Delta \le 0$, $\eta_2 \ge b$ $L < \overline{b} \le M$, $\Delta \le 0$ $L < \overline{b} \le M$, $\Delta > 0$, $\xi_0 \le a$
$\{u, v, w\}$	none	none	none
$\{a, v_a, w_a\}$	none	$\eta_0 = d - \dfrac{(d-w_a)^2}{d-a}$	$\eta_0 \le a$
$\{u_b, v_b, b\}$	none	$\eta_2 = M + \dfrac{(M-v_b)^2}{b-M}$	$\eta_2 \ge b$
$\{a, v_{a,b}, b\}$	none	$\eta_0 = L - \dfrac{(v_{a,b}-L)^2}{L-a}$ $\eta_2 = M + \dfrac{(M-v_{a,b})^2}{b-M}$	$\eta_0 \le a$ and $\eta_2 \ge b$

A *numerical algorithm* to evaluate the maximizing random variable is as follows :

Step 1 : Give in the standardized scale the values
$a < L < M < b, \quad r \in (0,1), \quad a < 0 < b, ab \le -1$

Step 2 : Find the finite set of triatomic random variables, which satisfy the conditons of Proposition 5.1

Step 3 : For the finite set of triatomic random variables found in step 2, check the QP-admissible conditions of Table 5.12. If a QP-admissible condition is fulfilled, the corresponding triatomic random variable is maximal.

Step 4 : Transform the result back to the non-standardized scale

6. Extremal expected values for symmetric random variables.

We have seen that the majorant/minorant quadratic polynomial method is a main tool to derive best bounds for expected values $E[f(X)]$ over the set of all random variables by known range, mean and variance. In case $f(x)$ is a piecewise linear function, a general algorithm to determine these extremal expected values has been formulated in Sections 2 and 3 and applied in Sections 4 and 5. In view of the following simple result, the same method remains valid if the random variables are additionally symmetric around the mean.

Lemma 6.1. Let $D_S(A):=D_S([-A,A];\mu,\sigma)$ be the set of all symmetric random variables defined on the interval $[-A,A]$ with known mean μ and variance σ^2. Suppose there exists a symmetric quadratic polynomial $q(x)=ax^2+c$ such that

(6.1) $$q(x)\ge(\le)\tfrac{1}{2}\{f(x)+f(-x)\},\ x\in[-A,A],$$

and a finite atomic random variable X^* (X_*) such that

(6.2) $$\Pr(q(X^*)=\tfrac{1}{2}\{f(X^*)+f(-X^*)\})=1,(\Pr(q(X^*)=\tfrac{1}{2}\{f(X^*)+f(-X^*)\})=1),$$

then one has $\max_{X\in D_S(A)}\{E[f(X)]\}=E[q(X^*)],\quad (\min_{X\in D_S(A)}\{E[f(X)]\}=E[q(X_*)]).$

Proof. Since $E[f(-X)]=E[f(X)]$ for $X\in D_S(A)$, this follows by majorization (minorization) using that $E[f(X)]=E[\tfrac{1}{2}\{f(X)+f(-X)\}]$.

As illustration the detailed optimization for a basic case is provided. Therefore the rest of the Section is devoted to the construction of the extremal stop-loss transforms for symmetric random variables. As seen in Section I.6, it suffices to consider the case of standard symmetric random variables defined on $[-a,a]$, where for the existence of such random variables we assume that $a\ge1$.

Theorem 6.1. The maximum stop-loss transform for standard symmetric random variables with range $[-a,a]$, $a\ge1$, is determined in Table 6.1.

Table 6.1 : maximum stop-loss transform for standard symmetric random variables

case	condition	maximum $\pi^*(d)$	extremal symmetric support
(1)	$-a \le d \le -\frac{1}{2}a$	$-d+\frac{1}{2}\left(\frac{a+d}{a^2}\right)$	$\{-a,0,a\}$
(2)	$-\frac{1}{2}a \le d \le -\frac{1}{2}$	$-d-\frac{1}{8d}$	$\{2d,0,-2d\}$
(3)	$-\frac{1}{2} \le d \le \frac{1}{2}$	$\frac{1}{2}(1-d)$	$\{-1,1\}$
(4)	$\frac{1}{2} \le d \le \frac{1}{2}a$	$\frac{1}{8d}$	$\{-2d,0,2d\}$
(5)	$\frac{1}{2}a \le d \le a$	$\frac{1}{2}\left(\frac{a-d}{a^2}\right)$	$\{-a,0,a\}$

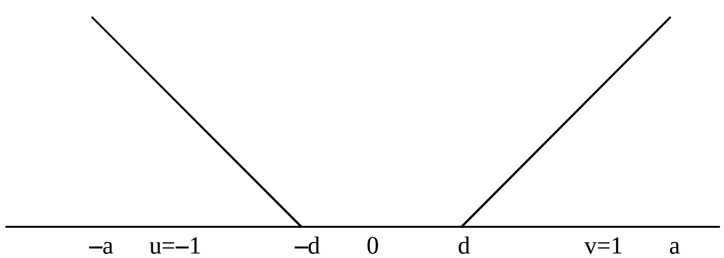
Proof. Since $X \in D_S([-a,a];0,1)$ is symmetric, the stop-loss transform satisfies the relation $\pi(d)=-d+\pi(-d)$. Therefore it suffices to consider the cases for which $d \ge 0$. In each case one constructs a symmetric quadratic polynomial majorant q(x) such that

$$q(x) \ge f(x) = \tfrac{1}{2}\{(x-d)_+ + (-x-d)_+\},\ x \in [-a,a],$$

and where equality is attained at the atoms of the extremal symmetric support. According to our algorithm for piecewise linear functions, the interval $I=[-a,a]$ has to be partitioned into the three pieces $I_0=[-a,-d], I_1=[-d,d], I_2=[d,a]$, such that on each piece f(x) coincides with the linear function $\ell_i(x), i=0,1,2$, defined by respectively $\ell_0(x)=-\frac{1}{2}(x+d),\ \ell_1(x)=0,\ \ell_2(x)=\frac{1}{2}(x-d)$. A case by case construction follows.

<u>Case (3)</u> : $0 \le d \le \frac{1}{2}$

One constructs $q(x) \ge f(x)$ as in the following figure :



According to Theorem 2.1, type (D1), a diatomic random variable $X^* = \{u, v\}$, $u \in I_0$, $v \in I_2$, can be extremal only if $(u,v) = (d_{02} - \sqrt{1+d_{02}^2}, d_{02} + \sqrt{1+d_{02}^2}) = (-1,1)$. The corresponding uniquely defined symmetric quadratic polynomial majorant is given by

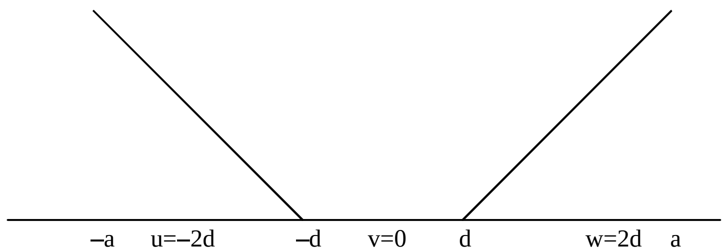
$$q(x) = \frac{1}{4}x^2 + \frac{1}{4} - \frac{1}{2}d.$$

The conditions of Theorem 3.1 are fulfilled, (C1) because $\beta_2 = \frac{1}{2} > \beta_0 = -\frac{1}{2}$, and (C2) because the discriminant of $Q_1(x) = q(x) - l_1(x)$, which equals $\frac{1}{2}d - \frac{1}{4}$, is always ≤ 0. The maximum stop-loss transform equals

$$\pi^*(d) = E\left[q(X^*)\right] = \frac{1}{2}(1-d).$$

<u>Case (4)</u> : $\frac{1}{2} \le d \le \frac{1}{2}a$

One starts with the following figure :



A triatomic extremum $X^* = \{u, v, w\}$ of type (T1) is only possible provided

$$u = d_{01} + d_{02} - d_{12} = -2d \in I_0 = [-a, -d],$$
$$v = d_{12} + d_{01} - d_{02} = 0 \in I_1 = [-d, d],$$
$$w = d_{02} + d_{12} - d_{01} = 2d \in I_2 = [d, a].$$

According to Lemma I.4.2 , the support $\{-2d, 0, 2d\}$ defines a feasible triatomic random variable exactly when $1 \le 2d \le a$, which is the defining condition in the present case (4). Observing that the condition (C1) for the type (T1) in Theorem 3.1 is fulfilled, one obtains the symmetric quadratic polynomial majorant

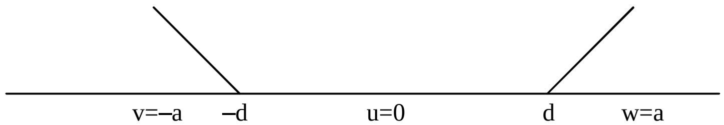
$$q(x) = \frac{x^2}{8d},$$

which leads to the maximum stop-loss transform

$$\pi^*(d) = E\left[q(X^*)\right] = \frac{1}{8d}.$$

<u>Case (5)</u> : $\frac{1}{2}a \le d \le a$

The following figure displays a symmetric quadratic polynomial majorant q(x) :



A triatomic extremum $X^* = \{u, v, w\}$ of type (T3) with $v = -a \in I_0$, $w = a \in I_2$ and $u \in I_1$ a double zero of $Q_1(x) = q(x) - \ell_1(x)$ is only possible if $u = \frac{1}{2}(v + w) = 0$ because

$$\frac{\nabla_{12}\ell(w)}{\nabla_{10}\ell(v)} = \frac{\ell_2(w) - \ell_1(w)}{\ell_0(v) - \ell_1(v)} = 1.$$

One obtains without difficulty that

$$q(x) = \frac{1}{2}\left(\frac{a-d}{a^2}\right)x^2.$$

The condition (C1) for type (T3) in Theorem 3.1 is fulfilled. First of all, one has $Q_1(x) \geq 0$ on I_1. Furthermore the second zero of $Q_0(x)$ equals $\eta_0 = \frac{-da}{a-d}$ and does not belong to $\overset{o}{I}_0 = (-a, -d)$, hence $Q_0(x) \geq 0$ on I_0. Similarly the second zero of $Q_2(x)$ equals $\eta_2 = \frac{ad}{a-d}$ and does not belong to $\overset{o}{I}_2 = (d, a)$, hence $Q_2(x) \geq 0$ on I_2. One concludes by noting that

$$\pi^*(d) = E\left[q(X^*)\right] = \frac{1}{2}\left(\frac{a-d}{a^2}\right). \quad \Diamond$$

The limiting case as $a \to \infty$ is of special interest.

Table 6.2 : maximum stop-loss transform for standard symmetric random variables with range $(-\infty, \infty)$

case	condition	maximum $\pi^*(d)$	extremal symmetric support
(1)	$d \leq -\frac{1}{2}$	$-d - \frac{1}{8d}$	$\{2d, 0, -2d\}$
(2)	$-\frac{1}{2} \leq d \leq -\frac{1}{2}$	$\frac{1}{2}(1-d)$	$\{-1, 1\}$
(3)	$d \geq \frac{1}{2}$	$\frac{1}{8d}$	$\{-2d, 0, 2d\}$

Furthermore, by appropriate transformations of variables, one recovers, in somewhat different presentation, the result by Goovaerts et al.(1984), p. 297, case C_{35}. In particular our

construction is an alternative to the method proposed by these authors on pp. 300-301, which has the advantage to be the application of a unified general algorithm.

Theorem 6.2. The maximum stop-loss transform for symmetric random variables with range $[A, B]$, symmetry center $C = \frac{1}{2}(A+B)$, and known variance σ^2 such that $0 \le \sigma \le E = \frac{1}{2}(B-A)$ is determined in Table 6.3.

Table 6.3 : maximum stop-loss transform for symmetric random variables with range $[A, B]$

case	condition	maximum $\pi^*(D)$	extremal symmetric support
(1)	$A \le D \le A + \frac{1}{2}E$	$(C-D) + \frac{1}{2}\left(\frac{D-A}{E^2}\right)\sigma^2$	$\{C-E, C, C+E\}$
(2)	$A + \frac{1}{2}E \le D \le C - \frac{1}{2}\sigma$	$(C-D) + \frac{\sigma^2}{8(C-D)}$	$\{C-2(C-D), C, C+2(C-D)\}$
(3)	$C - \frac{1}{2}\sigma \le D \le C + \frac{1}{2}\sigma$	$\frac{1}{2}(\sigma + C - D)$	$\{C - \frac{1}{2}\sigma, C + \frac{1}{2}\sigma\}$
(4)	$C + \frac{1}{2}\sigma \le D \le B - \frac{1}{2}E$	$\frac{\sigma^2}{8(D-C)}$	$\{C-2(D-C), C, C+2(D-C)\}$
(5)	$B - \frac{1}{2}E \le D \le B$	$\frac{1}{2}\left(\frac{B-D}{E^2}\right)\sigma^2$	$\{C-E, C, C+E\}$

Proof. Let X be a random variable, which satisfies the stated properties. Then the random variable $Z = \frac{X-C}{\sigma}$ is standard symmetric with range $\left[-\frac{E}{\sigma}, \frac{E}{\sigma}\right]$. It follows that Table 6.3 follows from Table 6.1 by use of the transformations

$$d \to \frac{D-C}{\sigma}, \quad a \to \frac{E}{\sigma}, \quad \text{atom } z \to \text{atom } x = C + \sigma z,$$

as well as the formula

$$\pi_X(D) = \sigma \cdot \pi_Z\left(\frac{D-C}{\sigma}\right). \ \Diamond$$

The construction of the best lower bound is equally simple. However, note that it has not been discussed in Goovaerts et al.(1984).

Theorem 6.3. The minimum stop-loss transform for standard symmetric random variables with range $[-a, a]$, $a \ge 1$, is determined in Table 6.4.

Table 6.4 : minimum stop-loss transform for standard symmetric random variables

case	condition	minimum $\pi_*(d)$	extremal symmetric support
(1)	$-a \le d \le -1$	$-d$	$\{d, 0, -d\}$
(2)	$-1 \le d \le 0$	$-d + \frac{1}{2}\left(\frac{1-d^2}{a-d}\right)$	$\{-a, d, -d, a\}$
(3)	$0 \le d \le 1$	$\frac{1}{2}\left(\frac{1-d^2}{a+d}\right)$	$\{-a, -d, d, a\}$
(4)	$1 \le d \le a$	0	$\{-d, 0, d\}$

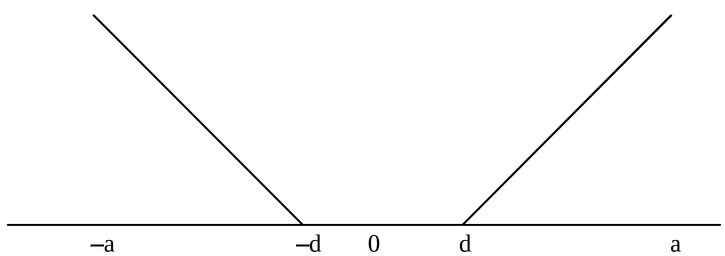
Proof. By symmetry, it suffices to consider the case $d \ge 0$. In each situation, one constructs a symmetric quadratic polynomial minorant

$$q(x) \le f(x) = \tfrac{1}{2}\{(x-d)_+ + (-x-d)_+\}, x \in [-a, a],$$

where equality is attained at the atoms of the minimal random variable X_* with the displayed extremal symmetric support.

Case (3) : $0 \le d \le 1$

There exists a unique symmetric quadratic polynomial such that $q(-a) = q(a) = \frac{1}{2}(a-d)$, $q(-d) = q(d) = 0$ as in the following figure :



It is given by $q(x) = \frac{1}{2}\left(\frac{x^2 - d^2}{a+d}\right)$. A calculation shows that

$$Q_0(x) = q(x) - l_0(x) = \frac{1}{2} \cdot \frac{(x+d)(x+a)}{(a+d)} \le 0 \quad \Leftrightarrow \quad x \in I_0 = [-a, -d],$$

$$Q_1(x) = q(x) \le 0 \quad \Leftrightarrow \quad x \in I_1 = [-d, d],$$

$$Q_2(x) = q(x) - l_2(x) = \frac{1}{2} \cdot \frac{(x-d)(x-a)}{(a+d)} \le 0 \quad \Leftrightarrow \quad x \in I_2 = [d, a].$$

Therefore one has $q(x) \le f(x)$ on $I = [-a, a]$. A look at the probabilities of the symmetric random variable X_* with support $\{-a, -d, d, a\}$ shows that it is feasible exactly when $d^2 \le 1$. The minimal stop-loss transform equals

$$\pi_*(d) = E[q(X_*)] = \frac{1}{2}\left(\frac{1-d^2}{a+d}\right).$$

<u>Case (4)</u> : $1 \le d \le a$

The symmetric random variable X_* with support $\{-d, 0, d\}$ is feasible and $q(x) \equiv 0$ is a minorant of f(x) such that $\Pr(q(X_*) = f(X_*)) = 1$. This implies that $\pi_*(d) = 0$. ◊

Similarly to Table 6.3, it is not difficult to obtain the minimal stop-loss transform for symmetric random variables with range $[A, B]$, symmetry center $C = \frac{1}{2}(A+B)$, and variance σ^2 such that $0 \le \sigma \le E = \frac{1}{2}(B-A)$.

Table 6.5 : minimum stop-loss transform for symmetric random variables with range $[A, B]$

case	condition	minimum $\pi_*(d)$	extremal symmetric support
(1)	$A \le D \le C - \sigma$	$C - D$	$\{D, C, 2C-D\}$
(2)	$C - \sigma \le D \le C$	$C - D + \frac{1}{2}\left(\frac{\sigma^2 - (C-D)^2}{B-D}\right)$	$\{A, D, 2C-D, B\}$
(3)	$C \le D \le C + \sigma$	$\frac{1}{2}\left(\frac{\sigma^2 - (C-D)^2}{B-D}\right)$	$\{A, 2C-D, D, B\}$
(4)	$1 \le d \le a$	0	$\{2C-D, C, D\}$

7. Notes.

In view of the historical and primordial importance in Probability and Statistics of Chebyshev's inequality, it can fairly be said that the origin of the majorant/minorant polynomial method goes back to Chebyshev, Markov and Possé. It has been first formulated as general principle by Isii(1960) and Karlin(1960) as mentioned by Karlin and Studden(1966). In the last monograph it appears as the main Theorem 2.1 in Chapter XII. More recent and theoretical views of the majorant/minorant method include Whittle(1992), Chapter 12.4.

Concerning the piecewise linear assumption made for the development of our general algorithm, it may be an additional motivation to note that in Financial Economics piecewise linear sharing rules can be solutions of certain equilibrium models of risk exchange, as shown in Hürlimann(1987), Proposition 1. Furthermore, the technique of bounding random functions by piecewise linear functions has certainly be applied in many fields (an example is Huang et al.(1977)).

The inequalities of Chebyshev type and their various generalizations and analogues have generated a vast and still active research field in Applied Probability and Statistics. Well-known surveys are Godwin(1955), Savage(1961), and recommended is Karlin and Studden(1966), Chapters XII to XIV, for a readable account. A recent source of abundant material is further found in Johnson and Kotz(1982/88). For a univariate event, which is the finite union of any subintervals of the real numbers, Godwin suggests it would be useful to have inequalities in terms of any number of moments, and states that Markov solved this problem, whose solution seems however to be lost. Since the indicator function of such an event is piecewise linear, it is in principle possible to retrieve this solution (at least by fixed mean and variance) through application of our general algorithm with perhaps the aid of a computer. Selberg did a first step in this direction. Our simple proof of Selberg's inequality in Section 4 may be viewed as a modern up-date, which turns out to be more comprehensive than the exposé by Karlin and Studden(1966), Example XII.3.1, p. 475-79. It is also worthwile to mention that other kinds of probabilistic inequalities may be reduced to inequalities of Chebyshev type. This assertion holds for the Bonferroni inequalities, which have been studied under this aspect by Samuels and Studden(1989), and Sibuya(1991). Sections 2, 3 and 5 are taken from Hürlimann(1996c/96d) with some minor changes.

Readers more specifically interested in Actuarial Science and Finance should note that there exists already a considerable amount of literature devoted to reinsurance contracts of stop-loss type and other financial derivatives. The excess-of-loss or stop-loss contract is long known to be an "optimal" reinsurance structure under divers conditions, as shown among others by Borch(1960), Kahn(1961), Arrow(1963/74), Ohlin(1969), Pesonen(1983), Hesselager(1993), Wang(1995) (see also any recent book on Risk Theory). Since in the real-world gross stop-loss premiums are usually heavily loaded and unlimited covers are often not available, the limited stop-loss contract is sometimes considered as alternative. Such a contract may be useful in the situation one wants to design reinsurance structures compatible with solvability conditions (e.g. Hürlimann(1993b/95a)). In case of unimodal distributions the corresponding optimization has been treated in Heijnen and Goovaerts(1987), and Heijnen(1990). A two-layers stop-loss contract has been shown "optimal" for any stop-loss order preserving criterion among the restricted set of reinsurance contracts generated by the n-layers stop-loss contracts with fixed expected reinsurance net costs (e.g. Van Heerwaarden(1991), p. 121, Kaas et al.(1994), Example VIII.3.1, p. 86-87). It appears also as optimal reinsurance structure in the theory developed by Hesselager(1993). It belongs to the class of perfectly hedged all-finance derivative contracts introduced by the author(1994/95b). As most stop-loss like treaty it may serve as valuable substitute in situations a stop-loss contract is not available, undesirable or does not make sense (for this last point see Hürlimann(1993b), Section 4, "Remarque"). For readers interested in mathematical genesis, it may be instructive to mention that our general algorithmic approach to "best bounds for expected values for piecewise linear random functions" has been made possible through a detailed study of the extremal problems for the two-layers stop-loss transform. The inequality of Bowers(1969) has inspired a lot of actuarial work, among which the earliest papers to be mentioned are by Gagliardi and Straub(1974) (see also Bühlmann(1974)), Gerber and Jones(1976), and Verbeek(1977). A short presentation of Verbeek's inequality is given by Kaas et al.(1994), Example X.1.1. Bivariate extensions of the inequality of Bowers are derived in Chapter V.

In other contexts, "best bounds for expected values" have been studied by many researchers including authors like Hoeffding(1955), Harris(1962), Stoyan(1973), etc. Similar extremal moment problems, which have not been touched upon in the present work, will require "non-linear methods". One may mention specializations to unimodal distributions by

Mallows(1956/63) and Heijnen and Goovaerts(1989). The latter authors apply the well-known Khintchine transform as simplifying tool. Inequalities of Chebyshev type involving conditional expected values have been discussed in particular by Mallows and Richter(1969). The derivation of best bounds, which take into account statistical estimation of the mean and variance parameters or other statistical inference methods, seems quite complex and has been illustrated among others by Guttman(1948) and more recently by Saw, Yang and Mo(1984) and Konijn(1987).

The similar problem of the maximization of variance values $Var[f(X)]$ by known range, mean and variance has not been touched upon in the present monograph. Certainly there exist a lot of literature dealing with this. A recent representative paper containing important references is Johnson(1993). The special stop-loss function has been dealt with by De Vylder and Goovaerts(1983), Kremer(1990), Birkel(1994) and Schmitter(1995). A unified approach to some results is contained in Hürlimann(1997a/b).

CHAPTER III

BEST BOUNDS FOR EXPECTED VALUES BY GIVEN RANGE AND KNOWN MOMENTS OF HIGHER ORDER

1. Introduction.

The construction of quadratic polynomial majorants and minorants for piecewise linear functions has been systematically discussed in Chapter II. A higher order theory has to include precise statements for polynomial majorants of arbitrary degree. Systematic results about the degree three and four are especially useful because many contemporary real-world stochastic phenomena, including the fields of Insurance and Finance, depend upon the coefficients of skewness and kurtosis of the corresponding distribution functions, which describe the phenomena of interest.

The most prominent example consists of the Chebyshev-Markov inequalities, which provide bounds on a distribution function when a finite number of its first moments are given. Our emphasis is on explicit analytical and numerical results for moments up to order four. In this important situation, an improved original presentation of these inequalities is presented in Section 4, which may be viewed as a modern elementary constructive account of the original Chebyshev-Markov inequalities.

The Chebyshev problem has generated a vast area of attractive research problems, known under the name "inequalities of Chebyshev type". From our point of view the recent actuarial research about bounds for stop-loss premiums in case of known moments up to the fourth order belong also to this area. An improved presentation of the relatively complex maximal stop-loss transforms is found in Section 5. Both the content of Sections 4 and 5 is a prerequisite for a full understanding of the new developments made in Chapter IV.

As a preliminary work, in Sections 2 and 3, it is shown how to construct polynomial majorants for the Heaviside indicator function $I_{(-\infty,t]}(x)$, which is 1 if $x\le t$ and 0 otherwise, and for the stop-loss function $(x-d)_+$, d the deductible. Both belong to the class of piecewise linear functions f(x) on an interval $I=[a,b]$, $-\infty \le a < b \le \infty$. For these simple but most important prototypes, one can decompose I into two disjoint adjacent pieces such that $I = I_1 \cup I_2$, and the function of interest is a linear function $f(x)=\ell_i(x)=\alpha_i+\beta_i x$ on each piece I_i, i=1,2. If q(x) is a polynomial of degree $n\ge 2$, then q(x)-f(x) is a piecewise polynomial function of degree n, which is denoted by Q(x) and which coincides on I_i with the polynomial $Q_i(x)=q(x)-\ell_i(x)$ of degree n. For the construction of polynomial majorants $q(x)\ge f(x)$ on I, one can restrict the attention to finite atomic random variables X with support $\{x_0=a, x_1,..., x_r, x_{r+1}=b\}\subset I$ such that $\Pr(q(X)=f(X))=1$ (e.g. Karlin and Studden(1966), Theorem XII.2.1). By convention if $a=-\infty$ then $x_0=a$ is removed from the support and if $b=\infty$ then $x_{r+1}=b$ is removed. In general, the fact that $x_0=a$ or/and $x_{r+1}=b$ does no belong to the support of X is technically achieved by setting the corresponding probabilities equal to zero. If an atom of X, say x_k, is an interior point of some I_i, then it must be a double zero of $Q_i(x)$. Indeed $q(x)\ge \ell_i(x)$ for $x\in I_i$ can only be fulfilled if the line $\ell_i(x)$ is tangent to q(x) at x_k, that is $q'(x_k)=\ell_i'(x_k)$. This simple observation allows one to derive finite exhaustive lists of all polynomials of a given degree, which can be used to construct polynomial majorants (see Tables 2.1 and 3.1) and minorants (see Remarks 2.1 and 3.1). A second step in the construction of best bounds for expected values consists in a detailed analysis of the algebraic moment problem for finite atomic random variables with support $\{x_0 = a, x_1, ..., x_r, x_{r+1} = b\}$, whose mathematical background has been introduced in Chapter I. The most useful results are based on the explicit analytical structure of di- and triatomic random variables by given range and known moments up to order four as presented in Section I.5.

2. Polynomial majorants and minorants for the Heaviside indicator function.

The indicator function is denoted by $f(x)=I_{(-\infty,t]}(x)$ as defined in the introductory Section 1. We decompose the interval $I=[a,b]$ into the pieces $I_1=[a,t]$, $I_2=[t,b]$, such that $f(x)=\ell_1(x)=1$ on I_1 and $f(x)=\ell_2(x)=0$ on I_2. For a fixed $m \in \{1,...,r\}$ the atom $x_m = t$ belongs always to the support of a maximizing finite atomic random variable X. We show that a polynomial majorant of fixed degree for $I_{(-\infty,t]}(x)$ is always among the finite many possibilities listed in Table 2.1 below.

Proposition 2.1. Let $\{x_0 = a, x_1, ..., x_m = t, ..., x_r, x_{r+1} = b\}$, $x_r < x_s$ for r<s, $m \in \{1,...,r\}$, be the ordered support of a random variable X defined on I, and let q(x) be a polynomial majorant such that $\Pr(q(X)=f(X))=1$ and $q(x) \geq f(x)$ on I. Then q(x) is a polynomial uniquely determined by the conditions in Table 2.1.

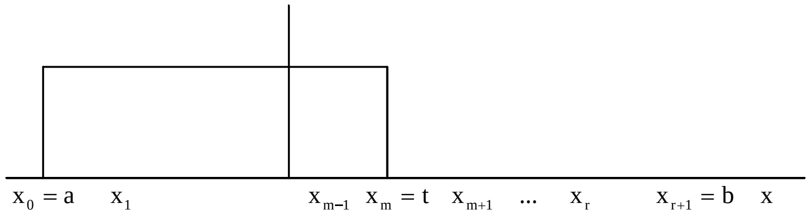
Table 2.1 : polynomial majorants for the Heaviside indicator function

case	support with $x_m = t$	$Q_j(x_i)=0,\ j=1,2$	$Q_j'(x_i)=0,\ j=1,2$	deg q(x)
(1)	$\{a, x_1, ..., x_r, b\}$	i=0,...,r+1	$i \neq 0$, m, r+1	2r
(2)	$\{a, x_1, ..., x_r\}$	i=0,...,r	$i \neq 0$, m	2r-1
(3)	$\{x_1, ..., x_r, b\}$	i=1,...,r+1	$i \neq$ m, r+1	2r-1
(4)	$\{x_1, ..., x_r\}$	i=1,...,r	$i \neq$ m	2r-2

Proof. We restrict our attention to the derivation of case(1). The other cases are shown by the same method and omitted for this reason. One must show the existence of a unique polynomial q(x) of degree n=2r as in Figure 2.1.

Figure 2.1 : polynomial majorant $q(x) \geq I_{(-\infty,t]}(x)$, $x \in [a,b]$

$y = I_{(-\infty,t]}(x)$



Consider the unique polynomial q(x) of degree n=2r such that

$$q(x_i) = \begin{cases} 1, & i = 0,\ldots,m \\ 0, & i = m+1,\ldots,r+1 \end{cases}$$
$$q'(x_i) = 0, \quad i \neq 0, m, r+1$$

By definition of $Q_j(x)$, j=1,2, the conditions of Table 2.1 under case (1) are fulfilled. By the theorem of Rolle, the derivaqtive q'(x) vanishes at least once on each of the r subintervals $(x_i, x_{i+1}), 0 \le i \le r, i \neq m$. It follows that there are exactly (r-1)+r=n-1 zeros of q'(x) on I. Furthermore one has q'(x) ≠ 0 on (x_m, x_{m+1}). More precisely one has q'(x)<0 on (x_m, x_{m+1}) because $q(x_m) = 1 > q(x_{m-1}) = 0$. It follows that q(x) is local minimal at all $x_i, i \neq 0, m, r+1$, and local maximal between each consecutive minima, as well as in the intervals (a, x_1) and (x_r, b). These properties imply the inequality q(x)≥ $I_{(-\infty,t]}(x)$. ◊

Remark 2.1. To construct a polynomial minorant of fixed degree for $I_{(-\infty,t]}(x)$, it suffices to construct a polynomial majorant for $1 - I_{(-\infty,t]}(x)$. Such a polynomial will a fortiori be a polynomial majorant for the indicator function

$$I_{[t,\infty)} = \begin{cases} 0, & x < t, \\ 1, & x \ge t, \end{cases}$$

which has been modified in x=t. The symmetry of the construction shows that the possible polynomial majorants for $I_{[t,\infty)}$ are exactly located at the same supports as those for $I_{(-\infty,t]}(x)$. In this situation Table 2.1 applies with the difference that the polynomials $Q_j(x)$ are replaced by $Q_1(x) = q(x)$ on I_1 and $Q_2(x) = q(x) - 1$ on I_2.

3. Polynomial majorants and minorants for the stop-loss function.

In this Section one sets $f(x) = (x - d)_+$, $d \in (a, b)$ the deductible, $I_1 = [a, d]$, $I_2 = [d, b]$. Then the stop-loss function f(x) may be viewed as the piecewise linear function defined by $f(x) = \ell_1(x) = 0$ on I_1, $f(x) = \ell_2(x) = x - d$ on I_2. By convention $m \in \{1,\ldots,r\}$ is fixed such that $x_m < d < x_{m+1}$. A polynomial majorant of fixed degree for f(x) belongs always to one of the finitely many types listed in Table 3.1 below. The notations q(x;ξ ,d) and Q_j(x;ξ,d) mean that these functions depend upon the parameter vector ξ =$(x_0,\ldots,x_{r+1})$ and the deductible d.

Proposition 3.1. Let $\{x_0 = a, x_1, ..., x_r, x_{r+1} = b\}$, $x_r < x_s$ for r<s, $x_m < d < x_{m+1}$, be the ordered support of a random variable X defined on I, and let q(x) be a polynomial majorant such that $\Pr(q(X) = f(X)) = 1$ and q(x)≥f(x) on I. Then q(x) is a polynomial uniquely determined by the conditions in Table 3.1.

Proof. There are essentially two typical cases for which a proof is required, say (1a) and (1b). The other cases are shown by the same method and omitted for this reason.

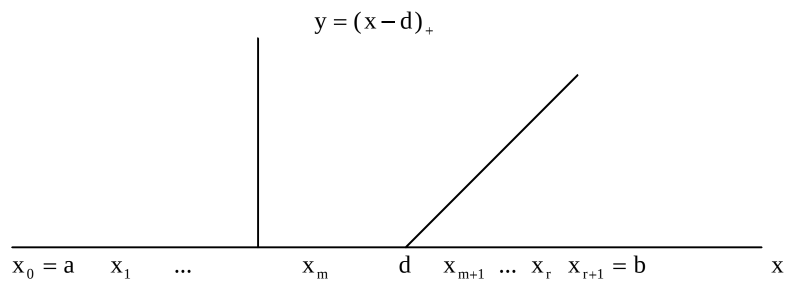
Case (1a) :

One shows the existence of a unique polynomial q(x) of degree n=2r as in Figure 3.1.

Table 3.1 : polynomial majorants for the stop-loss function

case	support $x_m < d < x_{m+1}$	$Q_j(x_i) = 0$, $j = 1,2$	$Q_j^{'}(x_i) = 0$, $j = 1,2$	deg q(x)	condition on deductible d
(1a)	$\{a, x_1, ..., x_r, b\}$	i = 0,...,r+1	i = 1,...,r–1	2r	$Q_2^{'}(x_r;\xi,d)=0$
(1b)	$\{a, x_1, ..., x_r, b\}$	i = 0,...,r+1	i = 1,...,r	2r+1	$Q_1(y;\xi,d)=0$, $y \in (-\infty, a]$
(1c)	$\{a, x_1, ..., x_r, b\}$	i = 0,...,r+1	i = 1,...,r	2r+1	$Q_2(z;\xi,d)=0$, $z \in [b, \infty)$
(2a)	$\{x_1, ..., x_r, b\}$	i = 1,...,r+1	i = 1,...,r	2r	$Q_2(z;\xi,d)=0$, $z \in [b, \infty)$
(2b)	$\{x_1, ..., x_r, b\}$	i = 1,...,r+1	i = 2,...,r	2r–1	$Q_1^{'}(x_1;\xi,d)=0$,
(3a)	$\{a, x_1, ..., x_r\}$	i = 0,...,r	i = 1,...,r	2r	$Q_1(y;\xi,d)=0$, $y \in (-\infty, a]$
(3b)	$\{a, x_1, ..., x_r\}$	i = 0,...,r	i = 1,...,r–1	2r–1	$Q_2^{'}(x_r;\xi,d)=0$
(4a)	$\{x_1, ..., x_r\}$	i = 1,...,r	i = 1,...,r–1	2r–2	$Q_2^{'}(x_r;\xi,d)=0$
(4b)	$\{x_1, ..., x_r\}$	i = 1,...,r	i = 1,...,r	2r-1	$Q_1(y;\xi,d)=0$, $y \in (-\infty, a]$
(4c)	$\{x_1, ..., x_r\}$	i = 1,...,r	i = 1,...,r	2r-1	$Q_2(z;\xi,d)=0$, $z \in [b, \infty)$

Figure 3.1 : polynomial majorant $q(x) \geq (x-d)_+$, $x \in [a,b]$, case (1a)



Consider the unique polynomial q(x) of degree n=2r such that

$$q(x_i) = \begin{cases} 0, & i = 0, \dots, m \\ x_i - d, & i = m+1, \dots, r+1 \end{cases}$$

$$q'(x_i) = \begin{cases} 0, & i = 1, \dots, m \\ 1, & i = m+1, \dots, r-1 \end{cases}$$

By definition of $Q_j(x)$, j=1,2, the conditions of Table 3.1 under case (1a) are fulfilled. In order that $q(x) \geq (x-d)_+$, the line $\ell_2(x) = x - d$ must be tangent of q(x) at the remaining atom $x = x_r$, that is $q'(x_r) = 1$ or $Q_2'(x_r;\xi,d)=0$. This condition is an implicit equation for the deductible d and restricts its range of variation (for precise statements a further analysis is required, as will be seen in Section 5). The theorem of Rolle implies the following facts :

(i) $Q_1'(x) = q'(x)$ vanishes at least once on each of the m subintervals (x_i, x_{i+1}), $i = 1,\dots,m$.

(ii) $Q_2'(x) = q'(x) - 1$ vanishes at least once on each of the r-m subintervals (x_i, x_{i+1}), $i = m+1,\dots,r$.

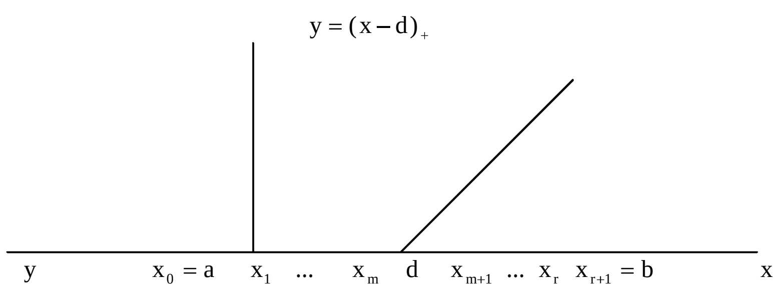
(iii) $Q_1'(x) \neq 0$ on $(x_m, d]$ and $Q_2'(x) \neq 0$ on $[d, x_{m+1})$. More precisely one has $Q_1'(x) > 0$ on $(x_m, d]$ since $Q_1(x_m) = 0 < Q_1(x_{m+1}) = x_{m+1} - d$, and $Q_2'(x) < 0$ on $[d, x_{m+1})$ since $Q_2(x_m) = d - x_m > 0 = Q_2(x_{m+1})$.

In particular there are exactly n-1 zeros of q'(x) on I. It follows that $Q_1(x)$ is local minimal at all x_i, $i = 1,\dots,m$, and local maximal between each consecutive minima, as well as in the interval (a, x_1). Similarly $Q_2(x)$ is local minimal at all x_i, $i = m+1,\dots,r$, and local maximal between each consecutive minima, as well as in the interval (x_r, b). These properties imply that $Q_1(x) \geq 0$ on I_1 and $Q_2(x) \geq 0$ on I_2, which together means that $q(x) \geq (x-d)_+$ on $I_1 \cup I_2 = I$.

<u>Case (1b)</u> :

One shows the existence of a unique polynomial q(x) of degree n=2r+1 as in Figure 3.2, where y is a further zero of q(x) in $(-\infty, a]$.

<u>Figure 3.2</u> : polynomial majorant $q(x) \geq (x-d)_+$, $x \in [a,b]$, case (1b)



Consider the unique polynomial q(x) of degree n=2r+1 such that

$$q(x_i) = \begin{cases} 0, & i = 0,\dots,m \\ x_i - d, & i = m+1,\dots,r+1 \end{cases}, \quad q'(x_i) = \begin{cases} 0, & i = 1,\dots,m \\ 1, & i = m+1,\dots,r \end{cases}$$

By definition of $Q_j(x)$, j=1,2, the conditions of Table 3.1 under case (1b) are fulfilled. In order that q(x) is a polynomial of odd degree, there must exist a further zero y of q(x) in $(-\infty, a]$, which yields the implicit equation $Q_1(y;\xi,d)=0$ for the deductible d (precise statements are analyzed later in Section 5). The rest of the proof similarly to case (1a). ◊

Remark 3.1. The construction of polynomial minorants for the stop-loss function is much easier. In fact a finite atomic random variable, which maximizes the probability distribution function also minimizes the stop-loss transform. A proof of this result is given by Kaas and Goovaerts(1986b), Theorem 1. Therefore, for polynomial minorants, Table 2.1 applies.

4. The Chebyshev-Markov bounds by given range and known moments to order four.

Based on Table 2.1 and Remark 2.1, it is possible to present an elementary proof of the Chebyshev-Markov inequalities for a distribution function defined over an arbitrary interval I=$[a,b]$, $-\infty \le a < b \le \infty$. For this it suffices to analyze in details the possible supports in Table 2.1, which must all be solutions of the moment problem for the interval $[a,b]$. Using the analytical-algebraic structure of the sets of di- and triatomic random variables, as presented in Section I.5, the bounds up to moments of the fourth order are obtained very simply. Even more, they are made explicit, a feature which will be required and exploited in Chapter V.

Theorem 4.1. Let F(x) be any standardized distribution function on $[a,b]$ such that a<0<b, 1+ab≤0. Then there exist extremal distributions $F_\ell(x) \le F(x) \le F_u(x)$ on $[a,b]$, and finite atomic random variables at which the bounds are attained, as given in Table 4.1.

Table 4.1 : Chebyshev-Markov extremal standard distributions for the range [a,b]

condition	$F_\ell(x)$	$F_u(x)$	extremal support
$a < x \le \bar{b}$	0	$\frac{1}{1+x^2}$	$\{x, \bar{x}\}$
$\bar{b} \le x \le \bar{a}$	$\frac{1+bx}{(b-a)(x-a)}$	$1-\frac{1+ax}{(b-a)(b-x)}$	$\{a, x, b\}$
$\bar{a} \le x < b$	$\frac{x^2}{1+x^2}$	1	$\{\bar{x}, x\}$

Proof. By Table 2.1 and Remark 2.1, quadratic polynomial majorants and minorants for the Heaviside indicator function are obtained either for ordered diatomic supports $\{x,\bar{x}\}$, $\{\bar{x},x\}$ (case (4)) or for a triatomic support $\{a,x,b\}$ (case (1)). With Proposition I.5.1, one has necessarily $x \in [a,\bar{b}]$ if $\{x,\bar{x}\}$ is the extremal support, and $x \in [\bar{a},b]$ if $\{\bar{x},x\}$ is the extremal support. Similarly the triatomic support $\{a,x,b\}$ is only feasible if $x \in [\bar{b},\bar{a}]$. The construction of the quadratic majorants for $I_{(-\infty,x]}(X)$ and the quadratic minorant for $I_{[x,\infty)}(X)$ implies immediately the displayed extremal values, which are equal to the expected values of $(1-E[I_{[x,\infty)}(X)])$ and $E[I_{(-\infty,x]}(X)]$ taken at the corresponding atomic extremal random variables. ◊

Since they are important in practical work, the results for the limiting intervals $(-\infty,\infty)$ and $[a,\infty)$ are stated in the separate Tables 4.1' and 4.1''.

Remark 4.1. The second case in Table 4.1'' must be viewed as a limiting case. There exists a sequence of extremal supports $\{a,x,b\}$ converging to the limiting random variable denoted

$\{a, x, \infty\}$ as $b \to \infty$. Only this interpretation guarantees that the variance of $\{a, x, \infty\}$ is one. Intuitively, the atom ∞ is endowed with an infinitesimal probability $p_\infty^{(3)}$ with the limiting property $p_\infty^{(3)} \cdot \infty = 1 + ax$. A rigorous proof of the Cantelli inequalities, in the theoretical sense of Mathematical Analysis, has been given by Royden(1953).

Table 4.1' : Chebyshev inequalities for a standard distribution on $(-\infty, \infty)$

condition	$F_\ell(x)$	$F_u(x)$	extremal support
$x \le 0$	0	$\frac{1}{1+x^2}$	$\{x, \bar{x}\}$
$x \ge 0$	$\frac{x^2}{1+x^2}$	1	$\{\bar{x}, x\}$

Table 4.1'' : Cantelli inequalities for a standard distribution on $[a, \infty)$

condition	$F_\ell(x)$	$F_u(x)$	extremal support
$a < x \le 0$	0	$\frac{1}{1+x^2}$	$\{x, \bar{x}\}$
$0 \le x \le \bar{a}$	$\frac{x}{(x-a)}$	1	$\{a, x, \infty\}$
$x \ge \bar{a}$	$\frac{x^2}{1+x^2}$	1	$\{\bar{x}, x\}$

Theorem 4.2. Let F(x) be any standard distribution function on $[a, b]$ with known skewness γ such that $a<0<b$, $1+ab\le 0$, $a + \bar{a} \le \gamma \le b + \bar{b}$. Then the extremal distributions $F_\ell(x) \le F(x) \le F_u(x)$ are given and attained as in Table 4.2.

Table 4.2 : Chebyshev-Markov inequalities for a standard distribution on $[a, b]$ by known skewness

condition	$F_\ell(x)$	$F_u(x)$	extremal support
$a < x \le c$	0	$p_x^{(3)}$	$\{x, \varphi(x,b), b\}$
$c \le x \le \varphi(a,b)$	$p_a^{(3)}$	$1 - p_{\varphi(a,x)}^{(3)}$	$\{a, x, \varphi(a,x)\}$
$\varphi(a,b) \le x \le \bar{c}$	$p_{\varphi(x,b)}^{(3)}$	$1 - p_b^{(3)}$	$\{\varphi(x,b), x, b\}$
$\bar{c} \le x < b$	$1 - p_x^{(3)}$	1	$\{a, \varphi(a,x), x\}$

Proof. By Lemma 2.1, Remark 2.1, and Proposition III.5.2, which characterizes triatomic standard random variables by known skewness, cubic polynomial majorants and minorants for the Heaviside indicator function can only be obtained at ordered triatomic supports of the four forms $\{a,x,\varphi(a,x)\}$, $\{a,\varphi(a,x),x\}$, $\{x,\varphi(x,b),b\}$, $\{\varphi(x,b),x,b\}$. The values of x, for which these forms define feasible extremal triatomic random variables, are determined as follows. If $\{a,x,\varphi(a,x)\}$ is the extremal support, then one has necessarily $\varphi(a,x)\in[\bar{c},b]$, $x=\varphi(a,\varphi(a,x))\in[c,\bar{c}]$. Since $\varphi(a,x)$ is strictly increasing in x, the inequality $\varphi(a,x)\le b$ implies further that $x=\varphi(a,\varphi(a,x))\le\varphi(a,b)$. But one has $\varphi(a,b)\le\bar{c}$, hence $x\in[c,\varphi(a,b)]$. It is immediate that $\{a,\varphi(a,x),x\}$ is feasible if $x\in[\bar{c},b]$, and that $\{x,\varphi(x,b),b\}$ is feasible if $x\in[a,c]$. If $\{\varphi(x,b),x,b\}$ is the extremal support, then one has $\varphi(x,b)\in[a,c]$, $x=\varphi(\varphi(x,b),b)\in[c,\bar{c}]$. The inequality $a\le\varphi(x,b)$ implies further that $\varphi(a,b)\le\varphi(\varphi(x,b),b)=x$. But one has $c\le\varphi(a,b)$, hence $x\in[\varphi(a,b),\bar{c}]$. By the polynomial majorant/minorant method, one obtains $F_\ell(x)=1-E\left[I_{[x,\infty)}(X)\right]$ and $F_u(x)=E\left[I_{(-\infty,x]}(X)\right]$, where expectation is taken at the extremal random variables. ◊

One observes that for the limiting interval $(-\infty,\infty)$, one recovers Table 4.1', which means that there is no improvement over the Chebyshev inequalities by additional knowledge of the skewness parameter. For the one-sided infinite interval $[a,\infty)$ the obtained limiting result is stated in Table 4.2''.

Table 4.2'' : Chebyshev-Markov inequalities for a standard distribution on $[a,\infty)$ by known skewness

condition	$F_\ell(x)$	$F_u(x)$	extremal support
$a<x\le c$	0	$\frac{1}{1+x^2}$	$\{x,\bar{x}\}$
$c\le x\le\bar{a}$	$p_a^{(3)}$	$1-p_{\varphi(a,x)}^{(3)}$	$\{a,x,\varphi(a,x)\}$
$\bar{a}\le x\le\bar{c}$	$\frac{x^2}{1+x^2}$	1	$\{\bar{x},x\}$
$x\ge\bar{c}$	$1-p_x^{(3)}$	1	$\{a,\varphi(a,x),x\}$

Theorem 4.3. Let F(x) be any standard distribution function on $[a,b]$ with known skewness γ, kurtosis $\gamma_2=\delta-3$, such that the following moment inequalities a<0<b, 1+ab≤0, $a+\bar{a}\le\gamma\le b+\bar{b}$, and $0\le\delta-\gamma^2-1\le\left(\frac{ab}{1+ab}\right)(\gamma-a-\bar{a})(b+\bar{b}-\gamma)$ are fulfilled. Then the extremal distributions $F_\ell(x)\le F(x)\le F_u(x)$ are given and attained as in Table 4.3.

Table 4.3 : Chebyshev-Markov inequalities for a standard distribution on $[a,b]$ by known skewness and kurtosis

condition	$F_\ell(x)$	$F_u(x)$	extremal support
$a<x\le b^*$	0	$p_x^{(3)}$	$\{x,\varphi(x,x^*),x^*\}$

$b^* \le x \le \varphi(a,a^*)$	$p_a^{(4)}(x,b)$	$p_a^{(4)}(x,b)+p_x^{(4)}(a,b)$	$\{a,x,\psi(x;a,b),b\}$
$\varphi(a,a^*) \le x \le \varphi(b,b^*)$	$1-p_x^{(3)}-p_{x^*}^{(3)}$	$1-p_{x^*}^{(3)}$	$\{\varphi(x,x^*),x,x^*\}$
$\varphi(b,b^*) \le x \le a^*$	$1-p_x^{(4)}(a,b)-p_b^{(4)}(a,x)$	$1-p_b^{(4)}(a,x)$	$\{a,\psi(x;a,b),x,b\}$
$a^* \le x < b$	$1-p_x^{(3)}$	1	$\{x^*,\varphi(x^*,x),x\}$

Proof. By Lemma 2.1, Remark 2.1, and Theorem I.5.3 and I.5.4, one observes that biquadratic polynomial majorants and minorants for the Heaviside indicator function can only be obtained at ordered tri-, respectively four- atomic supports of the five forms $\{x,\varphi(x,x^*),x^*\}$, $\{\varphi(x,x^*),x,x^*\}$, $\{x^*,\varphi(x^*,x),x\}$ (case (4) of Table 2.1), respectively $\{a,x,\psi(x;a,b),b\}$, $\{a,\psi(x;a,b),x,b\}$ (case (1) of Table 2.1). The values of x, for which these forms define feasible tri-, respectively four-atomic extremal random variables, are determined as follows. From Theorem I.5.3, it is immediate that $\{x,\varphi(x,x^*),x^*\}$ is only feasible if $x \in [a,b^*]$. Since * is an involution, $\{x^*,\varphi(x^*,x),x\}$ is only feasible if $x \in [a^*,b]$. In case $\{\varphi(x,x^*),x,x^*\}$ is feasible, one has $\varphi(x,x^*) \in [a,b^*]$. Moreover one has $\varphi(x,x^*)^* = x^*$ and $x = \varphi(\varphi(x,x^*),x^*)$. Since φ is strictly increasing in x, it follows from $a \le \varphi(x,x^*) \le b^*$ that $\varphi(a,a^*) \le \varphi(\varphi(x,x^*),\varphi(x,x^*)^*) = \varphi(\varphi(x,x^*),x^*) = x \le \varphi(b,b^*)$ as desired. By Theorem I.5.4, the support $\{a,x,\psi(x;a,b),b\}$ is feasible only if $x \in [b^*,\varphi(a,a^*)]$. Since $\psi(x;a,b)$ is strictly increasing in x, and $\psi(b^*;a,b) = \varphi(b,b^*)$ $\psi(\varphi(a,a^*);a,b) = a^*$, it follows that $\{a,\psi(x;a,b),x,b\}$ is feasible provided $x \in [\varphi(b,b^*),a^*]$. Finally the polynomial majorant/minorant method implies that $F_\ell(x) = 1 - E[I_{[x,\infty)}(X)]$ and $F_u(x) = E[I_{(-\infty,x]}(X)]$, where expectation is taken at the corresponding atomic extremal random variables. ◊

The bounds obtained for limiting intervals are stated in the Tables 4.3' and 4.3''.

Table 4.3' : Chebyshev-Markov inequalities for a standard distribution on $(-\infty,\infty)$ by known skewness and kurtosis

condition	$F_\ell(x)$	$F_u(x)$	extremal support
$x \le c$	0	$p_x^{(3)}$	$\{x,\varphi(x,x^*),x^*\}$
$c \le x \le \bar{c}$	$1-p_x^{(3)}-p_{x^*}^{(3)}$	$1-p_{x^*}^{(3)}$	$\{\varphi(x,x^*),x,x^*\}$
$x \ge \bar{c}$	$1-p_x^{(3)}$	1	$\{x^*,\varphi(x^*,x),x\}$

Table 4.3'' : Chebyshev-Markov inequalities for a standardized distribution on $[a,\infty)$ by known skewness and kurtosis

condition	$F_\ell(x)$	$F_u(x)$	extremal support
$a < x \le c$	0	$p_x^{(3)}$	$\{x,\varphi(x,x^*),x^*\}$
$c \le x \le \varphi(a,a^*)$	$p_a^{(3)}$	$p_a^{(3)} + p_x^{(3)}$	$\{a,x,\varphi(a,x),\infty\}$
$\varphi(a,a^*) \le x \le \bar{c}$	$1-p_x^{(3)} - p_{x^*}^{(3)}$	$1-p_{x^*}^{(3)}$	$\{\varphi(x,x^*),x,x^*\}$
$\bar{c} \le x \le a^*$	$1-p_x^{(3)}$	1	$\{a,x,\varphi(a,x),\infty\}$
$x \ge a^*$	$1-p_x^{(3)}$	1	$\{x^*,\varphi(x^*,x),x\}$

Proof of Tables 4.3' and 4.3''. Table 4.3' is derived along the same line as Table 4.3 with the difference that the only possible biquadratic polynomial majorants and minorants for the Heaviside indicator function are obtained at the ordered triatomic supports $\{x,\varphi(x,x^*),x^*\}$, $\{\varphi(x,x^*),x,x^*\}$, $\{x^*,\varphi(x^*,x),x\}$. Table 4.3'' is obtained as limiting case of Table 4.3 as $b\to\infty$. In particular, one applies formula (5.8) in Theorem I.5.3 to show that $\lim_{b\to\infty} b^* = c$ and $\lim_{b\to\infty}\psi(x;a,b) = \varphi(a,x)$. Furthermore, the limiting random variables $\{a,x,\varphi(a,x),\infty\}$ and $\{a,x,\varphi(a,x),\infty\}$ must be interpreted similarly to Remark 4.1 in order that the kurtosis constraint is fulfilled. ◊

Remark 4.2. It is well-known (e.g. Kaas and Goovaerts(1986a)) that the above results can be used to bound any probability integrals of the type

$$\int_a^x g(x)dF(x) = E\left[g(X)\cdot I_{(-\infty,x]}(X)\right],$$

where $g(x)$ is a non-negative real function defined on $[a,b]$. The upper bound is found by evaluation of this expectation at the maximizing atomic random variables. Similarly the lower bound is obtained from the expectation $E[g(X)] - E\left[g(X)\cdot I_{[x,\infty)}(X)\right]$.

5. The maximal stop-loss transforms by given range and known moments to order four.

We will derive the following structure for the maximal stop-loss transform on a given interval $[a,b]$, $-\infty \le a < b \le \infty$. There exists a finite partition $[a,b] = \bigcup_{i=1}^m [d_{i-1},d_i]$ with $d_0 = a, d_m = b$, such that in each subinterval one finds a monotone increasing function $d_i(x) \in [d_{i-1},d_i]$, the parameter x varying in some interval $[x_{i-1},x_i]$, which we will name a *deductible function*. Then the maximal stop-loss transform on $[d_{i-1},d_i]$ is attained at a finite atomic extremal random variable $X_i(x)$ with support $\{x_{i0}(x),...,x_{ir+1}(x)\}$ and probabilities $\{p_{i0}(x),...,p_{ir+1}(x)\}$, $x\in[x_{i-1},x_i]$, and is given implicitely by the formula

(5.1) $$\pi^*(d_i(x)) = \sum_{j=0}^{r+1} p_{ij}(x)\cdot(x_{ij}(x)-d_i(x))_+, \quad x\in[x_{i-1},x_i], \quad i=1,\dots,m.$$

Based on Table 3.1, it is possible to derive and present the results in the specified unified manner. As a novel striking fact, we observe that all deductible functions below can be written as weighted averages.

Theorem 5.1. The maximal stop-loss transform of a standard random variable on $[a,b]$ is determined in Table 5.1, where the monotone increasing deductible functions are "weighted averages of extremal atoms" given by the formulas

$$d_1(x)=\frac{(\bar{a}-x)a+(\bar{a}-a)\bar{a}}{(\bar{a}-x)+(\bar{a}-a)},\ d_2(x)=\frac{1}{2}(x+\bar{x}),\ d_3(x)=\frac{(b-\bar{b})\bar{b}+(x-\bar{b})b}{(b-\bar{b})+(x-\bar{b})}.$$

Table 5.1 : maximal stop-loss transform on $[a,b]$

case	range of parameter	range of deductible	$\pi^*(d_i(x))$	extremal support
(1)	$x\le a$	$a\le d_1(x)\le \frac{1}{2}(a+\bar{a})$	$p_{\bar{a}}^{(2)}\cdot(\bar{a}-d_1(x))$	$\{a,\bar{a}\}$
(2)	$a\le x\le \bar{b}$	$\frac{1}{2}(a+\bar{a})\le d_2(x)\le \frac{1}{2}(b+\bar{b})$	$p_{\bar{x}}^{(2)}\cdot(\bar{x}-d_2(x))=\frac{1}{2}(-x)$	$\{x,\bar{x}\}$
(3)	$x\ge b$	$\frac{1}{2}(b+\bar{b})\le d_3(x)\le b$	$p_{b}^{(2)}\cdot(b-d_3(x))$	$\{\bar{b},b\}$

Proof. By Table 3.1 quadratic polynomial majorants $q(X)\ge(X-d)_+$, d the deductible of the stop-loss function, can only be obtained at diatomic supports of the forms $\{a,\bar{a}\}$ (case (3a), $\{x,\bar{x}\}$ (case (4a)) or $\{\bar{b},b\}$ (case (2a)).

Case (1) : extremal support $\{a,\bar{a}\}$

The unique quadratic polynomial $q(X)=q(X;a,\bar{a},d)$ such that $q(a)=0, q(\bar{a})=\bar{a}-d$, $q'(\bar{a})=1$, is given by

$$q(X)=\frac{(d-a)(X-\bar{a})^2}{(\bar{a}-a)^2}+(X-d).$$

Solving the condition $Q_1(x;a,\bar{a},d)=q(x;a,\bar{a},d)=0$, $x\le a$, one finds for the deductible

$$d=\frac{\bar{a}^2-ax}{2\bar{a}-a-x}, \quad x\le a,$$

which implies without difficulty the desired results.

Case (2) : extremal support $\{x, \bar{x}\}$

By Theorem I.5.3 the ordered diatomic support $\{x, \bar{x}\}$ is feasible exactly when $x \in [a, \bar{b}]$. The unique quadratic polynomial $q(X) = q(X; x, \bar{x}, d)$ such that $q(x) = q'(x) = 0$, $q(\bar{x}) = \bar{x} - d$, is given by

$$q(X) = \frac{(\bar{x} - d)(X - x)^2}{(\bar{x} - x)^2}.$$

Solving the condition $Q_2'(\bar{x}; x, \bar{x}, d) = q'(\bar{x}) - 1 = 0$, one finds

$$d = \frac{1}{2}(x + \bar{x}), \quad a \le x \le \bar{b},$$

from which all statements follow.

Case (3) : extremal support $\{\bar{b}, b\}$

The unique quadratic polynomial $q(X) = q(X; \bar{b}, b, d)$ such that $q(\bar{b}) = q'(\bar{b}) = 0$, $q(b) = b - d$, is given by

$$q(X) = \frac{(b - d)(X - \bar{b})^2}{(b - \bar{b})^2}.$$

Solving the condition $Q_2(x; \bar{b}, b, d) = q(x) - (x - d) = 0, \quad x \ge b$, one finds

$$d = \frac{bx - \bar{b}^2}{x + b - 2\bar{b}}, \quad x \ge b,$$

and the stated results are immediately checked. ◊

In many applications one is especially interested in the limiting ranges $[a, \infty)$ and $(-\infty, \infty)$. For the range $[a, \infty)$ the case (3) is inapplicable and one obtains the same formulas as in the cases (1) and (2) above, with the difference that $a \le x < 0$ in case (2). For the range $(-\infty, \infty)$ both cases (1) and (3) are inapplicable and $-\infty < x < 0$ in case (2). Since the function $d = \frac{1}{2}(x + \bar{x})$ may be inversed such that $x = d - \sqrt{1 + d^2}$, one gets after calculation

$$\pi^*(d) = p_x^{(2)} \cdot (\bar{x} - d) = \frac{1}{2}(\sqrt{1 + d^2} - d), \quad d \in (-\infty, \infty),$$

a formula first derived by Bowers(1969).

Theorem 5.2. The maximal stop-loss transform of a standard random variable on $[a, b]$ by known skewness γ is determined in Table 5.2.

Table 5.2 : maximal stop-loss transform of a standard random variable on $[a,b]$ by known skewness γ

case	range of parameter	$\pi^*(d_i(x))$	extremal support
(1)	$x \le a$	$-d_1(x) + p_a^{(3)} \cdot (d_1(x) - a)$	$\{a, \varphi(a,b), b\}$
(2)	$a \le x \le c$	$-d_2(x) + p_x^{(3)} \cdot (d_2(x) - x)$	$\{x, \varphi(x,b), b\}$
(3)	$x \ge b$	$p_{\bar{c}}^{(2)} \cdot (\bar{c} - d_3(x))$	$\{c, \bar{c}\}$
(4)	$x \le a$	$p_{\bar{c}}^{(2)} \cdot (\bar{c} - d_4(x))$	$\{c, \bar{c}\}$
(5)	$\bar{c} \le x \le b$	$p_x^{(3)} \cdot (x - d_5(x))$	$\{a, \varphi(a,x), x\}$
(6)	$x \ge b$	$p_b^{(3)} \cdot (b - d_6(x))$	$\{a, \varphi(a,b), b\}$

The monotone increasing deductible functions are "weighted averages" and given by the following formulas :

$$d_1(x) = \frac{(b-a)(\varphi-a)x + (x-\varphi)^2 a + (b-a)(\varphi-x)\varphi}{(b-a)(\varphi-a) + (x-\varphi)^2 + (b-a)(\varphi-x)}, \quad \varphi = \varphi(a,b),$$
$$d_2(x) = \frac{2(b-x)x + (\varphi(x,b)-b)b}{2(b-x) + (\varphi(x,b)-b)},$$
$$d_3(x) = \frac{2(x-c)c + (\bar{c}-c)x}{2(x-c) + (\bar{c}-c)},$$
$$d_4(x) = \frac{(\bar{c}-c)x + 2(\bar{c}-x)\bar{c}}{(\bar{c}-c) + 2(\bar{c}-x)}$$
$$d_5(x) = \frac{(x-\varphi(a,x))a + 2(x-a)x}{(x-\varphi(a,x)) + 2(x-a)}$$
$$d_6(x) = \frac{(b-a)(x-\varphi)\varphi + (x-\varphi)^2 b + (b-a)(b-\varphi)x}{(b-a)(x-\varphi) + (x-\varphi)^2 + (b-a)(b-\varphi)}, \quad \varphi = \varphi(a,b).$$

Proof. The only standard diatomic random variable with known skewness has support $\{c, \bar{c}\}$. From Table 3.1 it follows that cubic polynomial majorants $q(X) \ge (X-d)_+$ can only be constructed for the displayed di- and triatomic extremal supports. Our application of the cubic polynomial majorant method consists in a partial generalization of the general quadratic polynomial majorant algorithm presented in Chapter II. Non defined notations and conventions are taken from that part. The letters u, v, w denote three ordered real numbers, $I = \cup I_i$ is a partition of $I=[a,b]$, and $f(x)$ is a piecewise linear function on I such that $f(x) = \ell_i(x)$ on I_i with $\ell_i(x) = \alpha_i + \beta_i x$. We assume that $\beta_i \ne \beta_j$ if $i \ne j$, and the point of intersection of two non-parallel lines $\ell_i(x)$ and $\ell_j(x)$ is denoted by

$d_{ij} = d_{ji} = \dfrac{\alpha_i - \alpha_j}{\beta_j - \beta_i}$. The piecewise cubic function Q(x)=q(x)–f(x) coincides on I_i with the cubic polynomial $Q_i(x) = q(x) - \ell_i(x)$. We use the backward functional operator $\nabla_{ij}\ell(x) = \ell_j(x) - \ell_i(x)$. In the present proof we set $I_i = [a, d]$, $I_j = [d, b]$, $\ell_i(x) = 0$, $\ell_j(x) = x - d$, $d = d_{ij}$. Relevant are two main symmetric cases, where in each case three types may occur as described below.

Case (I) : $u \in I_i$, $v \in I_j$, $w \in I_j \cup (b, \infty)$

Since a cubic polynomial is uniquely determined by four conditions, there exists a unique q(x) such that

$Q_i(u) = 0$ (u is a simple zero),
$Q_j(v) = Q_j'(v) = 0$ (v is a double zero),
$Q_j(w) = 0$ (w is a simple zero).

It is given by

$$q(x) = \nabla_{ij}\ell(u) \cdot \frac{(x-v)^2 (x-w)}{(v-u)^2 (w-u)} + \ell_j(x), \quad \nabla_{ij}\ell(u) = (\beta_j - \beta_i)(u - d_{ij}).$$

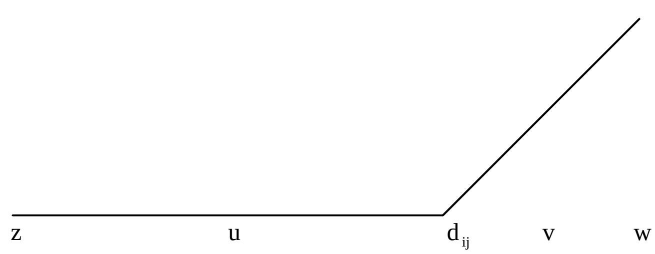
To obtain it, set it equal to a cubic form

$$q(x) = c_{ij}^3(u) \cdot (x-v)^2 (x-w) + c_{ij}^2(u) \cdot (x-v)^2 + \ell_j(x),$$

and show that

$$c_{ij}^2(u) = -\frac{\nabla_{ij}\ell(u)}{(v-u)^2}, \text{ from the fact that } Q_j(w) = 0,$$
$$c_{ij}^3(u) = -\frac{c_{ij}^2(u)}{(w-u)}, \text{ from the fact that } Q_i(u) = 0.$$

Type (1) : u, w rand points of I_i, I_j as in the following figure :



There exists a second zero $z \in (-\infty, u]$ of $Q_i(x)$. The condition $q(z) = \ell_i(z)$ is equivalent with the formulas

$$d_{ij} = \frac{(z-v)^2(z-w)u + (v-u)^2(w-u)z}{(z-v)^2(z-w) + (v-u)^2(w-u)} = \frac{(z-v)^2 u + (w-u)(v^2-uz)}{(z-v)^2 + (w-u)(2v-u-z)}$$
$$= \frac{(w-u)(v-u)z + (v-z)^2 u + (w-u)(v-z)v}{(w-u)(v-u) + (v-z)^2 + (w-u)(v-z)}.$$

The third equality is a weighted average rearrangement of the second one. The latter expression is obtained from the first one by expanding the first term in the numerator as

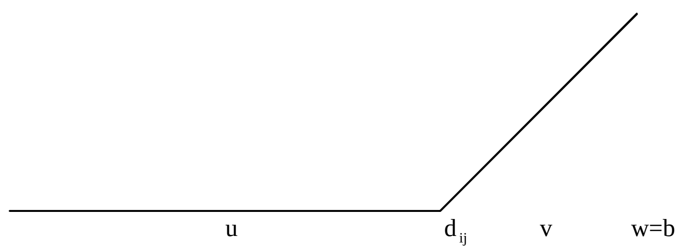
$$(z-v)^2(z-w)u = (z-v)^2(z-u)u + (z-v)^2(u-w)u,$$

and the second term in the denominator as

$$(v-u)^2(w-u) = \left[(v-z)^2 + (z-u)(2v-u-z)\right]\cdot(w-u).$$

Setting u=a, $v=\varphi(a,b)$, w=b, $z = x \in (-\infty, a]$, one obtains the expression for the deductible function $d_1(x)$, which is shown to be monotone increasing for $x \le a$. The maximal value $\pi^*(d_1(x))$ is immediate.

Type (2) : u double zero of $Q_i(x)$, w=b rand point of I_j as in the following figure:

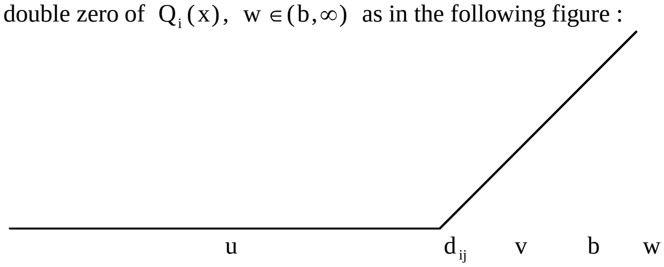


The condition $Q_i^{'}(u) = 0$ implies the formulas

$$d_{ij} = \frac{(u+v)w - 2u^2}{v + 2w - 3u} = \frac{2(w-u)u + (v-u)w}{2(w-u) + (v-u)}.$$

Setting u=x, $v=\varphi(x,b)$, w=b, one obtains $d_2(x)$. From the characterization theorem for triatomic distributions, it follows that the extremal support is feasible only if $x \in [a,c]$.

Type (3) : u double zero of $Q_i(x)$, $w \in (b,\infty)$ as in the following figure :



As for the type (2) one obtains the value

$$d_{ij} = \frac{2(w-u)u + (v-u)w}{2(w-u)+(v-u)}.$$

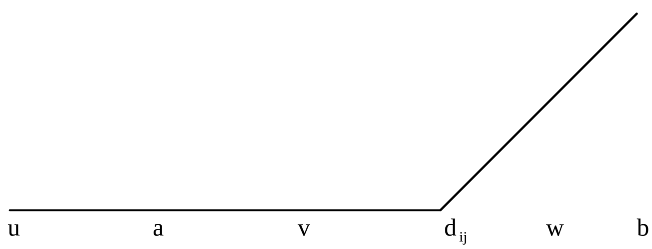
The formula for $d_3(x)$ follows by setting u=c, $v=\bar{c}$, $w = x \in (b,\infty)$.

<u>Case (II)</u> : $u \in (-\infty, a) \cup I_i$, $v \in I_i$, $w \in I_j$

All formulas below can be obtained from case (I) by symmetry. It suffices to exchange u and w, i and j and replace b by a. For completeness the main steps are repeated. First, there exists a unique cubic polynomial q(x) such that $Q_i(u)=0$, $Q_i(v)=Q_i^{'}(v)=0$, $Q_j(w)=0$. It is given by

$$q(x) = \nabla_{ij}\ell(w)\cdot\frac{(x-v)^2(x-u)}{(w-v)^2(w-u)} + \ell_i(x).$$

<u>Type (4)</u> : w double zero of $Q_j(x)$, $u \in (-\infty, a)$ as in the following figure :

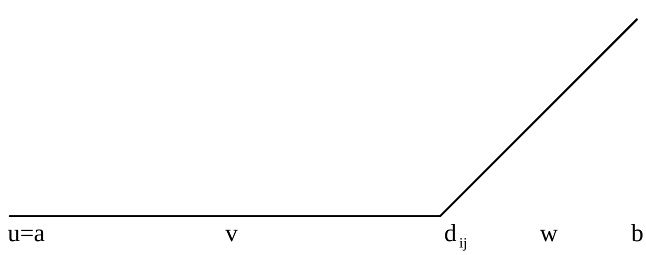


From the condition $Q_j^{'}(w)=0$, or by symmetry with type (3), one obtains

$$d_{ij} = \frac{2(w-u)w + (w-v)u}{2(w-u)+(w-v)}.$$

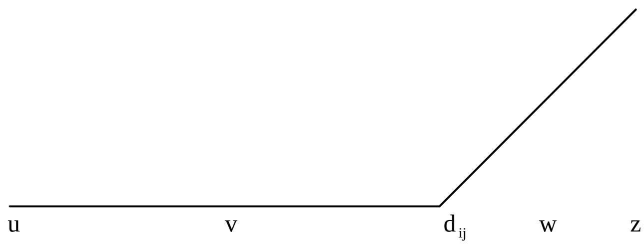
The deductible function $d_4(x)$ follows by setting $u = x \in (-\infty, a)$, v=c, $w=\bar{c}$.

<u>Type (5)</u> : w double zero of $Q_j(x)$, u=a rand point of I_i as in the following figure:



The same formula for d_{ij} as in type (4) holds. Setting u=a, $v = \varphi(a,x)$, w=x, one obtains $d_5(x)$, which is symmetric to $d_2(x)$. The extremal support is feasible only if $x \in [\bar{c}, b]$.

<u>Type (6)</u> : u, w rand points of I_i, I_j as in the following figure :



This is symmetric to type (1). The second zero $z \in [w, \infty)$ of $Q_j(x)$ leads to the weighted average formula

$$d_{ij} == \frac{(w-u)(w-v)z + (z-v)^2 w + (w-u)(z-v)v}{(w-u)(w-v) + (z-v)^2 + (w-u)(z-v)}.$$

Setting u=a, $v = \varphi(a,b)$, w=b, $z = x \in [b, \infty)$, one obtains $d_6(x)$. ◊

Let us look at the limiting ranges. If $b \to \infty$ then the cases (3) and (6) in Table 5.2 are inapplicable. Using that $\lim_{b\to\infty} \varphi(x,b) = \bar{x}$ one obtains the following quite tractable result.

Table 5.2'' : maximal stop-loss transform of a standard random variable on $[a, \infty)$ by known skewness γ

case	range of parameter	range of deductible	$\pi^*(d_i(x))$	extremal support
(1)	$x \le a$	$\left[a, \frac{1}{2}(a+\bar{a})\right]$	$p_{\bar{a}}^{(2)} \cdot (\bar{a} - d_1(x))$	$\{a, \bar{a}\}$
(2)	$a \le x \le c$	$\left[\frac{1}{2}(a+\bar{a}), \frac{1}{2}(c+\bar{c})\right]$	$p_{\bar{x}}^{(2)} \cdot (\bar{x} - d_2(x)) = \frac{1}{2}(-x)$	$\{x, \bar{x}\}$
(3)	$x \le a$	$\left[\frac{1}{2}(c+\bar{c}), d_3(a)\right]$	$p_{\bar{c}}^{(2)} \cdot (\bar{c} - d_3(x))$	$\{c, \bar{c}\}$
(4)	$x \ge \bar{c}$	$[d_3(a) = d_4(\bar{c}), \infty)$	$p_x^{(3)} \cdot (x - d_4(x))$	$\{a, \varphi(a,x), x\}$

The monotone increasing deductible functions take the weighted average forms :

$$d_1(x) = \frac{(\bar{a}-x)a + (\bar{a}-a)\bar{a}}{(\bar{a}-x) + (\bar{a}-a)}, \; d_2(x) = \frac{1}{2}(x+\bar{x}), \; d_3(x) = \frac{(\bar{c}-c)x + 2(\bar{c}-x)\bar{c}}{(\bar{c}-c) + 2(\bar{c}-x)},$$

$$d_4(x) = \frac{(x-\varphi(a,x))a + 2(x-a)x}{(x-\varphi(a,x)) + 2(x-a)}.$$

Let further $a \to -\infty$ in Table 5.2". Then the cases (1) and (3) are inapplicable. Since $\varphi(a,x) \to \bar{x}$, one sees that $d_4(x) = d_2(x)$ and $\pi^*(d_4(x)) = \pi^*(d_2(x)) = \frac{1}{2}(-x)$. One recovers the best upper bound by Bowers(1969). As for the Chebyshev-Markov inequality over $(-\infty,\infty)$, one observes that the additional knowledge of the skewness does not improve the best upper bound for standard random variables on $(-\infty,\infty)$.

Theorem 5.3. The maximal stop-loss transform of a standard random variable on $[a,b]$ by known skewness γ and kurtosis $\gamma_2 = \delta - 3$ is determined in Table 5.3.

The monotone increasing deductible functions are defined by the following "weighted averages" :

$$d_1(x) = \frac{(a^*-x)^2[2\varphi - a - x]a + (\varphi - a)^2[(a^*-x)a + (a^*-a)a^*]}{(a^*-x)^2[2\varphi - a - x] + (\varphi - a)^2[2a^* - a - x]}, \quad \varphi = \varphi(a,a^*)$$

$$d_2(x) = \frac{1}{2}\left\{\frac{[\varphi(x,x^*) - x](x + x^*) + 2(x^*-x)x}{[\varphi(x,x^*) - x] + (x^*-x)}\right\}$$

$$d_3(x) = \frac{(b-b^*)(x-b^*)(b^* + \varphi(b^*,b)) + (\varphi(b^*,b) - b^*)(x + b - 2b^*)b^*}{2(b-b^*)(x-b^*) + (\varphi(b^*,b) - b^*)(x + b - 2b^*)}$$

$$d_4(x) = \frac{(\varphi-x)^2[(\varphi-b^*)b^* + (b-b^*)(b^*+\varphi)] + (b-b^*)^2[(\varphi-x)(b^*+\varphi) + (\varphi-b^*)\varphi]}{(\varphi-x)^2[(\varphi-b^*) + 2(b-b^*)] + (b-b^*)^2[2(\varphi-x) + (\varphi-b^*)]}, \quad \varphi = \varphi(b^*,b)$$

$$d_5(x) = \frac{(\psi-a)^2[2(b-x)x + (\psi-x)b] + (b-x)^2[2(\psi-a)\psi + (\psi-x)a]}{(\psi-a)^2[2(b-x) + (\psi-x)] + (b-x)^2[2(\psi-a) + (\psi-x)]}, \quad \psi = \psi(x;a,b)$$

$$d_6(x) = \frac{(x-\varphi)^2[(a^*-\varphi)a^* + (a^*-a)(\varphi+a^*)] + (a^*-a)^2[(x-\varphi)(\varphi+a^*) + (a^*-\varphi)\varphi]}{(x-\varphi)^2[(a^*-\varphi) + 2(a^*-a)] + (a^*-a)^2[2(x-\varphi) + (a^*-\varphi)]}, \quad \varphi = \varphi(a,a^*)$$

$$d_7(x) = \frac{(a^*-a)(a^*-x)(\varphi(a^*,a) + a^*) + (a^* - \varphi(a^*,a))(2a^* - a - x)a^*}{2(a^*-a)(a^*-x) + (a^* - \varphi(a^*,a))(2a^* - a - x)}$$

$$d_8(x) = \frac{1}{2}\left\{\frac{(x^*-x)[\varphi(x,x^*) + x] + 2(x^* - \varphi(x,x^*))\varphi(x,x^*)}{(x^*-x) + (x^* - \varphi(x,x^*))}\right\}$$

$$d_9(x) = \frac{(a^*-\varphi)^2[(x-a)a^* + (a^*-a)a] + (x-a)^2[a^* + x - 2\varphi]a^*}{(a^*-\varphi)^2[x + a^* - 2a] + (x-a)^2[a^* + x - 2\varphi]}, \quad \varphi = \varphi(a,a^*)$$

Table 5.3 : maximum stop-loss transform on $[a,b]$ by known skewness and kurtosis

case	range of parameter	$\pi^*(d_i(x))$	extremal support
(1)	$x \le a$	$p^{(3)}_{\varphi(a,a^*)}(\varphi(a,a^*) - d_1(x)) + p^{(3)}_{a^*}(a^* - d_1(x))$	$\{a, \varphi(a,a^*), a^*\}$
(2)	$a \le x \le b^*$	$p^{(3)}_{\varphi(x,x^*)}(\varphi(x,x^*) - d_2(x)) + p^{(3)}_{x^*}(x^* - d_2(x))$	$\{x, \varphi(x,x^*), x^*\}$

(3)	$x \geq b$	$p^{(3)}_{\varphi(b^*,b)}(\varphi(b^*,b)-d_3(x))+p^{(3)}_b(b-d_3(x))$	$\{b^*,\varphi(b^*,b),b\}$
(4)	$x \leq a$	$p^{(3)}_{\varphi(b^*,b)}(\varphi(b^*,b)-d_4(x))+p^{(3)}_b(b-d_4(x))$	$\{b^*,\varphi(b^*,b),b\}$
(5)	$b^* \leq x \leq \varphi(a,a^*)$	$p^{(4)}_{\psi}(\psi-d_5(x))+p^{(4)}_b(b-d_5(x))$	$\{a,x,\psi,b\}$, $\psi=\psi(x;a,b)$, $\varphi=\varphi(a,b)$
(6)	$x \geq b$	$p^{(3)}_{a^*}(a^*-d_6(x))$	$\{a,\varphi(a,a^*),a^*\}$
(7)	$x \leq a$	$p^{(3)}_{a^*}(a^*-d_7(x))$	$\{a,\varphi(a,a^*),a^*\}$
(8)	$a \leq x \leq b^*$	$p^{(3)}_{x^*}(x^*-d_8(x))$	$\{x,\varphi(x,x^*),x^*\}$
(9)	$x \geq b$	$p^{(3)}_b(b-d_9(x))$	$\{b^*,\varphi(b^*,b),b\}$

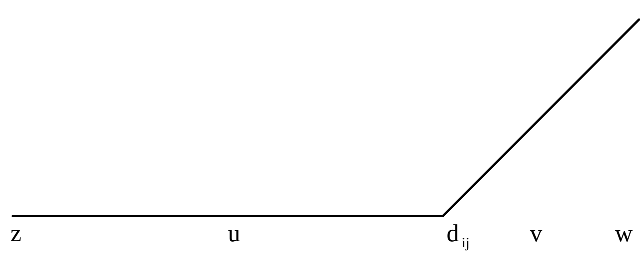
Proof. From Table 3.1 it follows that biquadratic polynomial majorants $q(X) \geq (X-d)_+$ can only be constructed for the displayed extremal supports. We proceed along the same line as in the proof of Theorem 5.2. There are four main cases, of which two can be obtained by symmetry.

Case (I) : $u \in I_i$ double zero of $Q_i(x)$, $v, w \in I_j$ double zeros of $Q_j(x)$

Since a biquadratic polynomial is uniquely determined by five conditions, there exists a unique q(x) with the required conditions, namely

$$q(x)=\frac{\nabla_{ji}\ell(u)}{(v-u)^2(w-u)^2}(x-v)^2(x-w)^2+\ell_j(x)$$

Type (1) : $z \in (-\infty,u]$ is a further zero of $Q_i(x)$ as in the following figure :

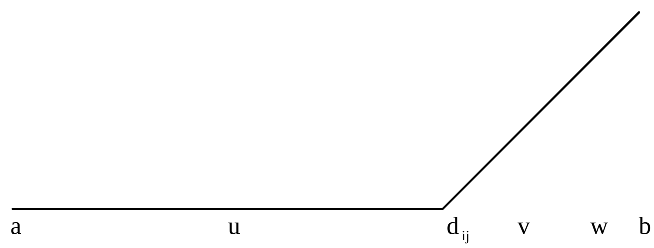


The condition $q(z)=\ell_i(z)$ implies the following formulas

$$d_{ij}=\frac{(z-v)^2(z-w)^2u-(v-u)^2(w-u)^2z}{(z-v)^2(z-w)^2-(v-u)^2(w-u)^2}$$
$$=\frac{(w-z)^2(2v-u-z)u+(v-u)^2\left[(w-z)u+(w-u)w\right]}{(w-z)^2(2v-u-z)+(v-u)^2(2w-u-z)},$$

of which the second equality is a weighted average rearrangement. Setting $u=a$, $v=\varphi(a,a^*)$, $w=a^*$, $z=x\in(-\infty,a]$, one obtains the deductible function $d_1(x)$, which is shown to be monotone increasing for $x\le a$. The value of $\pi^*(d_1(x))$ is immediate.

Type (2) : u is a double zero of $Q_i(x)$ as in the following figure :



Solving the condition $Q_i^{'}(u)=0$ one obtains

$$d_{ij}=\frac{1}{2}\left\{\frac{vw+u(v+w)-3u^2}{v+w-2u}\right\}=\frac{(v-u)(u+w)+2(w-u)u}{2(v-u)+2(w-u)}$$

Setting $u=x$, $v=\varphi(x,x^*)$, $w=x^*$, one gets $d_2(x)$. The characterization theorem for triatomic random variables shows that the extremal support is feasible provided $x\in\left[a,b^*\right]$.

Case (II) : $u\in I_i$ double zero of $Q_i(x)$, $v\in I_j$ double zero of $Q_j(x)$, $w\in I_j$ double zero of $Q_j(x)$

The unique $q(x)$ with the required conditions takes the form

$$q(x)=c_{ij}^4(w)(x-u)^2(x-v)^2+c_{ij}^3(v)(x-u)^2(x-v)+c_{ij}^2(v)(x-u)^2+\ell_i(x),$$

where the coefficients are given by

$$c_{ij}^2(v)=\frac{\nabla_{ij}\ell(v)}{(v-u)^2}=\frac{(\beta_j-\beta_i)(v-d_{ij})}{(v-u)^2},\ \text{from the condition } Q_j(v)=0,$$
$$c_{ij}^3(v)=\frac{(\beta_j-\beta_i)(v-u)-2\nabla_{ij}\ell(v)}{(v-u)^3}=\frac{(\beta_j-\beta_i)(2d_{ij}-u-v)}{(v-u)^3},\ \text{from } Q_j^{'}(v)=0,$$

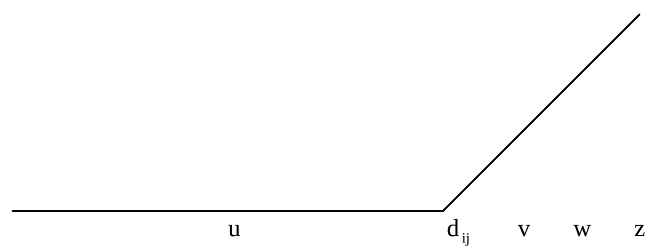
$$c_{ij}^4(v) = \frac{\nabla_{ij}\ell(w) - c_{ij}^2(v)(w-u)^2 - c_{ij}^3(v)(w-u)^2(w-v)}{(w-u)^2(w-v)^2}$$
$$= \frac{(\beta_j - \beta_i)\left[(u+v)w - 2u^2 - (2w+v-3u)d_{ij}\right]}{(w-u)^2},$$

from the condition $Q_j(w) = 0$. The last equality follows by calculation using the identities

$$(v-u)^3 + (w-u)^2(2w+u-3v) = (w-v)^2(2w+v-3u),$$

$$w(v-u)^3 + v(w-u)^2(2w+u-3v) - (v-u)(w-v)(w-u)^2$$
$$= w(v-u)^3 + (w-v)^2((u+v)w - 2v^2)$$
$$= (w-v)^2((u+v)w - 2u^2).$$

<u>Type (3)</u> : $z \in [w, \infty)$ is a further zero of $Q_j(x)$ as in the following figure :

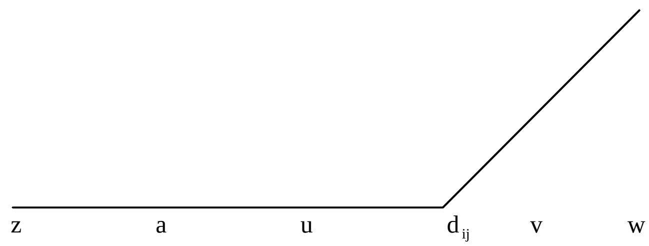


Solving the condition $q(z) = \ell_j(z)$ one finds

$$d_{ij} = \frac{\left[(u+v)w - 2u^2\right](z-u)^2(z-v)^2 + (w-u)^2\left\{\left[2v^2 - (u+v)z\right](z-u)^2 - (v-u)^3 z\right\}}{\left[2w+v-3u\right](z-u)^2(z-v)^2 + (w-u)^2\left\{\left[3v-u-2z\right](z-u)^2 - (v-u)^3\right\}}$$
$$= \frac{\left[(u+v)w - 2u^2\right](z+w-2u) - (u+v)(w-u)^2}{(2w+v-3u)(z+w-2u) - 2(w-u)^2}$$
$$= \frac{(w-u)(z-u)(u+v) + (v-u)\left[(z-u)u + (w-u)u\right]}{2(w-u)(z-u) + (v-u)(z+w-2u)},$$

from which the expression for $d_3(x)$ follows by setting $u = b^*$, $v = \varphi(b^*, b)$, w=b, $z = x \in [b, \infty)$.

<u>Type (4)</u> : $z \in (-\infty, a]$ is a further zero of $Q_i(x)$ as in the following figure :



Solving the condition $q(z) = \ell_i(z)$ one obtains

$$d_{ij} = \frac{[(u+v)w - 2u^2](v-z)^2 + (w-u)^2[2v^2 - (u+v)z]}{[2w + v - 3u](v-z)^2 + (w-u)^2[3v - u - 2z]}$$

$$= \frac{(v-z)^2[(v-u)u + (w-u)(u+v)] + (w-u)^2[(v-z)(u+v) + (v-u)v]}{(v-z)^2[(v-u) + 2(w-u)] + (w-u)^2[2(v-z) + (v-u)]},$$

from which $d_4(x)$ follows by setting $u = b^*$, $v = \varphi(b^*, b)$, w=b, $z = x \in (-\infty, a]$.

<u>Type (5)</u> : z=a is a further zero of $Q_i(x)$ as in the figure of type (4)

Setting z=a, u=x, $v = \psi(x; a, b)$, w=b in the formula of type (4), one obtains $d_5(x)$. From Theorem I.5.4 one knows that the extremal support is feasible provided $x \in [b^*, \varphi(a, a^*)]$.

<u>Case (III)</u> : $u \in I_i$ simple zero of $Q_i(x)$, $v \in I_i$ double zero of $Q_i(x)$, $w \in I_j$ double zero of $Q_j(x)$

By exchanging u and w as well as i and j, and a and b, this case is seen to be symmetric to case (II). Since $d_{ij} = d_{ji}$ one obtains by symmetry the unique q(x) with the required conditions as

$$q(x) = c_{ij}^4(u)(x-v)^2(x-w)^2 + c_{ij}^3(v)(x-w)^2(x-v) + c_{ij}^2(v)(x-w)^2 + \ell_j(x),$$

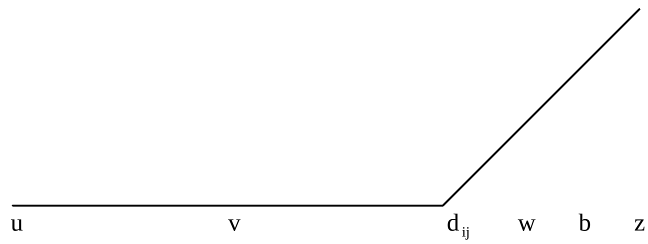
with the coefficients

$$c_{ij}^2(v) = \frac{(\beta_j - \beta_i)(d_{ij} - v)}{(w-v)^2},$$

$$c_{ij}^3(v) = \frac{(\beta_j - \beta_i)(2d_{ij} - v - w)}{(w-v)^3},$$

$$c_{ij}^4(v) = \frac{(\beta_j - \beta_i)[2w^2 - (v+w)u - (3w - v - 2u)d_{ij}]}{(w-u)^2}.$$

Type (6) : $z \in [b, \infty)$ is a further zero of $Q_j(x)$ as in the following figure :

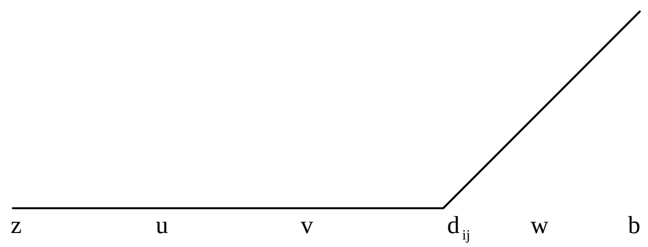


By symmetry to type (4) the condition $q(z) = \ell_j(z)$ yields

$$d_{ij} = \frac{(z-v)^2[(w-v)w + (w-u)(v+w)] + (w-u)^2[(z-v)(v+w) + (w-v)v]}{(z-v)^2[(w-v) + 2(w-u)] + (w-u)^2[2(z-v) + (w-v)]},$$

from which $d_6(x)$ follows by setting u=a, $v = \varphi(a, a^*)$, $w = a^*$, $z = x \in [b, \infty)$.

Type (7) : $z \in (-\infty, a]$ is a further zero of $Q_i(x)$ as in the following figure :



By symmetry to type (3) the condition $q(z) = \ell_i(z)$ yields

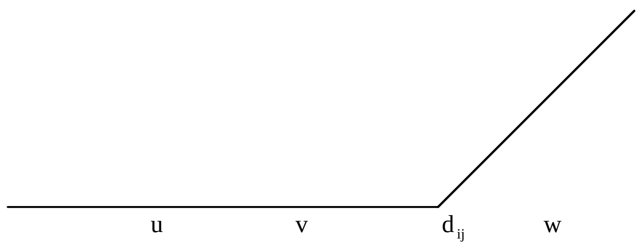
$$d_{ij} = \frac{(w-u)(w-z)(v+w) + (w-v)[(w-z)w + (w-u)w]}{2(w-u)(w-z) + (w-v)(2w-u-z)},$$

from which $d_7(x)$ follows by setting u=a, $v = \varphi(a, a^*)$, $w = a^*$, $z = x \in (-\infty, a]$.
Case (IV) : $u, v \in I_i$ double zeros of $Q_i(x)$, $w \in I_j$ simple zero of $Q_j(x)$

By exchanging u and w as well as i and j, and a and b, this case is seen to be symmetric to case (I). The unique q(x) with the required conditions is

$$q(x) = \frac{\nabla_{ij}\ell(w)}{(w-u)^2(w-v)^2}(x-u)^2(x-v)^2 + \ell_i(x).$$

Type (8) : w is a double zero of $Q_j(x)$ as in the following figure :

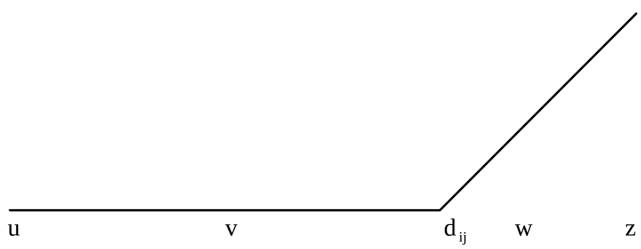


By symmetry to type (2), the condition $Q_j^{'}(w)=0$ yields

$$d_{ij}=\frac{(w-v)(u+w)+2(w-u)w}{2(w-v)+2(w-u)},$$

from which $d_8(x)$ follows by setting u=x, $v=\varphi(x,x^*)$, $w=x^*$. The extremal support is feasible provided $x\in[a,b^*]$.

Type (9) : $z\in[w,\infty)$ is a further zero of $Q_j(x)$ as in the following figure :



By symmetry to type (1) the condition $q(z)=\ell_j(z)$ yields the formula

$$d_{ij}=\frac{(z-u)^2(w+z-2v)w+(w-v)^2[(z-u)w+(w-u)u]}{(z-u)^2(w+z-2v)+(w-v)^2(z+w-2u)},$$

from which $d_9(x)$ follows by setting $u=b^*$, $v=\varphi(b^*,b)$, w=b, $z=x\in[b,\infty)$. ◊

It is instructive and useful to state the maximal stop-loss transforms for the limiting ranges $[a,\infty)$ and $(-\infty,\infty)$. The formulas simplify considerably, especially in the case $(-\infty,\infty)$, and are thus quite tractable in applications.

Proof of Table 5.3'. The same method as in the proof of Table 5.3 applies. The proof simplifies considerably due to the fact that the only feasible extremal support is $\{x,\varphi(x,x^*),x^*\}$, where two cases can occur according to whether $\varphi(x,x^*)\geq d$ or not. Applicable are thus only the types (2) and (8) in Table 5.3. Case (2) applies with the parameter range $(-\infty,c]$ and an equivalent formula for $\pi^*(d(x))$ obtained from the fact that $p_x^{(3)} x + p_{\varphi(x,x^*)}^{(3)} \varphi(x,x^*) + p_{x^*}^{(3)} x^* = 0$. In case (8) one changes x to x^*. By the involution property, the new parameter range is then $[\bar{c}, \infty)$. ◊

Table 5.3' : maximum stop-loss transform on $(-\infty,\infty)$ by known skewness and kurtosis

case	range of parameter	range of deductible	$\pi^*(d(x))$	extremal support
(1)	$x \le c$	$d(x) \le \frac{1}{2}\gamma$	$-d(x) + p_x^{(3)}(d(x) - x)$	$\{x, \varphi(x,x^*), x^*\}$
(2)	$x \ge \bar{c}$	$d(x) \ge \frac{1}{2}\gamma$	$p_x^{(3)}(x - d(x))$	$\{x^*, \varphi(x^*,x), x\}$

The monotone increasing deductible function is defined by the weighted average

$$d(x) = \frac{[\varphi(x,x^*) - x](x + x^*) + 2(x^* - x)x}{2[\varphi(x,x^*) - x] + 2(x^* - x)}.$$

Table 5.3'' : maximum stop-loss transform on $[a,\infty)$ by known skewness and Kurtosis

case	range of parameter	$\pi^*(d_i(x))$	extremal support
(1)	$x \le a$	$p_{\varphi(a,a^*)}^{(3)}(\varphi(a,a^*) - d_1(x)) + p_{a^*}^{(3)}(a^* - d_1(x))$	$\{a, \varphi(a,a^*), a^*\}$
(2)	$a \le x \le c$	$p_{\varphi(x,x^*)}^{(3)}(\varphi(x,x^*) - d_2(x)) + p_{x^*}^{(3)}(x^* - d_2(x))$	$\{x, \varphi(x,x^*), x^*\}$
(3)	$x \le a$	$p_{\bar{c}}^{(3)}(\bar{c} - d_3(x))$	$\{c, \bar{c}, \infty\}$
(4)	$c \le x \le \varphi(a,a^*)$	$p_{\varphi(a,x)}^{(3)}(\varphi(a,x) - d_4(x))$	$\{a, x, \varphi(a,x), \infty\}$
(5)	$x \le a$	$p_{a^*}^{(3)}(a^* - d_5(x))$	$\{a, \varphi(a,a^*), a^*\}$
(6)	$a \le x \le c$	$p_{x^*}^{(3)}(x^* - d_6(x))$	$\{x, \varphi(x,x^*), x^*\}$

The monotone increasing deductible functions are defined by the following "weighted averages" :

$$d_1(x) = \frac{(a^* - x)^2[2\varphi - a - x]a + (\varphi - a)^2[(a^* - x)a + (a^* - a)a^*]}{(a^* - x)^2[2\varphi - a - x] + (\varphi - a)^2[2a^* - a - x]}, \quad \varphi = \varphi(a,a^*)$$

$$d_2(x) = \frac{1}{2}\left\{\frac{[\varphi(x,x^*)-x](x+x^*)+2(x^*-x)x}{[\varphi(x,x^*)-x]+(x^*-x)}\right\}$$

$$d_3(x) = \frac{(\bar{c}-c)x+2(\bar{c}-x)\bar{c}}{(\bar{c}-c)+2(\bar{c}-x)}$$

$$d_4(x) = \frac{2(\varphi(a,x)-a)\varphi(a,x)+(\varphi(a,x)-x)a}{2(\varphi(a,x)-a)+(\varphi(a,x)-x)}$$

$$d_5(x) = \frac{(a^*-a)(a^*-x)(\varphi(a^*,a)+a^*)+(a^*-\varphi(a^*,a))(2a^*-a-x)a^*}{2(a^*-a)(a^*-x)+(a^*-\varphi(a^*,a))(2a^*-a-x)}$$

$$d_6(x) = \frac{1}{2}\left\{\frac{(x^*-x)[\varphi(x,x^*)+x]+2(x^*-\varphi(x,x^*))\varphi(x,x^*)}{(x^*-x)+(x^*-\varphi(x,x^*))}\right\}$$

Proof of Table 5.3''. We let $b \to \infty$ in Table 5.3. The cases (3), (6) and (9) are inapplicable. The cases (1) and (7) apply without change. With the aid of formula (5.8) in Theorem I.5.3 , one shows that $\lim_{x\to\infty} x^* = c$, $\lim_{x\to\infty} \varphi(x,x^*) = \bar{c}$. Therefore in case (2) the parameter range $[a,b^*]$ must be replaced by $[a,c]$. In case (4) the extremal support $\{b^*,\varphi(b^*,b),b\}$ must be replaced by the limiting triatomic support $\{c,\bar{c},\infty\}$. Using that $\lim_{b\to\infty} \psi(x;a,b) = \varphi(a,x)$, the extremal support $\{a,x,\psi,b\}$ in case (5) must be replaced by the limiting support $\{a,x,\varphi(a,x),\infty\}$. Again the limiting random variables have to satisfy the kurtosis constraint (see proof of Table 4.3"). ◊

6. Extremal stop-loss transforms for symmetric random variables by known kurtosis.

As seen in Section I.6, it suffices to restrict ourselve to the case of standard symmetric random variables with range $[-a,a]$, $a \geq 1$, and kurtosis $\gamma_2 = \delta - 3$. Furthermore the stop-loss transform of a symmetric random variable satisfies the relation $\pi(d) = -d + \pi(-d)$, hence only the case of a non-negative deductible $d \geq 0$ has to be discussed.

Theorem 6.1. The maximal stop-loss transform for standard symmetric random variables with range $[-a,a]$, $a \geq 1$, and kurtosis $\gamma_2 = \delta - 3$ is determined in Table 6.1.

Examples 6.1.

The different dependence upon kurtosis by varying deductible is illustrated through straightforward calculation.

(i) If $a=3$, $\delta=3$, one obtains from case (1) in Table 6.1 that

$$\pi^*(0) = \frac{1}{2}\cdot\left(\frac{20+7\sqrt{3}}{12+13\sqrt{3}}\right) \approx 0.465.$$

If a=3, δ=6, one obtains similarly

$$\pi^*(0) = \frac{1}{6} \cdot \left(\frac{50 + 33\sqrt{\frac{3}{2}}}{12 + 25\sqrt{\frac{3}{2}}} \right) \approx 0.354 .$$

It is a bit surprising that for the higher kurtosis 6 the maximal stop-loss transform value is closer to the stop-loss transform value $1/\sqrt{2\pi} \approx 0.399$ of a standard normal. However it is known from other risk-theoretical situations that the stop-loss transform at this deductible is often well approximated by $1/\sqrt{2\pi}$ (e.g. Benktander(1977), Hürlimann(1996a)).

(ii) From case (4) one gets immediately that $\pi^*(\frac{1}{2}\sqrt{\delta}) = \frac{1}{4\sqrt{\delta}}$, $\pi^*(\frac{3}{4}\sqrt{\delta}) = \frac{1}{8\sqrt{\delta}}$. If δ=3 one has $\pi^*(\frac{3}{4}\sqrt{3}) = \frac{1}{8\sqrt{3}} \approx 0.072$ while if $\delta = \frac{27}{4}$ one has $\pi^*(\frac{3}{4}\sqrt{3}) = \frac{1}{6\sqrt{3}} \approx 0.096$. For the same deductible, one obtains a higher maximal stop-loss transform by higher kurtosis. This is opposite to the behaviour observed in situation (i).

Table 6.1 : maximal stop-loss transform for standard symmetric random variables by known kurtosis

case	condition	deductible $d_i(x)$	maximum $\pi^*(d_i(x))$	extremal support
(1)	$0 \le x \le \frac{aa^s(a+2a^s)}{2(a+a^s)^2}$	x	$\frac{1-2xa^s}{4a^s} + \frac{(a^s)^2(2+a^2+2aa^s)-\delta}{4a^s(a+a^s)^2}$	$\{-a,-a^s,a^s,a\}$
(2)	$0 \le x \le \delta-(a^s)^2$	$\frac{a\sqrt{\delta-x}(a+2\sqrt{\delta-x})}{2(a+\sqrt{\delta-x})^2}$	$\frac{(a+\sqrt{\delta-x})^2+\delta-2x}{4(a+\sqrt{\delta-x})^2}$	$\{-a,-\sqrt{\delta-x}, 0,\sqrt{\delta-x},a\}$
(3)	$x \ge a$	$\frac{\sqrt{\delta}x(2\sqrt{\delta}+x)}{2(\sqrt{\delta}+x)^2}$	$\frac{\sqrt{\delta}-d_3(x)}{2\delta}$	$\{-\sqrt{\delta},0,\sqrt{\delta}\}$
(4)	$\frac{1}{2} \le x \le \frac{3}{4}$	$\sqrt{\delta}\cdot x$	$\frac{1-x}{2\sqrt{\delta}}$	$\{-\sqrt{\delta},0,\sqrt{\delta}\}$
(5)	$\sqrt{\delta} \le x \le a$	$\frac{(x^s)^2+3x^2}{4x}$	$\frac{(x^s)^4-2(x^s)^2+\delta}{8x(x^2-(x^s)^2)}$	$\{-x,-x^s,x^s,x\}$
(6)	$x \ge a$	$\frac{1}{(x+a)}\cdot\left\{ax+\frac{a^2x^2-(a^s)^4}{(x^2+a^2-2(a^s)^2)}\right\}$	$\frac{(a^s)^4-2(a^s)^2+\delta}{2(a^2-(a^s)^2)}\cdot\{a-d_6(x)\}$	$\{-a,-a^s,a^s,a\}$

Proof of Theorem 6.1. In each case one constructs a symmetric biquadratic polynomial majorant q(x) such that

$$q(x) \ge (x-d)_+ + (-x-d)_+, \quad x \in [-a,a],$$

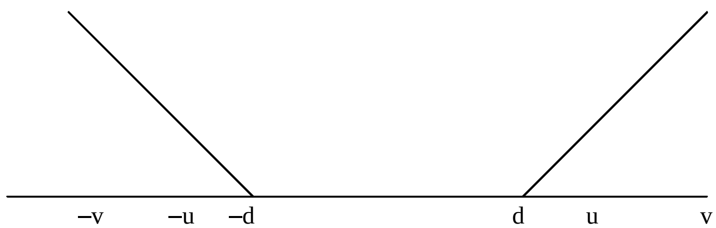
and where equality is attained at the atoms of the displayed extremal support. The possible types of extremal supports are taken from Theorem I.6.1. The maximal stop-loss transform is then given by

$$\pi^*(d) = \max_{X \in D_S(a;\delta)} E\left[\tfrac{1}{2}(X-d)_+ + \tfrac{1}{2}(-X-d)_+\right] = \tfrac{1}{2}E[q(X)],$$

as seen in Lemma II.6.1. Proofs of the several cases in Table 6.1 are now given.

Case (1) :

Setting $u = a^s, v = a$ one constructs q(x) such that $q(u) = q(-u) = u - d$, $q'(u) = -q'(-u) = 1$, $q(v) = q(-v) = v - d$ as in the following figure :



It is appropriate to consider the biquadratic form

$q(x) = A(x^2 - u^2)^2 + B(x^2 - u^2) + C$, with first derivative $q'(x) = 2x \cdot \{2A(x^2 - u^2) + B\}$.

The required conditions imply that the coefficients are equal to

$$C = u - d, \quad B = \frac{1}{2u}, \quad A = -\frac{1}{2u(u+v)^2}.$$

Through calculation one obtains

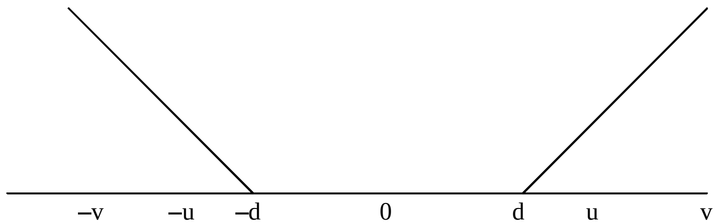
$$q(x) = \frac{1}{2u(u+v)^2} \cdot \left\{x^2\left[(u+v)^2 + 2u^2\right] - x^4 + u^2 v(v+2u) - 2du(u+v)^2\right\}.$$

Since q'(0)=0, $q''(0) = \dfrac{(u+v)^2 + 2u^2}{u(u+v)^2} > 0$, the polynomial q(x) is local minimal at x=0. A necessary condition for $q(x) \ge 0$ on the interval $[-d,d]$ is thus $q(0) \ge 0$. This implies the condition

$$0 \le d \le \frac{uv(v+2u)}{2(u+v)^2}.$$

Case (2) :

Set $u = u(x) = \sqrt{\delta - x}$, $x \in \left[0, \delta - (a^s)^2\right]$, $v = a$ and construct $q(x)$ such that $q(u) = q(-u) = u - d$, $q'(u) = -q'(-u) = 1$, $q(v) = q(-v) = v - d$, $q(0) = q'(0) = 0$ as in the following figure :



From case (1) one obtains that $q(x)$ must be of the form

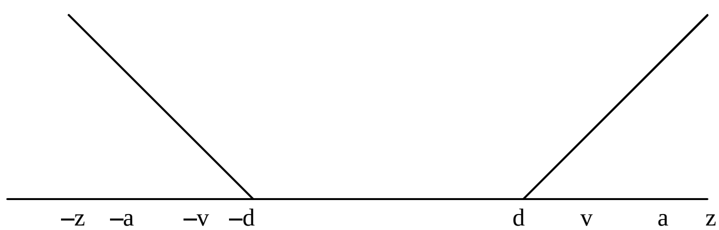
$$q(x) = \frac{1}{2u(u+v)^2} \cdot \left\{ x^2\left[(u+v)^2 + 2u^2\right] - x^4 + u^2 v(v+2u) - 2du(u+v)^2 \right\}.$$

The additional constraint $q'(0)=0$ is fulfilled, while $q(0)=0$ implies that the deductible function equals

$$d = d(x) = \frac{u(x)v(v+2u(x))}{2(u(x)+v)^2}.$$

Cases (3) and (4) :

Set $u = 0$, $v = \sqrt{\delta}$ and construct $q(x)$ such that $q(0) = q'(0) = 0$, $q(v) = q(-v) = v - d$, $q'(v) = -q'(-v) = 1$ as in the following figure (valid for case (3)) :



Construct $q(x)$ of the form

$q(x) = A(x^2 - v^2)^2 + B(x^2 - v^2) + C$, with first derivative $q'(x) = 2x \cdot \left\{ 2A(x^2 - v^2) + B \right\}$.

The required conditions imply that the coefficients are equal to

$$C = v - d, \quad B = \frac{1}{2v}, \quad A = -\frac{2d - v}{2v^4}.$$

Inserting these coefficients one finds the polynomial

$$q(x) = \frac{x^2}{2v^4}\left\{(2d - v)x^2 + (3v - 4d)v^2\right\}.$$

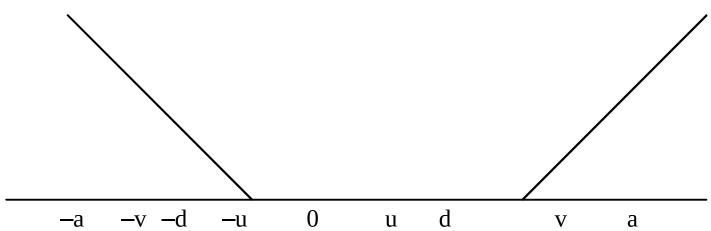
If $2d - v < 0$ one has $q(x) \to -\infty$ as $x \to \pm\infty$. In this situation there exists $z \geq a$ such that $q(z) = q(-z) = z - d$, hence

$$d = d(z) = \frac{vz(z + 2v)}{2(z + v)^2}, \quad z \geq a.$$

This settles case (3). If $2d - v \geq 0$ one must have $q(x) \geq 0$ for all x. This condition implies that the discriminant of the quadratic polynomial set in curly bracket in the expression of $q(x)$, which equals $4(2d - v)(4d - 3v)$, must be non-positive. It follows that in this situation $\frac{1}{2}v \leq d \leq \frac{3}{4}v$, which settles case (4).

Case (5) :

Set $u = u(x) = x^s$, $v = v(x) = x$, $x \in \left[\sqrt{\delta}, a\right]$, and construct $q(x)$ such that $q(u) = q(-u) = u - d$, $q'(u) = q'(-u) = 0$, $q(v) = q(-v) = v - d$, $q'(v) = -q'(-v) = 1$ as in the following figure :



It is immediate that

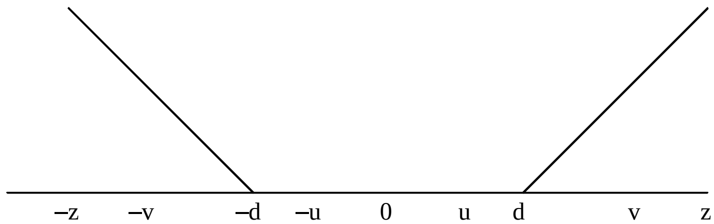
$$q(x) = \frac{(x^2 - u^2)^2}{4v(v^2 - u^2)}$$

satisfies the conditions $q(u) = q(-u) = u - d$, $q'(u) = q'(-u) = 0$, $q'(v) = -q'(-v) = 1$. The remaining condition $q(v) = q(-v) = v - d$ implies that

$$d = \frac{u^2 + 3v^2}{4v}.$$

<u>Case (6)</u> :

Setting $u = a^s, v = a$ one constructs q(x) such that $q(u) = q(-u) = u - d$, $q'(u) = q'(-u) = 0$, $q(v) = q(-v) = v - d$, as in the following figure :



A biquadratic polynomial, which satisfies the given conditions, is given by

$$q(x) = \frac{(x^2 - u^2)^2 (v - d)}{v^2 - u^2} .$$

Since $q(x) \to -\infty$ as $x \to \pm\infty$ and v is not a double zero of $q(x) - (x - d)$, there exists a $z \geq a$ such that $q(z) = q(-z) = z - d$. Solving this condition yields after an elementary but labourious calculation the deductible function

$$d = d(z) = \frac{(z^2 + v^2 - 2u^2) vz + v^2 z^2 - u^4}{(v + z)(z^2 + v^2 - 2u^2)}, \quad z \geq a. \ \Diamond$$

Theorem 6.2. The minimal stop-loss transform for standard symmetric random variables with range $[-a, a]$, $a \geq 1$, and kurtosis $\gamma_2 = \delta - 3$ is determined in Table 6.2.

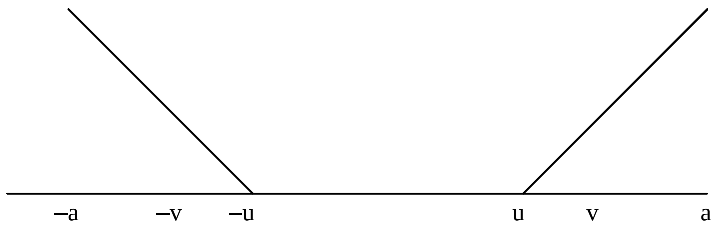
Table 6.2 : minimal stop-loss transform for standard symmetric random variables by known kurtosis

case	condition	minimum $\pi_*(x)$	extremal support
(1)	$0 \leq x \leq a^s$	$\frac{1}{2} \cdot \left(\frac{1 - x^2}{x + x^s} \right)$	$\{-x^s, -x, x, x^s\}, x^s = \sqrt{\frac{\delta - x^2}{1 - x^2}}$
(2)	$a^s \leq x \leq \sqrt{\delta}$	$\frac{1}{2a^2} \cdot \left(\frac{\delta - x^2}{a + x} \right)$	$\{-a, -x, 0, x, a\}$
(3)	$\sqrt{\delta} \leq x \leq a$	0	$\{-x, -x^s, x^s, x\}, x^s = \sqrt{\frac{x^2 - \delta}{x^2 - 1}}$

Proof. We proceed as for Table 6.1 and construct in each case a symmetric biquadratic polynomial minorant $q(x) \le (x-d)_+ + (-x-d)_+$, $x \in [-a,a]$, for which equality is attained at the atoms of the displayed extremal support.

Case (1) :

Setting $v = d^s, u = d \in [0, a^s]$ one constructs q(x) such that $q(u) = q(-u) = 0$, $q(v) = q(-v) = v - d$, $q'(v) = -q'(-v) = 1$ as in the following figure :



Consider the biquadratic form

$q(x) = A(x^2 - v^2)^2 + B(x^2 - v^2) + C$, with first derivative $q'(x) = 2x \cdot \{2A(x^2 - v^2) + B\}$.

The required conditions imply that the coefficients are equal to

$$C = v - d, \quad B = \frac{1}{2v}, \quad A = -\frac{1}{2v(u+v)^2},$$

and q(x) is a minorant. The minimum stop-loss transform equals

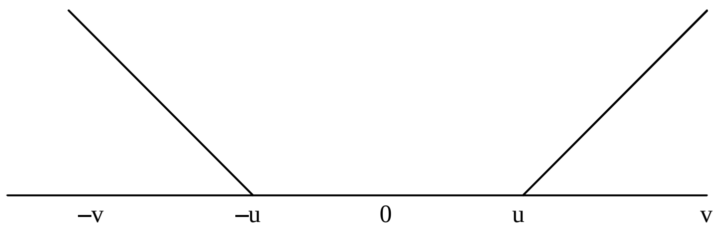
$$\pi_*(d) = p_{d^s}^{(4)} \cdot (d^s - d) = \frac{1}{2} \cdot \frac{(1-d^2)^2}{\delta - 1 + (1-d^2)^2} \cdot \left(\sqrt{\frac{\delta - d^2}{1-d^2}} - d \right)$$

$$= \frac{1}{2} \cdot \frac{(1-d^2)^2}{(\delta - d^2) - d^2(1-d^2)} \cdot \left(\frac{\sqrt{\delta - d^2} - d\sqrt{1-d^2}}{\sqrt{1-d^2}} \right) = \frac{1}{2} \cdot \frac{(1-d^2)^{\frac{3}{2}}}{\sqrt{\delta - d^2} + d\sqrt{1-d^2}}$$

$$= \frac{1}{2} \cdot \left(\frac{1-d^2}{d + d^s} \right)$$

Case (2) :

Set $v = a$, $u = d \in [a^s, \sqrt{\delta}]$, and construct q(x) such that $q(u) = q(-u) = 0$, $q(v) = q(-v) = v - d$, $q(0) = q'(0) = 0$ as in the following figure :



Consider the biquadratic form

$q(x) = A(x^2 - u^2)^2 + B(x^2 - u^2) + C$, with first derivative $q'(x) = 2x \cdot \{2A(x^2 - u^2) + B\}$.

The required conditions imply that the coefficients are equal to

$$C = 0, \quad B = \frac{u^2}{u^2(u+v)}, \quad A = \frac{1}{u^2(u+v)},$$

and $q(x)$ is a minorant. The minimum stop-loss transform equals

$$\pi_*(d) = p_d^{(5)} \cdot (a-d) = \frac{1}{2} \cdot \frac{\delta - d^2}{a^2(a^2 - d^2)} \cdot (a-d) = \frac{1}{2a^2} \cdot \left(\frac{\delta - d^2}{a+d} \right).$$

Case (3) :

The symmetric random variable with support $\{-d, -d^s, d^s, d\}, d \in [\sqrt{\delta}, a]$, is feasible and $q(x) \equiv 0$ is clearly a minorant, hence $\pi_*(d) = 0$. ◊

7. Notes.

The historical origin of the Chebyshev-Markov inequalities dates back to Chebyshev(1874), who has first formulated this famous problem and has proposed also a solution without proof, however. Proofs were later given by Markov(1884), Possé(1886) and Stieltjes(1883/1894/95). Twentieth century developments include among others Uspensky(1937), Shohat and Tamarkin(1943), Royden(1953), Krein(1951), Karlin and Studden(1966) and Krein and Nudelman(1977). A short account is also found in Whittle(1971), p.110-118. It seems that the Chebyshev-Markov inequalities have been stated in full generality for the first time by Zelen(1954). Explicit analytical results for moments up to order four have been given in particular by Zelen(1954), Simpson and Welch(1960), and Kaas and Goovaerts(1986a). Our method, which uses the complete algebraic-analytical structure of the lower dimensional sets of finite atomic random variables derived in Chapter I, improves in this point the mentioned previous works. For an introduction into the vast subject entitled "inequalities of Chebyshev type", the reader is recommended Karlin and Studden(1966), chap. XII to XIV.

The determination of the maximal stop-loss transforms by given moments up to the order four is based on Jansen et al.(1986), which has been the completion of the work started by Bowers(1969), DeVylder and Goovaerts(1982), and Mack(1984/85). Our improvements concern mainly the analytical unified presentation. In particular the formulas are made more explicit with the introduction of tractable "deductible functions".